Supersymmetry and the Goldstino-Like Mode in Bose-Fermi Mixtures

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Supersymmetry is assumed to be a basic symmetry of the world in many high-energy theories, but none of the superpartners of any known elementary particle have been observed yet. We argue that supersymmetry can also be realized and studied in ultracold atomic systems with a mixture of bosons and fermions, with properly tuned interactions and single particle dispersion. We further show that in such nonrelativistic systems supersymmetry is either spontaneously broken or explicitly broken by a chemical potential difference between the bosons and fermions. In both cases the system supports a *sharp* fermionic collective mode similar to the *Goldstino* mode in high-energy physics, due to supersymmetry. We also discuss possible ways to detect this mode experimentally.

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Supersymmetry, which is a symmetry that relates bosons and fermions, became a topic of strong interest in elementary particle physics after Wess and Zumino constructed the first "realistic" model [1]. It may play a fundamental role because the supersymmetry algebra is the only graded Lie algebra of the symmetries of the S matrix consistent with relativistic quantum field theory [2]. The supersymmetric string theory is a unique theory expected to give a unified description of all interactions in nature [3]. However, none of the superpartners (which have identical properties except for opposite statistics) of any known elementary particles have been found in experiments thus far. Therefore, it is extremely important to study the breaking of supersymmetry.

Recent experimental progress in mixtures of ultracold Bose and Fermi atoms [4] provides an opportunity to realize and study supersymmetry in such atomic systems. Theoretically, several different models that exhibit supersymmetry have been studied. An ultracold superstring model was constructed [5]. The physical behavior of an exactly soluble model of one-dimensional Bose-Fermi mixture was investigated [6]. A general formalism to study such supersymmetric systems based on coherent state path integral was set up in Ref. [7]. In a recent work [8], we studied a supersymmetric Hubbard model and focused on the Mott insulator phase for bosons.

In this Letter we study some general properties of supersymmetric Bose-Fermi mixtures, in which bosons and fermions are supersymmetric partners of each other. We show that in the presence of time-reversal symmetry, supersymmetry is always broken, either spontaneously or by a chemical potential difference between bosons and fermions. For a truly supersymmetric grand canonical Hamiltonian, we find the system contains bosons only even though the grand canonical Hamiltonian is invariant under a supersymmetry transformation that turns a boson into a fermion; in this case supersymmetry is spontaneously broken. To support a finite density of fermions, one needs a chemical potential difference between the bosons and fermions: $\Delta\mu = \mu_F - \mu_B > 0$, which breaks supersymmetry of grand canonical Hamiltonian explicitly but keeps the canonical Hamiltonian supersymmetric. We find in both cases the system supports a sharp fermionic collective excitation similar to the Goldstino mode in supersymmetric high-energy theories, which is gapless in the former case while it has a gap equal to $\Delta\mu$ for the latter. For simplicity we refer to this Goldstino-like mode as "Goldstino" in the following. We will also discuss its possible experimental detection.

While our results are general, in the following we illustrate them by considering a simple lattice model (in the grand canonical ensemble) with mixture between a single species of bosons and a single species of fermions:

$$H_G = H - \mu_F N_F - \mu_B N_B; \tag{1}$$

$$H = \hat{T} + \hat{V}; \tag{2}$$

$$\hat{T} = -\sum_{i\neq j}(t^B_{ij}a^\dagger_i a_j + t^F_{ij}f^\dagger_i f_j); \tag{3}$$

$$\hat{V} = \sum_{(ij)}[U^{BB}_{ij}n^a_i n^a_j + U^{BF}_{ij}n^a_i n^f_j + U^{FF}_{ij}n^f_i n^f_j]. \tag{4}$$

Here a_i and f_i are the boson and fermion operators on site i, and n^a_i and n^f_i are the corresponding number operators; $N_B = \sum_i n^a_i$ and $N_F = \sum_i n^f_i$. In the presence of time-reversal symmetry, all the hopping matrix elements (t's) are real; we further assume they are all non-negative so the hoppings are not frustrated as is usually the case.

