Classical Perfect Diamagnetism: Expulsion of Current from the Plasma Interior

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The vanishing of generalized helicity is shown to be the necessary and sufficient condition for a perfect conductor to display perfect diamagnetism, considered to be the defining attribute of a conventional superconductor. Although conventional superconductivity is brought about by quantum correlations in classical systems, prepared in the state of zero initial helicity (helicity is a constant of the motion for a perfect conductor), it can mimic the superconductor's behavior.

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A conventional superconductor (SC) is a perfect conductor that expels the magnetic flux from its interior (Meissner-Ochsenfeld effect). For a canonical SC, the flux expulsion property is brought about by quantum phenomena that cause electrons to form Cooper pairs (a correlated electron pair of charge $(-2e)$ and mass $(2m_e)$ which exhibit different properties from the normal electrons. From now on, these correlated electrons, the carriers of superconductivity, will be called superelectrons.

The electrodynamics of perfect diamagnetism, thus, is contained in the dynamics of superelectrons; its expression is the London equation [1], first proposed phenomenologically and then derived from the microscopic theories of superconductivity [2]. In its simplest form, the system consists of the steady-state Maxwell equation

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \tag{1}$$

and the constitutive relationship

$$\nabla \times \mathbf{J} = -\frac{c}{4\pi}\frac{\mathbf{B}}{\lambda_s^2}, \tag{2}$$

where $\lambda_s = c/\omega_{\mathrm{ps}}$ is the skin depth associated with superelectrons $(\omega_{\mathrm{ps}} = (4\pi n_s e_s^2/m_s)^{1/2}$ is the corresponding plasma frequency). Equation (2) readily follows from the condition that the current $\mathbf{J} = -n_s e_s \mathbf{v}_s$ is entirely due to superelectrons stipulated to have zero canonical momentum $m_s\mathbf{v}_s - (e_s/c)\mathbf{A} = 0$.

Equations (1) and (2) are combined to derive the standard form of the London equation

$$\nabla^2\mathbf{B} = \frac{\mathbf{B}}{\lambda_s^2}, \tag{3}$$

which, for a system with dimension $\ell \gg \lambda_s$, confines the magnetic flux to a distance λ_s near the edge; in this narrow skin, the field literally jumps from a value 0 to its external value. Naturally both $\mathbf{B}$ and $\mathbf{J}$ are restricted to the same skin depth. Within the London framework, then, the current and flux expulsion are indistinguishable and equivalent; either expulsion could define the canonical superconductivity.

Before proceeding further, a clarifying statement is in order. This Letter is not a paper on "quantum mechanics" or on the origins of standard superconductivity. It is certainly not an attempt at a classical derivation of standard superconductivity whose origin is indisputably quantum [3]. What I am undertaking is a search for the electrodynamic "signature" of the quantum correlations that yields (2) and (3) for the conventional superconductor and also to explore if such an electrodynamic signature could be forged for a classical gaseous plasma system.

I will begin with comparing the electrodynamics of an SC with that of a magnetized plasma in order to identify the similarities and difference between the two. The hope is that this comparison will lead to a better understanding (electrodynamically speaking) of conventional superconductivity and, in the process help in determining whether a classical magnetized plasma can mimic or display, partially or fully, the magnetic behavior peculiar to a superconductor. Two essential steps in this investigation will be (i) to isolate and identify the physical determinants that control the overall magnetic behavior, and (ii) to suggest possible experiments to test the theoretical underpinnings of the analysis.

Do classical plasma states have anything to match the London state (3)? It turns out that one of the most important and highly investigated plasma states [4,5]

$$\nabla \times \mathbf{B} = \alpha^{-1}\mathbf{B} \Rightarrow \nabla^2\mathbf{B} = -\frac{\mathbf{B}}{\alpha^2}, \tag{4}$$

where α is real, is exactly an antithesis of (3). In fact, in (4), the magnetic flux occupies the whole region (in one Cartesian dimension, $\mathbf{B}$ is oscillatory).

