Cusp Anomalous Dimension in Maximally Supersymmetric Yang-Mills Theory at Strong Coupling

B. Basso,[1] G. P. Korchemsky,[1] and J. Kotański[1,2]

[1]*Laboratoire de Physique Théorique, Université de Paris XI and CNRS (UMR 8627), 91405 Orsay Cédex, France*
[2]*Institute of Physics, Jagellonian University, Reymonta 4, PL-30-059 Cracow, Poland*

(Received 30 August 2007; published 3 March 2008)

We construct an analytical solution to the integral equation which is believed to describe logarithmic growth of the anomalous dimensions of high-spin operators in planar $\mathcal{N}=4$ super Yang-Mills theory and use it to determine the strong coupling expansion of the cusp anomalous dimension.

DOI: 10.1103/PhysRevLett.100.091601 PACS numbers: 11.15.Me, 11.25.Tq, 11.30.Pb

1. Introduction.—The cusp anomalous dimension is an important observable in four dimensional gauge theories ranging from QCD to maximally supersymmetric $\mathcal{N}=4$ Yang-Mills theory (SYM) since it governs the scaling behavior of various gauge invariant quantities like logarithmic growth of the anomalous dimensions of high-spin Wilson operators, Sudakov asymptotics of elastic form factors, the gluon Regge trajectory, infrared singularities of on-shell scattering amplitudes, etc. By definition [1], $\Gamma_{\rm cusp}(g)$ measures the anomalous dimension of a Wilson loop evaluated over a closed contour with a lightlike cusp in Minkowski space-time. It is a function of the gauge coupling only, and its expansion at weak coupling is known in QCD to three loops [2] and in $\mathcal{N}=4$ SYM theory to four loops [3]. Recently, significant progress has been achieved in determining $\Gamma_{\rm cusp}(g)$ at strong coupling in planar $\mathcal{N}=4$ SYM. Within the AdS/CFT correspondence [4], $\Gamma_{\rm cusp}(g)$ at strong coupling is related to the semiclassical expansion of the energy of folded string rotating in AdS_3 part of the target space [5] (see also [6])

$$\Gamma_{\rm cusp}(g) = 2g - \frac{3\ln 2}{2\pi} + O(1/g), \qquad g = \frac{\sqrt{\lambda}}{4\pi}, \tag{1}$$

with $\lambda = g^2_{\rm YM}N_c$ being 't Hooft coupling. On the gauge theory side, the Bethe ansatz approach to calculating $\Gamma_{\rm cusp}(g)$ in the weak coupling limit was developed in [7] based on integrability symmetry of planar Yang-Mills theory to one loop [8]. This approach was recently extended to all loops in planar $\mathcal{N}=4$ SYM theory. Comparing integrable structures present on both sides of the AdS/CFT correspondence, an all-loop asymptotic Bethe ansatz was proposed in [9]. It involves a nontrivial scattering phase satisfying the crossing symmetry [10] whose explicit form was found in [11]. This led to an integral equation for the all-loop cusp anomalous dimension [12,13], the Beisert-Eden-Staudacher (BES) equation,

$$\hat\sigma(t) = \frac{t}{e^t-1}\left[K(2gt,0) - 4g^2\int_0^\infty dt' K(2gt,2gt')\hat\sigma(t')\right], \tag{2}$$

with $\Gamma_{\rm cusp}(g) = 8g^2\hat\sigma(0)$. Here, the kernel $K(t,t')$ is expressed in terms of Bessel functions, $K(t,t') = \sum_{n,m=1}^\infty z_{nm}(g)J_n(t)J_m(t')/(tt')$, and its explicit form can be found in [12]. At weak coupling, the Neumann series solution to (2) yields perturbative expansion of $\Gamma_{\rm cusp}(g)$ in powers of g^2 which agrees with the known four-loop result [3]. At intermediate coupling, Eq. (2) was solved numerically in [14]. The obtained solution for $\Gamma_{\rm cusp}(g)$ was found to be a smooth function of g that matches for $g>1$ the string theory prediction (1) with high accuracy. Moreover, an exact analytical solution to Eq. (2) in the limit $g\to\infty$ was recently constructed in [15] leading to $\Gamma_{\rm cusp}(g) = 2g + O(g^0)$, in agreement with (1) and with the numerical analysis of [14]. Equation (2) was analyzed further in [16], but it has resisted an analytical solution so far. In parallel development, the computation of the two-loop $O(1/g)$ corrections to the string theory prediction (1) was initiated in [17]. Also, the result (1) was reproduced [18] from the quantum string Bethe ansatz for a folded string.

