Anisotropic Solitons in Dipolar Bose-Einstein Condensates

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Starting with a Gaussian variational ansatz, we predict anisotropic bright solitons in quasi-2D Bose-Einstein condensates consisting of atoms with dipole moments polarized *perpendicular* to the confinement direction. Unlike isotropic solitons predicted for the moments aligned with the confinement axis [Phys. Rev. Lett. **95**, 200404 (2005)], no sign reversal of the dipole-dipole interaction is necessary to support the solitons. Direct 3D simulations confirm their stability.

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The creation of two- and three-dimensional (2D and 3D) solitons is a challenge to experiment in optical and atomic physics [1]. Both "light bullets" in nonlinear optics [2] and 2D/3D matter-wave solitons are objects of profound significance. In optics, the best result was the creation of partially localized (quasi-2D) spatiotemporal solitons in crystals with quadratic nonlinearity [3]. As concerns matter waves, quasi-1D bright solitons were demonstrated in self-attractive Bose-Einstein condensates (BECs) of ^{7}Li and ^{85}Rb atoms [4], and gap solitons were created in a self-repulsive ^{87}Rb condensate trapped in an optical lattice (OL) [5]. A fundamental impediment to the creation of 2D and 3D bright solitons with attractive cubic nonlinearity is their instability to collapse. Theoretically considered stabilization schemes include the use of OLs [6] and periodic time modulation of the nonlinearity by means of a Feshbach resonance (FR) in ac magnetic field [7]. Another approach assumes using 2D or 3D OLs to create gap solitons in respective settings with repulsive nonlinearity [8]. So far, however, no experimental realization of matter-wave solitons in a 2D or 3D setting has been reported; hence, other stabilization mechanisms may be relevant.

One such mechanism may utilize the nonlocal interaction between dipolar atoms in the quantum Bose gas [9]. Potential implementations include BECs of magnetically polarized Cr atoms [10], permanent electric dipoles in quantum gases of heteronuclear molecules [11], and electric dipoles induced by laser illumination [12] or by strong dc electric field. In this context, it is natural to consider a nearly 2D "pancake" geometry, the simplest (isotropic) configuration having dipole moments aligned with the tight-confinement direction, denoted here as z. This approach to the creation of quasi-2D solitons in dipolar BECs was elaborated in Ref. [13], using a variational approximation based on the Gaussian ansatz $\psi_{\rm iso} = \pi^{-3/4}\alpha^{1/2}\gamma^{1/4}\exp\{-[\alpha(x^2+y^2)+\gamma z^2]/2\}$, where α and γ are the inverse squares of the radial and axial widths, respectively. This trial function generates the following Gross-Pitaevskii (GP) energy functional,

$$E\{\psi_{\rm iso}\} = \frac{1}{2}\left(\alpha + \frac{\gamma}{2} + \frac{1}{2\gamma}\right) + \alpha\sqrt{\frac{\gamma}{2\pi}}\left(\frac{g}{4\pi} + \frac{g_d}{3}f(\kappa)\right), \quad (1)$$

where $\kappa \equiv \sqrt{\gamma/\alpha}$ is the aspect ratio, and

$$f(\kappa) \equiv \frac{2\kappa^2+1}{\kappa^2-1} - \frac{3\kappa^2}{(\kappa^2-1)\sqrt{|\kappa^2-1|}}\begin{cases}\arctan\sqrt{\kappa^2-1}, & \kappa > 1\\ \frac{1}{2}\ln\left(\frac{1+\sqrt{1-\kappa^2}}{1-\sqrt{1-\kappa^2}}\right), & \kappa < 1\end{cases}. \quad (2)$$

Here and below, length, time, and energy are scaled as $\mathbf{r} \to \mathbf{r}/l_\perp$, $t \to \omega_\perp t$, and $E \to E/(\hbar\omega_\perp)$, where $\omega_\perp$ is the trap frequency in the z direction, and $l_\perp \equiv \sqrt{\hbar/m\omega_\perp}$. The interaction strengths are $g \equiv 4\pi N a_s/l_\perp$ and $g_d \equiv N d^2 m(\hbar^2 l_\perp)^{-1}$, where $a_s > 0$ is the s-wave scattering length, d and m the atomic dipole moment and mass, N the number of atoms, and the wave function is subject to normalization $\int|\psi(\mathbf{r})|^2 d\mathbf{r} = 1$.