We now introduce generators of supersymmetry [9]:

$$Q = \sum_i a^\dagger_i f_i; \qquad Q^\dagger = \sum_i a_i f^\dagger_i. \tag{5}$$

It is easy to verify that they satisfy the following relations: $Q^2 = (Q^\dagger)^2 = 0, \{Q, Q^\dagger\} = N = N_B + N_F$, and $[Q, N] = [Q^\dagger, N] = 0$. From these relations it is clear that Q is a *fermionic* operator. Physically Q turns a fermion into a boson, and $Q^\dagger$ does the opposite. While in (5) Q is defined in 2nd quantized notation, we also need to know its operation on a 1st quantized wave function

$$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_{N_B}; \mathbf{y}_1, \ldots, \mathbf{y}_{N_F}), \tag{6}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the coordinates of the bosons and fermions, respectively, so that ψ is symmetric under the exchange of $\mathbf{x}$'s while antisymmetric under the exchange of $\mathbf{y}$'s. We find (up to a normalization constant)

$$Q\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F}) = S_{\mathbf{x}_1,\ldots,\mathbf{x}_{N_B},\mathbf{y}_1}\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F}), \tag{7}$$

$$Q^\dagger\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F}) = A_{\mathbf{x}_1,\mathbf{y}_1,\ldots,\mathbf{y}_{N_F}}\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F}), \tag{8}$$

where $S_{\mathbf{x}_1,\ldots,\mathbf{x}_{N_B},\mathbf{y}_1}$ is the symmetrization operator for coordinates $\mathbf{x}_1, \ldots, \mathbf{x}_{N_B}, \mathbf{y}_1$, and $A_{\mathbf{x}_1,\mathbf{y}_1,\ldots,\mathbf{y}_{N_F}}$ is the antisymmetrization operator for coordinates $\mathbf{x}_1, \mathbf{y}_1, \ldots, \mathbf{y}_{N_F}$. Physically, S turns a fermion into a boson by symmetrizing one fermion coordinate with respect to all boson coordinates, and A turns a boson into a fermion by antisymmetrizing one boson coordinate with respect to all fermion coordinates.

If the bosons and fermions have the same dispersion and interaction strength, namely, $t^B_{ij} = t^F_{ij}$, and $U^{BB}_{ij} = U^{BF}_{ij} = U^{FF}_{ij}$, it is easy to show that the canonical ensemble Hamiltonian H is supersymmetric, i.e.,

$$[Q, H] = [Q^\dagger, H] = 0. \tag{9}$$

For completeness we also present a generic example of supersymmetric H in the continuum, in 1st quantization:

$$\begin{aligned} H'_{1\text{st}} = & \sum_{i\le N_B}[-\nabla^2_{\mathbf{x}_i} + V(\mathbf{x}_i)] + \sum_{i\le N_F}[-\nabla^2_{\mathbf{y}_i} + V(\mathbf{y}_i)] \\ & + \sum_{i<j\le N_B} U(\mathbf{x}_i - \mathbf{x}_j) + \sum_{i<j\le N_F} U(\mathbf{y}_i - \mathbf{y}_j) \\ & + \sum_{i\le N_B, j\le N_F} U(\mathbf{x}_i - \mathbf{y}_j), \end{aligned} \tag{10}$$

where V is single particle potential and U is two-body interaction. Of course the 2nd quantized, lattice Hamiltonian (2) also has a corresponding 1st quantized form:

$$H_{1\text{st}} = \hat{T}_{1\text{st}} + \hat{V}_{1\text{st}}, \tag{11}$$

where the form of $\hat{T}_{1\text{st}}$ and $\hat{V}_{1\text{st}}$ can be deduced from Eqs. (3) and (4). If $\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F})$ is an eigenwave function of the Hamiltonian $H_{1\text{st}}$ with N_B bosons and N_F fermions, then $Q\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F})$ is an eigenwave function of $H_{1\text{st}}$ with $N_B + 1$ bosons and $N_F - 1$ fermions with exactly the same eigenenergy. Similarly, $Q^\dagger\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F})$ is an eigenwave function of $H_{1\text{st}}$ with $N_B - 1$ bosons and $N_F + 1$ fermions, also with exactly the same eigenenergy.

If we further have the same chemical potential for the bosons and fermions, $\mu_F = \mu_B$, the grand canonical Hamiltonian is also supersymmetric:

$$[Q, H_G] = [Q^\dagger, H_G] = 0. \tag{12}$$

Given the great tunability of parameters in cold atom systems, we expect such conditions can be reached in a variety of systems. In the following we discuss consequences of such supersymmetry when present.