In this Letter, we describe an approach to finding a strong coupling expansion of the solution to Eq. (2). It allows us to determine exact analytical expressions for the coefficients in the $1/g$-expansion of the cusp anomalous dimension (1) to any desired order.

Let us introduce two functions $\gamma_\pm(-t) = \pm\gamma_\pm(t)$

$$\frac{e^t-1}{t}\hat\sigma(t) = \frac{\gamma_+(2gt)}{2gt} + \frac{\gamma_-(2gt)}{2gt}. \tag{3}$$

Following [15], we expand $\gamma_\pm(t)$ over the Bessel functions

$$\begin{aligned}\gamma_+(t) &= \sum_{k\geq 1}(-1)^{k+1}(2k)J_{2k}(t)\gamma_{2k},\\ \gamma_-(t) &= \sum_{k\geq 1}(-1)^{k+1}(2k-1)J_{2k-1}(t)\gamma_{2k-1},\end{aligned} \tag{4}$$

with the expansion coefficients $\gamma_k \sim \int_0^\infty dt'/t' J_k(t')\gamma_\sigma(t')$ ($\sigma = +/-$ for $k=$ even or odd). Substituting (3) into Eq. (2) and separating even or odd in t parts, we find that (2) is equivalent to the (infinite) system of equations

$$\begin{aligned}\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+(t)}{1-e^{-t/2g}} - \frac{\gamma_-(t)}{e^{t/2g}-1}\right]J_{2n}(t) &= 0,\\ \int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t)}{1-e^{-t/2g}} + \frac{\gamma_+(t)}{e^{t/2g}-1}\right]J_{2n-1}(t) &= \frac{1}{2}\delta_{n,1},\end{aligned} \tag{5}$$

with $n\geq 1$. The cusp anomalous dimension can be read from small-t expansion $\gamma_-(t) = t\Gamma_{\rm cusp}(g)/(8g^2) + O(t^2)$. At weak coupling, one finds from (5) that $\gamma_-(t) = J_1(t) + O(g^2)$

leading to $\Gamma_{\rm cusp}(g) = 4g^2 + O(g^4)$ in agreement with the known one-loop result [1].

2. Exact solution.—The system (5) has the following remarkable property. Introducing two functions $\Gamma_\pm(t) = \gamma_\pm(t) \mp \gamma_\mp(t)\coth\frac{t}{4g}$, or equivalently

$$2\gamma_\pm(t) = \left[1 - \mathrm{sech}\frac{t}{2g}\right]\Gamma_\pm(t) \pm \tanh\frac{t}{2g}\Gamma_\mp(t), \qquad (6)$$

we find from (5)

$$\int_0^\infty \frac{dt}{t}\left[\Gamma_-(t) + (-1)^n\Gamma_+(t)\right]J_n(t) = \delta_{n,1}, \qquad (7)$$

(with $n \geq 1$), and the cusp anomalous dimension is now given by $\Gamma_{\rm cusp}(g) = -2g\Gamma_+(0)$. Here, in comparison with (5), the dependence on g only resides in $\Gamma_\pm(t)$.

At large g, we expect from (6) that the functions $\Gamma_\pm(t)$ admit expansion in the Bessel function Neumann series

$$\begin{aligned}\Gamma_+(t) &= \sum_{k\geq 0}(-1)^{k+1}J_{2k}(t)\Gamma_{2k},\\ \Gamma_-(t) &= \sum_{k\geq 0}(-1)^{k+1}J_{2k-1}(t)\Gamma_{2k-1}.\end{aligned} \qquad (8)$$

In distinction to (4), the first series involves $J_0(t)$ term which ensures that $\Gamma_+(0) \neq 0$. Also, in virtue of $J_{-1}(t) = -J_1(t)$, the coefficient in front of $J_1(t)$ is given by $(\Gamma_1 + \Gamma_{-1})$ so that it is only the sum that is uniquely defined. We make use of this ambiguity to choose $\Gamma_{-1} = 1$.