For fixed γ in the absence of dipole-dipole (DD) interactions ($g_d = 0$), both kinetic and contact-interaction energies [the first and fourth terms in Eq. (1)] scale as the inverse square of the radial size in the 2D plane, resulting in the well-known instability of the *Townes solitons* [14]. Stabilization is shown by fixing $\gamma = 1$ in Eq. (1) and searching for a minimum of the energy as a function of α, which results in the necessary condition [13],

$$(2/3\sqrt{2\pi})g_d < 1 + g/(2\pi)^{3/2} < -(4/3\sqrt{2\pi})g_d, \quad (3)$$

that may only hold for $g_d < 0$. The interpretation of this condition reveals the origin of self-trapping in the proposed configuration. Since the DD interaction is repulsive and attractive, respectively, for "side-by-side" and "head-to-tail" dipole pairs, the overall dipolar energy varies from negative for a prolate structure, with $L_\rho \sim \alpha^{-1/2} \ll L_z \sim \gamma^{-1/2}$, through zero for the isotropic case, $L_\rho = L_z$, to positive for the oblate condensate with $L_\rho \gg L_z$ [this variation is evident in the sign change of function (2) at

$\kappa = 1$]. Fixing L_z, a *maximum* may be attained for the total energy as a function of L_ρ, provided that the DD term is sufficiently large to reverse the sign of the sum of the kinetic and contact-interaction energies, all scaling as $1/L_\rho^2$. This unstable maximum may be turned into a stable minimum if the sign of the dipolar interaction strength is reversed to $g_d < 0$. This goal may be attained by using rotating external fields [15], but in combination with the necessity to reduce contact interactions in order to make the DD interaction dominant, it introduces an essential complication to the experiment [13], and it is not possible for electric moments induced by the polarizing dc field.

Condition (3) does not guarantee a true minimum of $E(\alpha, \gamma)$ in the (α, γ) plane. For weak DD interactions, Eq. (3) does not hold, and all fixed-γ energy curves are monotonic in α, as shown in Fig. 1(a). For larger values of $|g_d|$, a shallow local minimum (elliptic point) does appear, as in Fig. 1(b). It gives rise to a quasi-2D soliton, which is isolated from collapsing in the direction of $\alpha, \gamma \to \infty$ by a saddle point. However, with further increase of the strength of the DD interactions, the elliptic and saddle points merge and disappear, resulting in an essentially 3D collapse, as seen in Fig. 1(c). This is the source of the "breathing instability" in Ref. [13].

Related works considered magnetization, spin-squeezing and entanglement [16], the stability of vortex lattices [17], and roton instability [18] in dipolar BECs with the same oblate geometry. Also relevant are theoretically elaborated examples of isotropic optical solitons supported by thermal and other nonlocal nonlinearities in the 2D geometry [19].

In this Letter, we aim to explore a new setting with dipole moments polarized in the 2D plane, i.e., *perpendicular* to the tight-confinement axis, which we now designate y, reserving label z for the dipolar axis. With this notation, the GP energy functional is

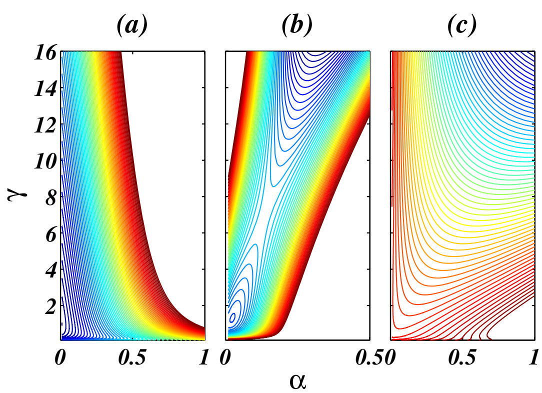

FIG. 1 (color online). Gross-Pitaevskii energy functional (1), as a function of variational parameters α and γ, with $g = 500$ and (a) $g_d/g = -0.10$, (b) $g_d/g = -0.20$, (c) $g_d/g = -0.40$. Weak dipolar interaction in (a) leads to expansion of the condensate, whereas strong attraction in (c) results in collapse. For intermediate strengths of the dipole interaction in (b), stable quasi-2D solitons are possible.

