Coulomb Blockade as a Probe for Non-Abelian Statistics in Read-Rezayi States

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We consider a quantum dot in the regime of the quantum Hall effect, particularly in Laughlin states and non-Abelian Read-Rezayi states. We find the location of the Coulomb blockade peaks in the conductance as a function of the area of the dot and the magnetic field. When the magnetic field is fixed and the area of the dot is varied, the peaks are equally spaced for the Laughlin states. In contrast, non-Abelian statistics is reflected in modulations of the spacing which depend on the magnetic field.

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The possibility that quasiparticles in certain quantum Hall systems satisfy non-Abelian statistics has been widely discussed in the last two decades [1–4]. However, this exciting theoretical possibility has no experimental support at present (in fact, only very recently first steps towards experimental tests of *Abelian* fractional statistics have been carried out [5]). Furthermore, only a few predictions have been put forward that may experimentally identify non-Abelian quasiparticles. The current Letter contributes to bridging this gap between theory and experiment by predicting signatures of non-Abelian statistics on Coulomb blockade peaks in transport through quantum dots.

Most experiments proposed so far for observing non-Abelian statistics [6–13] rely on interference of quasiparticle trajectories in a two point contact geometry that creates Fabry-Perot or Mach-Zehnder interferometers. The Fabry-Perot interferometer [14], sketched in Fig. 1, is a Hall bar with two quantum point contacts (QPCs) introducing backscattering through quasiparticle tunnelling from one edge to the other. In lowest order interference experiments [8–11], the sensitivity to the statistics of quasiparticles originates from the motion of the back-scattered current around quasiparticles that are localized in the bulk, in the "island" formed between two interfering trajectories. The number of these quasiparticles, $n_{\rm is}$, may be varied in a controlled way. For fixed $n_{\rm is}$, signatures of non-Abelian statistics manifest themselves in the interference term of the backscattered current. When the number $n_{\rm is}$ fluctuates in time, signatures of non-Abelian statistics may be present in the current noise [15].

In this work we study the limit of strong backscattering by the two point contacts (see Fig. 1). In this limit, the area between the point contacts becomes a quantum dot, weakly coupled to the rest of the Hall bar. The low-temperature conductance through the dot is suppressed by its charging energy, except in the degeneracy points that give rise to Coulomb blockade peaks [16]. We show that for non-Abelian quantum Hall states of the Read-Rezayi series [2], the position of the peaks in a two-parameter plane of the area of the dot, S, and the magnetic field, B, is sensitive to the non-Abelian statistics of the quasiparticles. Here the origin of the sensitivity is the effect of the localized quasiparticles on the spectrum of edge excitations. Such a sensitivity was already discovered for the $\nu = 5/2$ state in [8], and we will compare our results for the Read-Rezayi states to those of the $\nu = 5/2$ state.

The Read-Rezayi states are expected to occur in the filling factor range of $2 < \nu < 3$. We assume that the point contacts strongly back-scatter only the edge state of the uppermost, partially filled Landau level. The quantum dot is defined then by the edge state of the partially filled Landau level. At the end of the Letter we discuss the case in which all edge states are backscattered by the two point contacts.

For an almost closed quantum dot the number of electrons is quantized to an integer, and the low-voltage low-temperature conductance through the dot is suppressed unless the ground state energy of the dot with N electrons is degenerate with its ground state energy with $N + 1$ electrons. Thus, Coulomb blockade peaks of the conductance appear for those values of the area and magnetic field for which the following equation

$$E(N, S, B) = E(N + 1, S, B) \tag{1}$$

is satisfied for some integer N.

For a clean large ($N \gg 1$) dot in a metallic state at zero magnetic field, where the electronic density is determined by charge neutrality with a uniform positive background of density n_0, one expects the area that separates consecutive

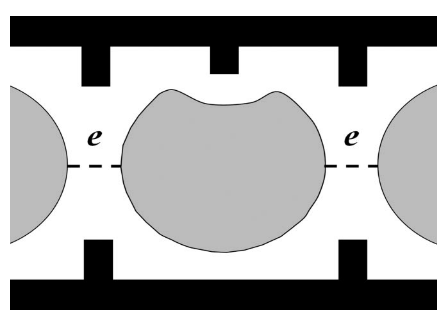

FIG. 1. Fabry-Perot interferometer in the limit of strong quasiparticle backscattering. The dot (a quantum Hall droplet) is coupled to the leads via electron tunneling, and its area may be varied using a side modulation gate.