We start by considering the case where H_G is supersymmetric: $[Q, H_G] = 0$. In this case we can prove that the ground state of H_G contains no or only one fermion. Let us *assume* the ground state contains more than one fermion, and has the wave function $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_{N_B}; \mathbf{y}_1, \ldots, \mathbf{y}_{N_F})$. Because of the time-reversal symmetry of H_G, ψ can be chosen to be real. Now construct a trial state:

$$\tilde{\psi}(\mathbf{x}_1, \ldots, \mathbf{x}_{N_B}; \mathbf{y}_1, \ldots, \mathbf{y}_{N_F}) = |\psi(\mathbf{x}_1, \ldots, \mathbf{y}_{N_F})|. \tag{13}$$

Obviously, $\tilde{\psi}$ is non-negative and different from ψ for $N_F > 1$, as the latter changes sign under the exchange of $\mathbf{y}$'s. In fact, $\tilde{\psi}$ is a two-component or "spin-1/2" boson wave function as it is also symmetric under the exchange of $\mathbf{y}$'s. Since $\tilde{\psi}$ and ψ differ in phase only, we find the potential or interaction energy does not change: $\langle\psi|\hat{V}_{1\text{st}}|\psi\rangle = \langle\tilde{\psi}|\hat{V}_{1\text{st}}|\tilde{\psi}\rangle$, because $\hat{V}_{1\text{st}}$ depends only on density but not the phase of wave function. The situation is different for $\hat{T}_{1\text{st}}$, which *is* sensitive to the wave function phase. Because $\tilde{\psi}$ and ψ are the same for certain configurations (when $\psi \ge 0$) and differ by a $-$ sign for others (when $\psi < 0$), we find the expectation value of every term in $\langle\hat{T}_{1\text{st}}\rangle$ is either the same for $|\psi\rangle$ and $|\tilde{\psi}\rangle$ or differ by a $-$ sign. Because $\tilde{\psi}$ is non-negative, and all t's in $\hat{T}$ are non-negative (meaning $\hat{T}$ only has negative matrix elements), we find every single term in $\langle\tilde{\psi}|\hat{T}_{1\text{st}}|\tilde{\psi}\rangle$ is non-positive. We thus have $\langle\psi|\hat{T}_{1\text{st}}|\psi\rangle \ge \langle\tilde{\psi}|\hat{T}_{1\text{st}}|\tilde{\psi}\rangle$, and as a result $\langle\psi|H_{1\text{st}}|\psi\rangle \ge \langle\tilde{\psi}|H_{1\text{st}}|\tilde{\psi}\rangle$. Since in general $|\tilde{\psi}\rangle$ is *not* an eigenstate of $H_{1\text{st}}$, and the ground state of such a spin-1/2 boson system is spin fully polarized [10,11], from variational theorem we find $\langle\psi|H_G|\psi\rangle > E_0$, where E_0 is the lowest H_G eigenvalue for the case with N bosons and no fermion (here we also used the fact that bosons and fermions have the same chemical potential). This is in contradiction with the assumption that $|\psi\rangle$ is the ground state of H_G. We thus conclude that in the ground state N_F can only be 0 or 1 [12].

We now show that the ground states have a double degeneracy. Assume $|\psi_0\rangle$ is the ground state with $N_F = 0$. We can then construct a different state with $N_F = 1$ and one fewer boson

$$|\psi_G\rangle = Q^\dagger|\psi_0\rangle, \tag{14}$$

which is also an exact eigenstate of H_G with exactly the same energy E_0, as guaranteed by supersymmetry (12). $|\psi_G\rangle$ can be viewed as a zero momentum, fermionic zero mode of the ground state, which is known as the Goldstino mode in the high-energy literature.

As a simple example illustrating the results presented above, consider the special case of noninteracting particles with all $U = 0$. For noninteracting bosons we always have $\mu_B = 0$ (we measure energy from the bottom of single particle dispersion) at zero temperature, regardless of boson number. Supersymmetry of H_G requires $\mu_F = \mu_B = 0$, as a result we can have either no fermion or a single fermion occupying the $\mathbf{k} = 0$ state (with zero energy) in the ground state of H_G.

It should be clear from the discussion above that not only the ground states but *all* eigenstates of H_G (except for vacuum with $N = 0$) come in degenerate pairs that have the same total number of particles but differ in fermion number by one due to (12). The vacuum is special in that it is the *only* state that is supersymmetric, as it is annihilated by *both* Q and $Q^\dagger$.

We can also construct Goldstino at finite momentum:

$$|\psi_G(\mathbf{q})\rangle = R_{\mathbf{q}}^\dagger|\psi_0\rangle, \tag{15}$$

by boosting $Q^\dagger$ to finite momentum:

$$R_{\mathbf{q}}^\dagger = \sum_i e^{-i\mathbf{q}\cdot\mathbf{x}_i} a_i f_i^\dagger = \sum_k a_{\mathbf{k}} f_{\mathbf{k}+\mathbf{q}}^\dagger. \tag{16}$$

Since the statistics of a single fermion created by $R_{\mathbf{q}}^\dagger$ has no physical consequence, $|\psi_G(\mathbf{q})\rangle$ is just like ferromagnetic spin-wave states for $SU(2)$ bosons [11]; they have exactly the same dispersion and differ only in statistics. While $|\psi_G(\mathbf{q})\rangle$ is not an exact eigenstate of H_G for finite $\mathbf{q}$, it approaches one in the long-wave length limit $\mathbf{q} \to 0$, and has quadratic dispersion: $E_{\mathbf{q}} - E_0 \propto |\mathbf{q}|^2$ [11].