$$E\{\psi\} = \frac{1}{2}\int[|\nabla\psi(\mathbf{r})|^2 + y^2|\psi(\mathbf{r})|^2 + g|\psi(\mathbf{r})|^4]d\mathbf{r} + \frac{g_d}{2} \times \iint\left[1 - \frac{3(z-z')^2}{|\mathbf{r}-\mathbf{r}'|^2}\right]|\psi(\mathbf{r}')|^2|\psi(\mathbf{r})|^2\frac{d\mathbf{r}d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \quad (4)$$

Substituting the general anisotropic ansatz, $\psi_{\rm aniso} = \pi^{-3/4}(\alpha\beta\gamma)^{1/4}\exp[-(1/2)(\alpha x^2 + \beta y^2 + \gamma z^2)]$, into (4), we find (with $\kappa_x \equiv \sqrt{\gamma/\alpha}$, $\kappa_y \equiv \sqrt{\gamma/\beta}$)

$$E(\psi_{\rm aniso}) = \frac{1}{4}(\alpha + \beta + \gamma) + \frac{1}{4\beta} + \sqrt{\frac{\alpha\beta\gamma}{2\pi}}\left[\frac{g}{4\pi} + \frac{g_d}{3}h(\kappa_x, \kappa_y)\right], \quad (5)$$

$$h(\kappa_x, \kappa_y) \equiv \int_0^1 \frac{3\kappa_x\kappa_y x^2 dx}{\sqrt{1 + (\kappa_x^2 - 1)x^2}\sqrt{1 + (\kappa_y^2 - 1)x^2}} - 1. \quad (6)$$

While $h(\kappa_x, \kappa_y)$ may be expressed in terms of elliptic integrals, its behavior is more readily seen directly from Eq. (6). We first consider the variation of the characteristic width, $L_z = 1/\sqrt{\gamma}$, along polarization axis z. At small L_z ($\gamma \gg \alpha, \beta$), we have $h(\kappa_x, \kappa_y) \to h(\infty, \infty) = 2$. On the other hand, $h(\kappa_x, \kappa_y) \to h(0, 0) = -1$ for $\gamma \ll \alpha, \beta$. Thus, for $g_d > 0$, i.e., for the *natural* sign of the DD interaction, Eq. (5) demonstrates a switch from repulsion at small L_z to attraction at large L_z, thus predicting a stable bound state as concerns the variation of L_z, provided that g_d is large enough.

While the above consideration makes stable self-trapping along the polarization axis evident, behavior along the other unconfined direction, x, is more subtle. Indeed, one might expect expansion of the condensate in this direction, as the side-by-side DD interaction is repulsive, However, examination of expression (6) yields the asymptotic limits, $h(0, \kappa_y) = -1$ and $h(\infty, \kappa_y) = (2\kappa_y - 1)/(\kappa_y + 1)$, so that for $g_d > 0$ and $\kappa_y < 1/2$, the DD interaction remains effectively *attractive* even for arbitrarily large values of L_x. For $\kappa_y > 1/2$ (i.e., for $L_z > 2L_y$), the DD interaction changes from the attraction for $\kappa_x \to 0$ to repulsion at $\kappa_x \to \infty$. Since the attractive part of the DD-interaction energy scales as $\sqrt{\alpha}$, whereas the kinetic energy in the x-direction scales as α, a local minimum can be provided by the interplay between them.

The DD interaction provides for effective attraction at large L_x, L_z and prevents expansion if it asymptotically overcomes the contact interaction. For $g_d > 0$, we thus obtain from (5) a threshold condition for the existence of the soliton, $g/g_d < 4\pi/3 \approx 4.19$. In Fig. 2, we plot equal-energy surfaces and their cross sections in the (α, β, γ) space, for $g/g_d = 0.911$. The expected local energy minimum exists, corresponding to an elongated soliton with dimensions $(L_x, L_y, L_z)_{\rm sol} = (2.71, 0.96, 5.20) \times l_\perp$.