Coulomb blockade peaks to be $\Delta S = e/n_0$, the area occupied by one electron. This would also be the situation for the quantum Hall state of noninteracting electrons at $\nu = 1$. Below, we start by showing that this is also the expected spacing for the Abelian Laughlin $\nu = 1/p$ states (with p odd). In contrast, for the Read-Rezayi series we find a much richer structure, that depends on B: while the *average* spacing between peaks remains e/n_0, the presence of non-Abelian quasiparticles in the bulk imposes modulations of the spacing which depend on their number, $n_{\rm is}$. The latter is determined by the magnetic field.

For all quantum Hall states the bulk is incompressible, and electronic transport takes place along the edge. The energies in (1) are then energies of edge modes. The edge of the Abelian Laughlin states is described [17] by the action of a chiral free boson (we take $\hbar = 1$)

$$S = -\frac{1}{4\pi\nu}\int dtdx[\partial_t\varphi\partial_x\varphi + v_c(\partial_x\varphi)^2], \qquad (2)$$

where v_c is the velocity of edge excitations. The bosonic field $\varphi(x)$ is normalized here such that the electronic creation operator is $e^{ip\varphi(x)}$, and the electron density along the edge is given by $\rho = \frac{1}{2\pi}\partial_x\varphi$. For the electron operator to be single valued the field φ must obey the quantization condition

$$\varphi(L) - \varphi(0) = 2\pi n/p. \qquad (3)$$

The total number of electrons on the edge is $N = n\nu = n/p$. Alternatively, the number of quasiparticles is n. Since the number of electrons in the dot is an integer, the total number of quasiparticles in the dot (edge and bulk together), which is $n + n_{\rm is}$, must be divisible by p.

When the magnetic field $B = B_0$ is such that the filling fraction is precisely $1/p$ there are no quasiparticles in the bulk. The energy is then $E = \frac{v_c}{4\pi\nu}\int dx(\partial_x\varphi)^2$, which is minimized by a space-independent $\varphi(x)$. As the area of the dot is varied continuously, the field φ is restricted by the quantization condition (3), and therefore may change only in a discrete way. Therefore, an infinitesimal increase in the area of the dot violates charge neutrality at the edge, with an associated energy cost. When the area grows sufficiently, it becomes energetically favorable to add a whole electron to the dot. The energy dependence on the area may then be incorporated into the description (2) by writing the edge energy as

$$E_c = \frac{v_c}{4\pi\nu}\int dx\left(\partial_x\varphi - 2\pi\nu\frac{B_0(S - S_0)}{L\phi_0}\right)^2, \qquad (4)$$

where ϕ_0 is the magnetic flux quantum. The total energy is minimized when the charge density is uniformly spread along the perimeter of the edge, i.e., when $\partial_x\varphi$ is x-independent. With $N = 0$ defined to be the number of electrons for a dot with area S_0, the energy for N electrons on the edge is

$$E_c(N) = \frac{\pi v_c}{\nu L}\left(N - \nu\frac{B_0(S - S_0)}{\phi_0}\right)^2. \qquad (5)$$

Equation (1) reduces to $E_c(N) = E_c(N+1)$, and the area separation ΔS between its solutions for consecutive values of N is $\Delta S = e/n_0$. This value is independent of the magnetic field, as long as the bulk is incompressible: as the magnetic field is changed from B_0 quasiparticles enter the bulk. The incompressibility of the bulk quantizes their number to an integer $n_{\rm is}$. As a consequence, the number of quasiparticles on the edge, n of Eq. (3), is not necessarily an integer multiple of p, but rather an integer of the form $\ell p - n_{\rm is}$, with ℓ an integer. That does not, however, change the area spacing ΔS. As we will now see, this is not the case for non-Abelian states.

While the free chiral boson field theory of Eq. (2) fully describes the edge of a $\nu = 1/p$ state, the edge of the Read-Rezayi non-Abelian states requires also a second field theory, whose properties we now review. The second theory is a conformal field theory (CFT) of a neutral field, and for the $\nu = 2 + k/(k+2)$ Read-Rezayi state (with $k = 2, 3, 4, \ldots$), it is the Z_k parafermionic CFT. Quasiparticles for this state have charge $\frac{e}{k+2}$. When the magnetic field is varied by one flux quantum, k quasiparticles appear; hence, the flux associated with a single quasiparticle is $\frac{2\pi}{ke}$.