Substitution of (8) into (7) yields an infinite system of finite-difference equations for the coefficients Γ_k. Applying standard methods, we were able to construct its solution for Γ_k (with $k \geq -1$) in the following form (detailed analysis will be published elsewhere)

$$\Gamma_k = -\frac{1}{2}\Gamma_k^{(0)} + \sum_{p=1}^\infty \frac{1}{g^p}\left[c_p^-\Gamma_k^{(2p-1)} + c_p^+\Gamma_k^{(2p)}\right], \qquad (9)$$

where $\Gamma_k^{(p)}$ are basis functions independent on g

$$\Gamma_{2m}^{(p)} = \frac{\Gamma(m+p-\frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2})}, \qquad \Gamma_{2m-1}^{(p)} = \frac{(-1)^p\Gamma(m-\frac{1}{2})}{\Gamma(m+1-p)\Gamma(\frac{1}{2})}, \qquad (10)$$

and the expansion coefficients $c_p^\pm$ given by series in inverse powers of the coupling, $c_p^\pm = \sum_{r\geq 0} g^{-r}c_{p,r}^\pm$. The sum over p in the r.h.s. of (9) describes the contribution of zero modes of (7). Their dependence on g is fixed by the additional condition of scaling behavior of γ_k (see Eqs. (15) and (16) below). Knowing the $c_p^\pm$-coefficients, we can determine the cusp anomalous dimension $\Gamma_{\rm cusp}(g) = -2g\Gamma_+(0) = 2g\Gamma_0$ as

$$\Gamma_{\rm cusp}(g) = 2g + \sum_{p=1}^\infty \frac{1}{g^{p-1}}\left[\frac{2c_p^-}{\sqrt{\pi}}\Gamma\left(2p - \frac{3}{2}\right) + \frac{2c_p^+}{\sqrt{\pi}}\Gamma\left(2p - \frac{1}{2}\right)\right]. \qquad (11)$$

Let us now establish the relation between the coefficients Γ_n and γ_n. To this end, we return to the relation (6) and apply the identities

$$\begin{aligned}\mathrm{sech}\,t - 1 &= \sum_{n\geq 1}(-1)^n a_{2n}t^{2n},\\ \tanh t &= \sum_{n\geq 1}(-1)^n a_{2n-1}t^{2n-1},\end{aligned} \qquad (12)$$

where a-coefficients with even (odd) indices are related to the Euler (Bernoulli) numbers. Replacing $\gamma_\pm(t)$ and $\Gamma_\pm(t)$ in (6) by the series (4) and (8), respectively, we make use of the Bessel function series for $(t/2)^pJ_m(t)$ to obtain

$$\begin{aligned}\gamma_{2m} &= \sum_{n=1}^m \sum_{j=0}^{m-n+1}\left[\Gamma_{2j-1}K_{2m,2j-1}^{2n-1} + \Gamma_{2j}K_{2m,2j}^{2n}\right],\\ \gamma_{2m-1} &= \sum_{n=1}^m \sum_{j=0}^{m-n}\left[\Gamma_{2j-1}K_{2m-1,2j-1}^{2n} + \Gamma_{2j}K_{2m-1,2j}^{2n-1}\right].\end{aligned} \qquad (13)$$

Here, the notation was introduced for the coefficients

$$K_{m,j}^n = -\frac{a_n/g^n}{2\Gamma(n)}\frac{\Gamma(\frac{1}{2}(m+j+n))\Gamma(\frac{1}{2}(m-j+n))}{\Gamma(\frac{1}{2}(m+j-n)+1)\Gamma(\frac{1}{2}(m-j-n)+1)}, \qquad (14)$$

Replacing Γ_j in (13) by their explicit expressions, Eqs. (9) and (10), we express γ_{2m} and γ_{2m-1} in terms of yet unknown coefficients $c_p^\pm$.