The existence and stability of the solitons predicted by the above analysis have been verified by simulations of the

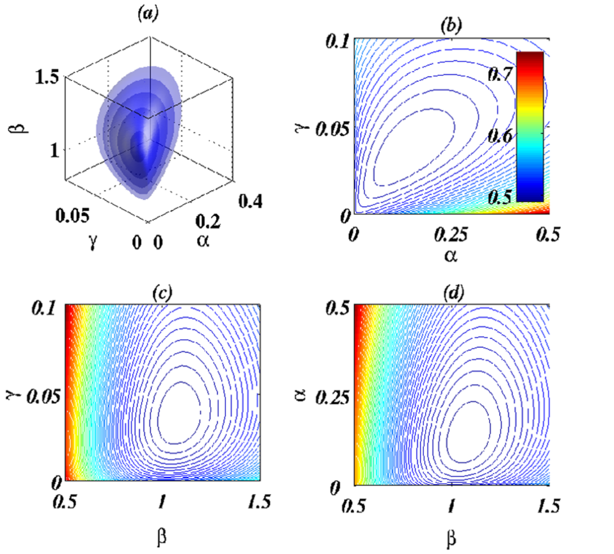

FIG. 2 (color online). (a) Equal-energy surfaces around the local minimum of Gross-Pitaevskii energy functional (5), in the variational parameter space of inverse squared widths α, β, γ, for $g = 10$ and $g/g_d = 0.911$. Energy values range from $E = 0.4907$ for the innermost surface to 0.4987 for the outermost one, in steps of $\Delta E = 2 \times 10^{-3}$. Panels (b), (c), (d) depict cross sections through the minimum-energy point, $(\alpha, \beta, \gamma)_{sol} = (0.136, 1.08, 0.037)$, in the (α, γ), (β, γ) and (α, β) planes, respectively. The color bar in (b) applies also to (c) and (d).

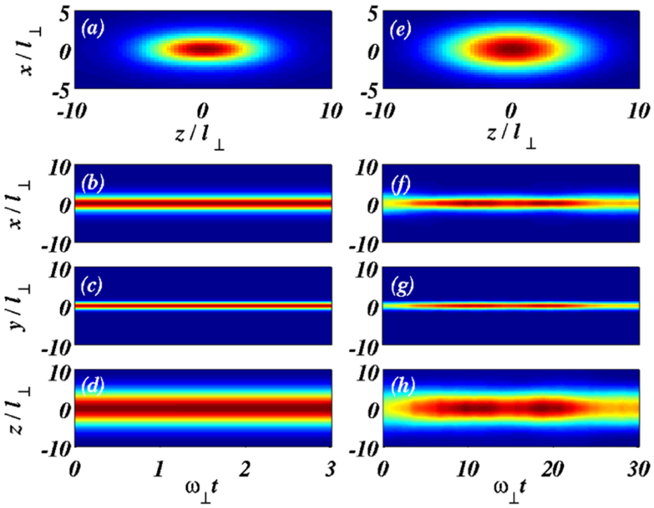

FIG. 3 (color online). Numerically simulated three-dimensional evolution: (a) Density of the numerical soliton solution at $y = 0$, obtained by the imaginary-time propagation of the Gaussian profile ψ_{aniso} with minimum-energy parameters $\alpha = 0.136$, $\beta = 1.08$, $\gamma = 0.037$ borrowed from Fig. 2. The 2D density profile of this Gaussian function at $y = 0$ is shown in (e). Results of the propagation in real time are presented in (b)–(d) and (f)–(h) for the initial functions (a) and (e), respectively. Density profiles along axes x, y, and z, with perpendicular planes going through the soliton's center, $(x_0, y_0, z_0) = (0, 0, 0)$, are shown as functions of time. Coefficients g and g_d are the same as in Fig. 2.

underlying 3D GP equation corresponding to energy functional (4). The soliton shown in Fig. 3(a) was obtained by the 3D propagation in imaginary time of the Gaussian minimum-energy ansatz depicted in Fig. 3(e). Forward propagation in real time, starting from this soliton solution [Figs. 3(b)–3(d), demonstrate perfect self-focusing. The stability is also demonstrated by the real-time 3D propagation starting with the Gaussian approximation, as shown in Figs. 3(f)–3(h). The observed long-lived, slowly damped oscillations indicate the excitation of an internal mode [20], which is typical to stable solitons in nonintegrable systems.

An interesting issue, which will be considered separately, is interactions between the solitons. Collisions between isotropic solitons supported by the inverted DD interactions were simulated in Ref. [13], demonstrating their fusion into a single soliton. In the present setting, anisotropy is expected to produce various outcomes of collisions, depending on the angle between the soliton's polarization and collision direction.