The creation operators of both an electron and a quasiparticle are then products of two factors. The first, $e^{i\alpha\varphi(z)}$, accounts for the flux and the charge associated with the electron ($\alpha = (k+2)/k$), and with the quasiparticle ($\alpha = 1/k$). The second part is a neutral field labeled by two quantum numbers Φ^l_m, obeying the restrictions $l \in \{0, 1, \ldots, k\}$, $\Phi^l_m = \Phi^l_{m+2k} = \Phi^{k-l}_{m-k}$ and $l + m \equiv 0(\text{mod}2)$. The integer m is known as the holomorphic charge (or Z_k charge) of the field Φ^l_m. Using the above identifications, the integer m may be restricted to the range $-l < m \leq l$. Fields that deserve special mention are the identity, $I = \Phi^0_0$, the parafermions $\psi_l = \Phi^0_{2l}$, and the parafermionic primary fields, also known as spin fields, $\sigma_l \equiv \Phi^l_l$, since the electron creation operator is $\psi_1 e^{i[(k+2)/k]\varphi}$ while the quasiparticle creation operator is $\sigma_1 e^{i(1/k)\varphi}$. The fusion rules for the parafermion CFT fields are given by [18,19]

$$\Phi^{l_\alpha}_{m_\alpha}\Phi^{l_\beta}_{m_\beta} = \sum_{l=|l_\alpha - l_\beta|}^{\min\{l_\alpha + l_\beta, 2k - l_\alpha - l_\beta\}} \Phi^l_{m_\alpha + m_\beta}. \qquad (6)$$

The conformal dimensions of the fields Φ^l_m, which will be crucial in determining the energy spectrum, are $h^l_m = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}$. The conformal dimension of the bosonic sector is $\nu\alpha^2/2$. The short-range product of two fields, known as the operator product expansion (OPE), is given by

$$\Phi^{l_\alpha}_{m_\alpha}(w)\Phi^{l_\beta}_{m_\beta}(z) = \sum_{l_\gamma} C_{\alpha\beta\gamma}(z - w)^{\Delta h}\Phi^{l_\gamma}_{m_\alpha + m_\beta}, \qquad (7)$$

where the fields appearing on the right hand side are determined by Eq. (6), $C_{\alpha\beta\gamma}$'s are constants, and $\Delta h = h^{l_\gamma}_{m_\alpha+m_\beta} - h^{l_\beta}_{m_\beta} - h^{l_\alpha}_{m_\alpha}$. As a consequence of that relation, when a field $\Phi^{l_\alpha}_{m_\alpha}$ goes around a field $\Phi^{l_\beta}_{m_\beta}$ and their fusion is to a field $\Phi^{l_\gamma}_{m_\alpha+m_\beta}$, the phase associated is $2\pi\Delta h$.

Let us now use this general input from the theory of CFT to calculate the spectrum of the edge. First we consider the case when the bulk of the dot does not include any quasiparticles ($n_{\rm is} = 0$). The fusion rules (6) imply that k parafermions ψ_1 fuse to the identity field. This property of the Z_k theory captures the clustering of the electrons in the Read-Rezayi states into groups of k electrons [2]. We imagine starting with the total number of electrons in the dot being divisible by k, and the system being relaxed into its ground state. As the number of electrons is varied, the remainder, which may assume any value between 1 and $k-1$ electrons, accumulates at the edge. The parafermionic state of the edge is then obtained by applying j operators ψ_1 to the vacuum, with $0 \le j \le k-1$. The energy of that state, denoted E_ψ, is calculated in the following way.

The Hilbert space of parafermionic states is constructed by acting with creation modes of the parafermion ψ_1 on the vacuum [19–21]. Although the 1 + 1 dimensional geometry of the edge may be thought of as a cylinder described by a single coordinate, $\xi = v_n t + ix$, where v_n is the velocity of the neutral sector as it propagates along the edge, it is easier to work on the punctured plane by performing a conformal transformation of the coordinates, $z = e^{2\pi\xi/L}$ [22]. On the plane, the parafermion ψ_1 is expanded in modes as follows:

$$\psi_1 = \sum_m z^{-m-h^0_2}\psi^1_m, \tag{8}$$

with $h^0_2 = 1 - 1/k$ being its conformal dimension. The allowed values of the index m are determined by the boundary conditions imposed on ψ_1 by the field it acts on [18,19]. In this case, since it acts on the vacuum, ψ_1 has periodic boundary conditions. Therefore, we must have $m \in \mathbb{Z} + 1/k$. However, if ψ_1 acts on an edge that already contains a parafermion, as in $\psi^1_{m_2}\psi^1_{m_1}|{\rm vac}\rangle$, then when it encircles the already existing parafermion, it accumulates also a phase of $2\pi(h^0_4 - 2h^0_2) = -4\pi/k$. Then, the allowed values of m_2 are $m_2 \in \mathbb{Z} + 3/k$. Similarly, for an edge that contains j parafermions, the allowed modes for the $j+1$ parafermion are $m_{j+1} \in \mathbb{Z} + (2j+1)/k$.