3. Quantization conditions.—In our approach, the coefficients $c_p^\pm$ are determined from the behavior of γ_{2m} and γ_{2m-1}, Eq. (13), at large m. To this end, we introduce the functions $z_\pm(x) \equiv \gamma_{2m-1} \pm \gamma_{2m}$ and examine their asymptotic behavior in the double-scaling limit

$$m, g \to \infty, \qquad x = \left(m - \frac{1}{4}\right)^2/g = \text{fixed}. \qquad (15)$$

Employing the approach of [14] and going through numerical analysis of $z_\pm(x)$, we found that in the limit (15), the solutions to (2) have the following remarkable scaling behavior

$$\begin{aligned}z_+(x) &= \frac{(gx)^{-1/4}}{g\sqrt{\pi}}\left[z_+^{(0)}(x) + \frac{z_+^{(1)}(x)}{gx} + O(1/g^2)\right],\\ z_-(x) &= \frac{(gx)^{-3/4}}{4g\sqrt{\pi}}\left[z_-^{(0)}(x) + \frac{z_-^{(1)}(x)}{gx} + O(1/g^2)\right],\end{aligned} \qquad (16)$$

where the functions $z_\pm^{(r)}(x)$ (with $r \geq 0$) do not depend on g and have faster-than-power decrease at large x. For $x \to 0$, small-x expansion of $z_\pm^{(r)}(x)$ runs in *integer positive* powers of x only. For $x \to \infty$, asymptotic behavior of $z_\pm^{(r)}(x)$ is controlled by the coefficients $c_p^\pm$. The quantization conditions for $c_p^\pm$ follow from the requirement $\int_0^\infty dx x^p z_+^{(r)}(x) =$ finite for any given $p, r \geq 0$.

Let us start with the leading term $z_\pm^{(0)}(x)$ in the expansion (16). From (9) and (13), we evaluate $z_\pm(x) = \gamma_{2m-1} \pm \gamma_{2m}$ in the scaling limit (15) and find that the sums in (13) receive dominant contribution from large j. This allows us to substitute the $K_{m,j}^n$-kernel in (13) by its leading asymptotic behavior and evaluate sum over large j in (13) by integration $\sum_j \mapsto \int dj$, leading to

$$z_+(x) = -\frac{g^{-5/4}}{2\sqrt{\pi}}\sum_{p\geq 0} c_p^+ \Gamma\left(p-\frac{1}{4}\right)\sum_{n\geq 1}\frac{a_n x^{n+p-5/4}}{\Gamma(n+p-\frac{1}{4})} + \ldots \tag{17}$$

where ellipses denote terms suppressed by powers of $1/g$. Then, taking the Laplace transform w.r.t. x, we obtain

$$\int_0^\infty dx e^{-x/s} z_+(x) = -\frac{(gs)^{-1/4}}{2g\sqrt{\pi}}\left[\sum_{p\geq 0} s^p c_p^+ \Gamma\left(p-\frac{1}{4}\right)\right] \times\left(\sum_{n\geq 1} a_n s^n\right) + \ldots \tag{18}$$

The sum over n can be evaluated with the help of (12) as $\sum_{n\geq 1} a_n s^n = -\frac{\sqrt{2}\sin(\frac{s}{2})}{\sin(\frac{\pi}{4}+\frac{s}{2})}$. As a function of s, it contains an infinite number of both poles and zeros on the real s-axis. Requiring that the integrals $z_p \equiv \int_0^\infty dx x^p z_+(x)$ should be finite for $p \geq 0$, we find that, firstly, $\int_0^\infty dx e^{-x/s} z_+(x)$ is an analytical function of s for $\mathrm{Re}\, s > 0$ and, secondly, it scales at large s as $z_0 - z_1/s + O(1/s^2)$. To satisfy these conditions in the r.h.s. of (18), it proves sufficient to take

$$\sum_{p\geq 0} s^p c_p^+ \Gamma\left(p-\frac{1}{4}\right) = \xi_+ \frac{\Gamma(1-\frac{s}{2\pi})}{\Gamma(\frac{3}{4}-\frac{s}{2\pi})} + O(1/g), \tag{19}$$

with $c_0^+ = -\frac{1}{2}$ and ξ_+ the normalization factor. Putting $s=0$ in both sides of (19), we get $\xi_+ = 2[\Gamma(\frac{3}{4})]^2$. Calculating the Laplace transform $\int_0^\infty dx e^{-x/s} z_-(x)$ in the similar manner and imposing the same conditions as for $z_+(x)$, we obtain the second quantization condition