Another dynamical behavior, unique to anisotropic solitons, may be attained by rotation of the external magnetic field used to align the dipole moments. Whereas the slow rotation of the field is adiabatically followed by the soliton, as shown in Fig. 4(a), it lags behind faster revolving fields, see Fig. 4(b). For instance, at $\omega_\perp t = 10$, the field is directed along x, whereas the long axis of the density profile is not parallel to it. This drag is accompanied by significant deformation and excitation, as evident from the expansion of the density distribution in Fig. 4(b). The amount of excitation around the rotating soliton (deviation from adiabaticity) smoothly increases with the rotation velocity, roughly like in the case of Landau-Zener tunneling. For slow rotation, it is clearly seen to be exponentially small in the rotation period.

To estimate the experimental feasibility of the proposed quasi-2D solitons in dipolar BECs, we notice that the above-mentioned threshold for the existence of the anisotropic solitons is $g_d/g > 0.24$. For ^{52}Cr with magnetic dipole moment $d = 6\mu_B$ and s-wave scattering length $a \approx 100a_0$ (μ_B and a_0 are the Bohr magneton and radius, respectively), this ratio is $g_d/g = \mu_0 d^2 m/(16\pi^2\hbar^2 a) = 0.036$ [10] (μ_0 is the vacuum permeability). Therefore, attenuation of the direct interaction between atoms may be necessary. A recent experiment has demonstrated that this interaction may be practically switched off by means of the FR in the condensate of Cr atoms, making the DD interactions absolutely dominant [21]. Alternatively, using BECs of dipolar molecules, with the electric dipole moment on the order of Debye, will certainly provide sufficiently strong DD interactions.

In conclusion, we have demonstrated, using the variational analysis and direct simulations of the GP equation in 3D, that the interplay of the ordinary repulsive contact interactions and long-range DD forces may give rise to *stable* quasi-2D anisotropic solitons in the dipolar BEC, if atomic moments are polarized in the 2D plane. We note

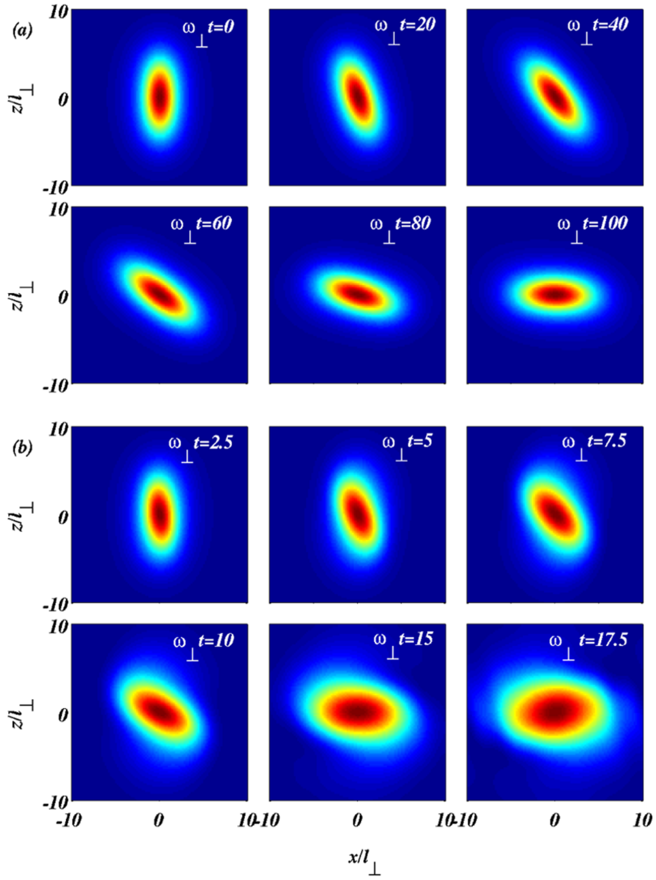

FIG. 4 (color online). The rotation of the soliton, driven by the polarization field revolving at angular velocity Ω, starting with the anisotropic soliton shown in Fig. 3(a), for $\Omega = \pi\omega_\perp/200$ (a) and $\Omega = \pi\omega_\perp/20$ (b). The interaction strengths are the same as in Figs. 2 and 3.

that similar elliptic optical solitons were observed using the thermal nonlinearity and anisotropic boundaries [22]. Unlike the previously considered isotropic solitons polarized perpendicular to the plane [13], the stability of the anisotropic solitons does not require artificial inversion of the sign of the dipole-dipole interaction. The anisotropic soliton may adiabatically follow slow rotation of the polarizing field.

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