Since $[\mathcal{L}_0, \psi^1_m] = -m\psi^1_m$, where $\mathcal{L}_0$ is the Virasoro algebra generator proportional to the Hamiltonian $H = \frac{2\pi v_n}{L}\mathcal{L}_0$, states created by repetitive applications of ψ^1 modes on the vacuum are eigenstates of the Hamiltonian. A general state with j parafermions is of the form

$$\psi^1_{-p_j+(2j-1)/k}\psi^1_{-p_{j-1}+(2j-3)/k}\cdots\psi^1_{-p_1+1/k}|{\rm vac}\rangle, \tag{9}$$

where the p's are integers. The eigenvalue of $\mathcal{L}_0$ for such a state is $\sum_{i=1}^{j}(p_i - (2i-1)/k)$. States with negative eigenvalues have zero norm and are unphysical.

In Refs. [20,21] it was shown that by choosing the integers p in Eq. (9) such that $p_{i+1} \ge p_i \ge 1$, the set of states obtained is free of zero norm vectors. Therefore the lowest energy state with j parafermions is obtained by choosing $p_i = 1$ for all i. Under these constraints, the lowest allowed value for the energy is therefore

$$E_\psi(j) = \frac{2\pi v_n}{L}\frac{j(k-j)}{k}. \tag{10}$$

To obtain the energy of the state with j electrons on the edge, we must sum E_ψ and the contribution of the bosonic field, E_c, given by Eq. (5) with $N = j$ and $\nu = k/(k+2)$ (the filling fraction of the uppermost partially filled Landau level).

Given the expression (10), together with Eqs. (1) and (5), we can extract the area spacing ΔS,

$$\Delta S = \frac{e}{n_0} + \frac{eL\nu}{2n_0\pi v_c}[E_\psi(N+2) - 2E_\psi(N+1) + E_\psi(N)]. \tag{11}$$

The second term, which is central to our discussion, adds a k dependent modulation to the average spacing e/n_0, and will have two possible values: since j of Eq. (10) is restricted to be in the range $0 \le j \le k-1$ while N is not, the spacing is given by

$$\Delta S_1 = \frac{e}{n_0}\left(1 - \nu\frac{v_n}{v_c}\frac{2}{k}\right) \tag{12}$$

as long as $[N+1]_k \ne 0$. When $[N+1]_k = 0$, the spacing is larger and given by

$$\Delta S_2 = \frac{e}{n_0}\left(1 + \nu\frac{v_n}{v_c}\left(2 - \frac{2}{k}\right)\right). \tag{13}$$

The pattern observed will be a bunching of the Coulomb blockade peaks into groups of k peaks. Within a group, the peaks are separated by ΔS_1, while the area spacing between consecutive groups will be ΔS_2. This k periodicity of the area spacing reflects the construction of the Read-Rezayi states from clusters of k electrons.

The effect of $n_{\rm is}$ bulk quasiparticles on the bosonic part of the edge theory is, similar to the Abelian case, a change in the boundary conditions on φ. That change does not affect $E_c(N+1) - E_c(N)$. The parafermionic energy E_ψ depends on $n_{\rm is}$, since the presence of quasiparticles in the bulk changes the boundary conditions for the field ψ_1 on the edge, and hence its spectrum. We now analyze this effect in detail, and show that it makes ΔS depend on $n_{\rm is}$.

According to the fusion rules (6), $n_{\rm is}$ quasiparticles in the bulk, each created by the operator $\sigma_1 e^{i(1/k)\varphi}$, will fuse to a combination of fields of the form $\Phi^a_{\tilde n} e^{i(n_{\rm is}/k)\varphi}$, where $\tilde n = [n_{\rm is}]_k \equiv n_{\rm is}({\rm mod}k)$ and the possible values of a are determined by (6). Since we start with $N({\rm mod}k) = 0$, the ground state has $a = \tilde n$.