$$\sum_{p\geq 0} s^p\left[c_p^- \Gamma\left(p-\frac{3}{4}\right) + 2c_p^+\left(p-\frac{1}{4}\right)\Gamma\left(p+\frac{1}{4}\right)\right] = \xi_- \frac{\Gamma(1-\frac{s}{2\pi})}{\Gamma(\frac{1}{4}-\frac{s}{2\pi})} + O(1/g), \tag{20}$$

with $c_0^- = 0$ and $c_0^+ = -\frac{1}{2}$. In comparison with (18), the Laplace transform of $z_-(x)$ contains the factor $\sum_{n=1}^\infty a_n(-s)^n = \sqrt{2}\sin(\frac{s}{2})/\sin(\frac{3\pi}{4}+\frac{s}{2})$ that leads to (20). As before, putting $s=0$ in both sides of (20), we fix the normalization factor $\xi_- = \frac{1}{4}[\Gamma(\frac{1}{4})]^2$. Then, expanding both sides of the quantization conditions (19) and (20) around $s=0$ and matching the coefficients in front of powers of s, we determine the coefficients $c_p^\pm$ (with $p\geq 1$) to the leading order in $1/g$. In this way,

$$c_1^+ = -\frac{3\ln 2}{\pi} + \frac{1}{2} + O(1/g), \quad c_1^- = \frac{3\ln 2}{4\pi} - \frac{1}{4} + O(1/g). \tag{21}$$

Substituting these relations into (11), we obtain $\Gamma_{\rm cusp}(g)$ which coincides with the string theory prediction (1).

To calculate subleading strong coupling corrections to $\Gamma_{\rm cusp}(g)$, or equivalently to determine the coefficients $c_p^\pm$, we expand further the Laplace transforms $\int_0^\infty dx e^{-x/s} z_\pm(x)$ in powers of $1/g$ and require each term of the expansion to verify the same analyticity conditions as the leading term. This can be done systematically by applying the Euler-Maclaurin formula to the sums over j in the r.h.s. of (13). In this manner, we obtain the following all-order quantization conditions

$$\begin{aligned}
\sum_{p\geq 0} s^p\left[c_p^+ Q_p^+\left(\frac{1}{gs}\right) + \frac{1}{gs} c_p^- Q_{p-1/2}^-\left(\frac{1}{gs}\right)\right] &= \frac{\Gamma(1-\frac{s}{2\pi})}{\Gamma(\frac{3}{4}-\frac{s}{2\pi})}\sum_{k\geq 0}(gs)^{-k}\xi_k^+(1/g), \\
\sum_{p\geq 0} s^p\left[c_p^- Q_{p-1/2}^+\left(\frac{1}{gs}\right) + c_p^+ Q_p^-\left(\frac{1}{gs}\right)\right] &= \frac{\Gamma(1-\frac{s}{2\pi})}{\Gamma(\frac{1}{4}-\frac{s}{2\pi})}\sum_{k\geq 0}(gs)^{-k}\xi_k^-(1/g),
\end{aligned} \tag{22}$$

where $\xi_k^\pm(1/g) = \sum_{r\geq 0}\xi_{k,r}^\pm g^{-r}$ and the (g-independent) functions $Q_p^+(x) = \sum_{k,l\geq 0} x^{k+l} Q_{k,p}^{2l}$, $\quad Q_p^-(x) = \sum_{k,l\geq 0} x^{k+l} Q_{k,p}^{2l+1}$. Explicit expressions for the coefficients $Q_{k,p}^l$ follow univocally from the Euler-Maclaurin summation formula, and they are too lengthy to present them here. For $g\to\infty$, the relations (22) coincide with (19) and (20) for $\xi_0^\pm(0) = \xi_\pm$, $Q_p^+(0) = \Gamma(p-\frac{1}{4})$, and $Q_p^-(0) = 2(p-\frac{1}{4})\Gamma(p+\frac{1}{4})$. Expanding both sides of (22) in powers of $1/g$ and s and matching the expansion coefficients, we can determine the functions $\xi_k^\pm(g)$ and $c_p^\pm(g)$ to arbitrary order in $1/g$. Substitution of the resulting expression for $c_p^\pm(g)$ into (11) yields the strong coupling expansion of the cusp anomalous dimension.