It is appropriate at this point to discuss the relation between the sharp fermionic collective mode we call Goldstino here and the Goldstino in high-energy context. In high-energy context, Goldstino refers to the Goldstone fermion arising from spontaneous breaking of the global supersymmetry; it is a Weyl spinor with spin-1/2. In our nonrelativistic model, the gapless Goldstone fermion mode is also the result of the spontaneous breaking of the global supersymmetry. In this sense, they are quite similar. In fact, if we had a two-component boson system instead of a Bose-Fermi mixture, we would have a pseudospin ferromagnet that breaks $SU(2)$ symmetry, with a branch of gapless spin-wave mode; the spin-wave mode is the Goldstone boson associated with breaking of $SU(2)$ symmetry. In the Bose-Fermi mixture, the second component is fermionic, and the $SU(2)$ symmetry becomes supersymmetry that is generated by the fermionic operator Q. The difference here is that the Goldstino is a spinless fermion with quadratic (instead of linear) dispersion, due to the absence of Lorentz symmetry.

We now turn to the more interesting case in which there is a *finite* density of fermions in the ground state $|\psi_0\rangle$. In order to sustain this we must have a higher chemical potential for the fermions, thus $\Delta\mu = \mu_F - \mu_B > 0$. $\Delta\mu$ can be viewed as a chemical potential for Goldstino; in its presence the fermionic Goldstino modes are "filled up" to some Fermi wave vector k_F.

In the presence of $\Delta\mu > 0$, H_G is no longer supersymmetric; it instead has a nonzero commutation relation with the supersymmetry generators:

$$\begin{aligned}[][Q, H_G] &= -[Q, \mu_F N_F + \mu_B N_B] = -\Delta\mu Q;\\ [Q^\dagger, H_G] &= -[Q^\dagger, \mu_F N_F + \mu_B N_B] = \Delta\mu Q^\dagger.\end{aligned} \tag{17}$$

From (17) we can immediately conclude the following: (i) $Q|\psi_0\rangle$ is an *exact* excited state with excitation energy $\Delta\mu$, and (ii) $Q^\dagger|\psi_0\rangle = 0$, because if it were not null, it would be a state with *negative* excitation energy $-\Delta\mu$. We thus find even though in this case supersymmetry is *explicitly* broken by $\Delta\mu$, we still have a *sharp* zero momentum fermionic collective mode generated by Q, which is now *gapped*. The situation is somewhat similar to what happens to a ferromagnet in an external magnetic field: the field breaks rotation symmetry and opens a spin-wave gap, but the spin wave remains a sharp collective mode. This is a "holelike" Goldstino mode since it is created by Q instead of $Q^\dagger$, which creates a hole in the occupied Goldstino Fermi sea. The analogous [to Eq. (15)] finite momentum states are $R_{\mathbf{q}}|\psi_0\rangle$, which are expected to have *downward* quadratic dispersion of the form $E_{\mathbf{q}} - E_0 \approx \Delta\mu - \alpha|\mathbf{q}|^2$.

Again due to (17), all eigenstates of H_G except for vacuum come in pairs whose energies differ by $\Delta\mu$ and fermion numbers differ by one. This is because the canonical Hamiltonian remains supersymmetric.

We now turn the discussion to possible experimental detection of the Goldstino modes. Normally one would expect that these modes can only be detected in processes in which a boson is turned to a fermion or vice versa, as that is what $Q^\dagger$ or Q does. There is no such process that can be easily engineered in cold atom systems that we are aware of, except for possible cotunneling processes in which a fermion leaves the system and a boson enters the system simultaneously, or vice versa. In the following we show that in the presence of a Bose condensate, the Goldstino mode contributes a *finite* spectral weight to the spectral function of single fermion Green's function; as a result it can be detected through processes that involve a *single* fermion. Physically this is possible because in the presence of a Bose condensate, the boson number is not fixed in the ground state; as a result a single fermion (hole) can grab a boson from the condensate and propagate as the Goldstino mode. To demonstrate this we calculate the overlap between the fermionic single hole state $f_{\mathbf{q}=0}|\psi_0\rangle$ with the normalized Goldstino mode $(1/\sqrt{N})Q|\psi_0\rangle$:

$$\begin{aligned}\frac{1}{\sqrt{N}}\langle\psi_0|Q^\dagger f_{\mathbf{q}=0}|\psi_0\rangle &= \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\langle\psi_0|f_{\mathbf{k}}^\dagger a_{\mathbf{k}} f_{\mathbf{q}=0}|\psi_0\rangle = \frac{1}{\sqrt{N}}\left[\langle\psi_0|f_0^\dagger f_0 a_0|\psi_0\rangle + \sum_{\mathbf{k}\neq 0}\langle\psi_0|f_{\mathbf{k}}^\dagger a_{\mathbf{k}} f_{\mathbf{q}=0}|\psi_0\rangle\right]\\ &= \sqrt{\frac{N_B^0}{N}}n_{\mathbf{q}=0}^f + \sqrt{\frac{1}{N}}\sum_{\mathbf{k}\neq 0}\langle\psi_0|f_{\mathbf{k}}^\dagger a_{\mathbf{k}} f_{\mathbf{q}=0}|\psi_0\rangle,\end{aligned} \tag{18}$$

where N_B^0 is the number of bosons in the condensate and $n^f_{\mathbf{q}=0} = \langle\psi_0|f^\dagger_{\mathbf{q}=0} f_{\mathbf{q}=0}|\psi_0\rangle \lesssim 1$. When N_B^0 is macroscopic (as is the case in the presence of a condensate), $a_0|\psi_0\rangle = \sqrt{N_B^0}|\psi_0\rangle$, and the first term in (18) dominates the second. We thus find the zero momentum fermion Green's function has *finite* weight on the Goldstino, and the weight is approximately N_B^0/N, proportional to the condensate density. As a result the fermion spectral function $A(\mathbf{q} = 0, \omega)$ has a *sharp* δ-function peak at $\omega = \Delta\mu$. This is highly unusual as in electronic or other fermionic systems, one normally expects sharp (coherent) spectral peak for fermions with momentum near k_F and corresponding energy near zero in a Fermi liquid, corresponding to Landau quasiparticles which are well defined only near the Fermi surface. The sharp, coherent fermion peak at *zero* momentum and *finite* energy we find here is a remarkable consequence of the combination of supersymmetry and Bose condensation, which is unique to such supersymmetric Bose-Fermi mixtures. Another remarkable property is that this spectral peak remains sharp at finite temperature T as long as T is below the Bose condensation temperature T_c. This is because (17) guarantees that the Goldstino mode is sharp at any T, while for $T < T_c$ the condensate density is nonzero, so the fermion spectral function has a weight (proportional to condensate density) on the Goldstino mode. We are not aware of any fermionic mode (like Landau quasiparticles) that remains sharp at finite T in other contexts. The fermion spectral function at small but nonzero momentum $\mathbf{q}$ will also have finite weight on the finite $\mathbf{q}$ Goldstino mode for $T < T_c$. This will result in a spectral peak at $\omega = \Delta\mu - \alpha|\mathbf{q}|^2$, whose width goes to zero as $|\mathbf{q}| \to 0$.

In electronic condensed matter systems, single electron spectral function can be measured using electron tunneling or photoemission. In cold atom systems we do not have equivalent methods (yet). On the other hand, alternative ways to measure single particle Green's function or spectral function are currently being developed, like stimulated Raman spectroscopy [13]. Once these methods become available, the Goldstino mode with the specific properties discussed above can be probed through the fermion spectral function, as long as there is a Bose condensate (which is what bosons tend to form, except at certain commensurate lattice filling). We are thus highly hopeful that supersymmetric Bose-Fermi mixtures can be studied and the Goldstino mode can be detected experimentally, and our predictions can be tested.

We close by stating that we have studied possible realization of supersymmetry in cold atom Bose-Fermi mixtures, in a *nonrelativistic* setting. Our study gives examples of supersymmetry breaking, both spontaneous and explicit. In both cases the system supports a sharp, collective fermionic excitation which is a Goldstino-like mode; we have discussed ways to detect it experimentally as an unambiguous signature of supersymmetry. We hope such a study can provide hints to the mechanism of supersymmetry breaking in relativistic quantum field theories, in which the supersymmetry algebra is usually a graded Poincaré algebra because of the linear dispersion of the relativistic fermion and that the boson is of the Klein-Gordon type. As a result the anticommutation relation between the supercharge and its Hermitian conjugate is the Hamiltonian in that case, instead of the total particle number in the present work.

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