When the parafermionic part of the bulk quasiparticles fuse to $\Phi^{\tilde n}_{\tilde n} = \sigma_{\tilde n}$, the edge is not in a vacuum state even when all electrons on the dot are clustered to clusters of k.

Rather, the state of the edge is $\Phi_{k-\tilde{n}}^{k-\tilde{n}}|\text{vac}\rangle = \sigma_{k-\tilde{n}}|\text{vac}\rangle$. The boundary conditions on a ψ_1 operating on this highest weight state are then

$$\psi_1(ze^{2\pi i}) = e^{2\pi i[(\tilde{n}-k)/k]}\psi_1(z). \quad (14)$$

Accordingly, the modes m in the expansion (8) are $m \in \mathbb{Z} + (k+1-\tilde{n})/k$. Again, the lowest energy state with a single ψ_1 mode is created by the creation operator with the smallest value of $|m|$, with m itself being nonpositive. Similarly to the $n_{\rm is} = 0$ case, the allowed values of m change with the number of parafermions on the edge, and for the jth parafermion become $m \in \mathbb{Z} + (2j - 1 + k - \tilde{n})/k$, where the value of j is limited by $k-1$. Because of the presence of a nontrivial Z_k charge of the highest weight state $\sigma_{k-\tilde{n}}|\text{vac}\rangle$, there will be another restriction on the integers $p_1, \ldots, p_j$ of Eq. (9): for $i > \tilde{n} > 0$ we must choose $p_i \geq 2$ [20].

Again, the energy E_ψ for $j = [N]_k$ parafermions is determined by the sum of the indices m_i for $i = 1, \ldots, j$. This sum depends on $\tilde{n}$ and therefore on $n_{\rm is}$,

$$E_\psi(j,\tilde{n}) = \frac{2\pi v_n}{L}h_\sigma + \frac{2\pi v_n}{L}\begin{cases} \frac{j(\tilde{n}-j)}{k} & j \leq \tilde{n} \\ \frac{(j-k)(\tilde{n}-j)}{k} & j > \tilde{n} \end{cases}, \quad (15)$$

where h_σ is the zero point energy of the spectrum, determined by the conformal dimension of the relevant primary field $\sigma_{k-\tilde{n}}$ acting on the vacuum.

Substituting Eq. (15) in Eq. (11) we may study the spacings between Coulomb blockade peaks through the properties of the spectrum. We again find that the peaks bunch into groups; however, this time they do not bunch into groups of k, but rather into alternating groups of $\tilde{n}$ and $k - \tilde{n}$ peaks. The spacing that separates peaks within a group is again given by Eq. (12), while the spacing that separates two consecutive groups is

$$\Delta S_2 = \frac{e}{n_0}\left(1 + \nu\frac{v_n}{v_c}\left(1 - \frac{2}{k}\right)\right). \quad (16)$$

Therefore, for an odd value of k, the only possible period of the peak structure is k, while when k is even and $[n_{\rm is}]_k = k/2$ we find a periodicity of $k/2$.

For $k = 2$ this result reproduces the even-odd effect predicted to occur at $\nu = 5/2$ in Ref. [8]: for odd $n_{\rm is}$ the periodicity will be $k/2 = 1$, while for even $n_{\rm is}$ it will become $k = 2$.

The reflection of non-Abelian statistics in the magnetic field dependence of ΔS carries over to the case when the point contacts back-scatter also the two edge states of the two filled Landau levels. In that case the peaks we analyzed are superimposed on peaks associated with tunnelling to the edge states of the filled levels. The spacing ΔS of the latter does not depend on magnetic field, however, and may therefore be separated from the ones of the partially filled level [8].

To summarize, we calculated the position of the Coulomb blockade peaks on the two-parameter plane of the area and magnetic field. For a fixed value of the magnetic field, we found that for the Laughlin states, the spacing between peaks is $\Delta S = e/n_0$. For Read-Rezayi states the peaks form groups of k peaks, where each group splits into two subgroups, one containing $[n_{\rm is}]_k$ peaks and the other containing $k - [n_{\rm is}]_k$ peaks. Having a period of k for ΔS is a consequence of the clustering of electrons, similar to the case of a superconductor, where the spacings between Coulomb blockade peaks alternate between two values due to the energy cost associated with having an unpaired electron. The dependence on $n_{\rm is}$ and the periodicity of $k/2$ occurring for even k and $[n_{\rm is}]_k = k/2$ are unique aspects of the non-Abelian nature of the quasiparticles.

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