4. Strong coupling expansion.—Solving the quantization conditions (22), we calculated $\Gamma_{\rm cusp}(g)$ to order $O(1/g^{40})$. The first few terms of the expansion are

$$\begin{aligned}
\Gamma_{\rm cusp}(g+c_1) = 2g\Big[& 1 - c_2 g^{-2} - c_3 g^{-3} - (c_4 + 2c_2^2)g^{-4} - (c_5 + 23c_2c_3)g^{-5} - \left(c_6 + \frac{166}{7}c_2c_4 + 54c_3^2 + 25c_2^3\right)g^{-6} \\
& - \left(c_7 + \frac{1721}{29}c_2c_5 + \frac{1431}{7}c_3c_4 + 457c_2^2c_3\right)g^{-7} \\
& - \left(c_8 + \frac{6352}{107}c_2c_6 + \frac{12606}{29}c_3c_5 + \frac{7916}{49}c_4^2 + \frac{6864}{7}c_2^2c_4 + 4563c_2c_3^2 + 374c_2^4\right)g^{-8} + O(g^{-9})\Big],
\end{aligned} \tag{23}$$

where the expansion coefficients are given by

$$c_1 = \frac{3\ln 2}{4\pi}, \qquad c_2 = \frac{1}{16\pi^2}K, \qquad c_3 = \frac{27}{2^{11}\pi^3}\zeta(3),$$
$$c_4 = \frac{21}{2^{10}\pi^4}\beta(4), \qquad c_5 = \frac{43065}{2^{21}\pi^5}\zeta(5), \qquad c_6 = \frac{1605}{2^{15}\pi^6}\beta(6),$$
$$c_7 = \frac{101303055}{2^{30}\pi^7}\zeta(7), \qquad c_8 = \frac{1317645}{2^{22}\pi^8}\beta(8), \qquad (24)$$

with $\zeta(x)$ the Riemann zeta function, $\beta(x) = \sum_{n\geq 0}(-1)^n(2n+1)^{-x}$ the Dirichlet beta function, and $K = \beta(2)$ the Catalan's constant. We verified that the coefficients (24) are in excellent agreement with the numerical values obtained within the approach of [14].

The reason why in (23) we expanded $\Gamma_{\rm cusp}(g + c_1)$ instead of $\Gamma_{\rm cusp}(g)$ is that the c_1-dependent terms inside $\Gamma_{\rm cusp}(g)$ can be resummed to all orders in $1/g$ by simply replacing $g \to g + c_1$. A distinguished feature of the series (23) is that the coefficients in front of $1/g^n$ are given by a linear combination of the product of $\zeta(2p+1)$ and $\beta(2r)$ such that the sum of their arguments equals n. Let us compare this with the weak coupling expansion of $\Gamma_{\rm cusp}(g)$. The latter runs in even powers of g, and the expansion coefficients only involve products of ζ-functions of both even and odd arguments such that the sum of their arguments equals the order in g [12,19].

We found that, up to order $O(1/g^{40})$, all expansion coefficients of $\Gamma_{\rm cusp}(g)$ except the first one are negative. In addition, at large orders in $1/g$, they grow factorially and the asymptotic expansion is not Borel summable

$$\Gamma_{\rm cusp}(g) \sim -g\sum_k \frac{\Gamma(k-\frac{1}{2})}{(2\pi g)^k} = g\int_0^\infty \frac{du\,u^{-1/2}e^{-u}}{u - 2\pi g}, \qquad (25)$$

with the Stieltjes integral having a pole at $u = 2\pi g$. This indicates that $\Gamma_{\rm cusp}(g)$ receives nonperturbative correction $\sim g^{1/2}e^{-2\pi g}$ proportional to the residue at the pole.

Our prediction for the cusp anomalous dimension (23) relies on the strong coupling expansion of the solution to the BES equation (2). Eventual verification of (23) remains a challenge for the string theory. We would like to mention that our result for $c_2 = K/(4\pi)^2$ is in a structural agreement with the (revised) two-loop superstring result of [17] and in precise agreement with a new superstring computation (R. Roiban and A. A. Tseytlin, to appear).

We are most grateful to A. Gorsky for collaboration at an early stage of the project. We would like to thank A. Belitsky, J. Maldacena, A. Manashov, A. Tseytlin, and A. Turbiner for interesting discussions and correspondence. This research has been supported in part by the French Agence Nationale de la Recherche, Contract No. ANR-06-BLAN-0142 (B. B., G. K.), by the Foundation for Polish Science, and by the MEiN Grant No. P03B-024-27 (J. K.).

Note added.—After this Letter was submitted, we learned [J. Maldacena (private communication)] that our result for nonperturbative corrections to $\Gamma_{\rm cusp}(g)$ is in perfect agreement with the findings of Ref. [20]. As was shown in [20], $\Gamma_{\rm cusp}(g)$ has the interpretation of an energy density of a certain flux configuration, and, as such, it receives correction proportional to m^2 with $m \sim g^{1/4}e^{-\pi g}$ being the mass gap in the $O(6)$ sigma model.

[1] A. M. Polyakov, Nucl. Phys. **B164**, 171 (1980); G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. B **171**, 459 (1986); G. P. Korchemsky, Mod. Phys. Lett. A **4**, 1257 (1989).

[2] A. Vogt, S. Moch, and J. A. M. Vermaseren, Nucl. Phys. **B691**, 129 (2004).

[3] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, Phys. Rev. D **75**, 085010 (2007); F. Cachazo, M. Spradlin, and A. Volovich, Phys. Rev. D **75**, 105011 (2007).

[4] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. 2 (1999) 253 .

[5] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. **B636**, 99 (2002).

[6] S. Frolov and A. A. Tseytlin, J. High Energy Phys. 06 (2002) 007; M. Kruczenski, J. High Energy Phys. 12 (2002) 024.

[7] G. P. Korchemsky, Nucl. Phys. **B462**, 333 (1996); A. V. Belitsky, A. S. Gorsky, and G. P. Korchemsky, Nucl. Phys. **B748**, 24 (2006).

[8] For reviews, see A. V. Belitsky, V. M. Braun, A. S. Gorsky, and G. P. Korchemsky, Int. J. Mod. Phys. A **19**, 4715 (2004); N. Beisert, Phys. Rep. **405** 1 (2004).

[9] G. Arutyunov, S. Frolov, and M. Staudacher, J. High Energy Phys. 10 (2004) 016; M. Staudacher, J. High Energy Phys. 05 (2005) 054; N. Beisert and M. Staudacher, Nucl. Phys. **B727**, 1 (2005).

[10] R. A. Janik, Phys. Rev. D **73**, 086006 (2006).

[11] N. Beisert and A. A. Tseytlin, Phys. Lett. B **629**, 102 (2005); R. Hernandez and E. Lopez, J. High Energy Phys. 07 (2006) 004; N. Beisert, R. Hernandez, and E. Lopez, J. High Energy Phys. 11 (2006) 070.

[12] B. Eden and M. Staudacher, J. Stat. Mech. 0611 (2006) P014; N. Beisert, B. Eden, and M. Staudacher, J. Stat. Mech. 0701 (2007) P021.

[13] A. V. Belitsky, Phys. Lett. B **643**, 354 (2006).

[14] M. K. Benna, S. Benvenuti, I. R. Klebanov, and A. Scardicchio, Phys. Rev. Lett. **98**, 131603 (2007).

[15] L. F. Alday, G. Arutyunov, M. K. Benna, B. Eden, and I. R. Klebanov, J. High Energy Phys. 04 (2007) 082.

[16] I. Kostov, D. Serban, and D. Volin, Nucl. Phys. **B789**, 413 (2008); M. Beccaria, G. F. De Angelis, and V. Forini, J. High Energy Phys. 04 (2007) 066.

[17] R. Roiban, A. Tirziu, and A. A. Tseytlin, J. High Energy Phys. 07 (2007) 056.

[18] P. Y. Casteill and C. Kristjansen, Nucl. Phys. **B785**, 1 (2007).

[19] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. **B769**, 217 (2007).

[20] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 11 (2007) 019.