Equation (4), pertaining to a perfectly conducting fluid, was derived in ideal magnetohydrodynamics (MHD) by minimizing the magnetic energy $\langle\rangle = \int d^3x$

$$E_m = \langle \mathbf{B}^2/8\pi \rangle, \tag{5}$$

subject to the constraint that the magnetic helicity

$$h_m = \frac{1}{8\pi}\langle \mathbf{A}\cdot\mathbf{B}\rangle \tag{6}$$

is conserved. Since the unconstrained minimization would lead to the trivial solution $\mathbf{B} = 0$, the recognition of magnetic helicity as an invariant was a major factor in our understanding of the structure of the magnetic fields [6], and the subsequent discovery and development of self-organized states accessible in ideal MHD ([4,5,7]). The helicity, a measure of the structural- topological complexity of a solenoidal vector field, is easily generalizable to systems more complicated than MHD; the constancy of the generalized helicities will be, for instance, exploited in this study to generate new and interesting field configurations.

We will now attempt to chart out the ''electrodynamic differences'' that lead one perfect conductor to exhibit perfect diamagnetism (3), while the other displays precisely the opposite behavior (4).

A perfectly conducting superelectron gas (charge $-e^*$ and mass m^*) obeys

$$\frac{\partial \mathbf{P}}{\partial t} \equiv \frac{\partial}{\partial t}\left(\mathbf{A} - \frac{cm^*}{e^*}\mathbf{u}\right) = \mathbf{u} \times \Omega + \nabla\left(\frac{u^2}{2} + g\right), \quad (7)$$

where

$$\mathbf{\Omega} = \nabla \times \mathbf{P} = \mathbf{B} - \frac{cm^*}{e^*}\nabla \times \mathbf{u}, \quad (8)$$

is called the generalized vorticity (GV), $\mathbf{u}$ is the fluid mechanical velocity, $\mathbf{P}$ is proportional to the canonical momentum, and the last term represents gradient forces (pressure, electrostatic potential). In deriving (7), the vector identity $(\mathbf{u} \cdot \nabla \mathbf{u}) = \nabla u^2/2 - \mathbf{u} \times (\nabla \times \mathbf{u})$ has been invoked. Taking the curl of Eq. (7) converts it to the vortex dynamical form:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{\Omega}). \quad (9)$$

One immediately notices that

$$\mathbf{\Omega} = 0, \quad (10)$$

is a very special solution of (9), because if (10) holds at any time, it is guaranteed to hold for all times. Thus if GV is zero, it is also a constant of motion.

The condition $\mathbf{\Omega} = 0$ (with $\mathbf{J} = -ne^*\mathbf{u}$) is precisely the constitutive relation (2) that yields the London equation. One is then free to interpret that quantum mechanical correlations create this gas of superelectrons and set its generalized vorticity to zero everywhere, i.e., the quantum transitions simply supply an initial condition that assures perfect diamagnetism.

It would appear that, in $\mathbf{\Omega} = 0$, we may have already unearthed the electromagnetic signature of the superconducting state. What I meant by the signature, however, goes a step deeper; it is to identify the controlling physical quantity that ''chooses'' the special singular constitutive law $\mathbf{\Omega} = 0$, and not the more general $\mathbf{\Omega} = \mu\mathbf{u}$, permitted as an equilibrium solution of (9). Surely for a classical system, the latter will be the general solution with $\mathbf{\Omega} = 0$ as a possible limiting case.

In order to identify the said physical quantity, and to investigate the class of states accessible to a classical plasma, we go back to analyze the dynamics of an ideal (perfectly conducting) plasma confined by a uniform magnetic field. The accessible states will be derived through a variational principle.

Let the system consist of several dynamic species (the standard superconductor has only one—the superelectrons), and be embedded in a strong confining magnetic field $\mathbf{B}_0 = \hat{e}_z B_0$ implying that the total magnetic field $\mathbf{B}_T = \mathbf{B} + B_0\hat{e}_z$, where $\mathbf{B}$ is the magnetic field produced by the plasma currents. The equation of motion for each of these perfectly conducting components, derived for a fluid with constant density and isotropic pressure, is [8]

$$\frac{\partial}{\partial t}\mathbf{P}_\alpha = \mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha + B_0(\mathbf{v}_\alpha \times \hat{e}_z) - \nabla\psi_\alpha, \quad (11)$$

where $\mathbf{P}_\alpha = \mathbf{A} + (m_\alpha c/q_\alpha)\mathbf{v}_\alpha$, $\mathbf{\Omega}_\alpha = \nabla \times \mathbf{P}_\alpha = \mathbf{B} + (m_\alpha c/q_\alpha)\nabla \times \mathbf{v}_\alpha$ is the generalized vorticity for the species α with mass (charge) $m_\alpha(q_\alpha)$, and $\psi_\alpha = c/q_\alpha(p_\alpha/n_\alpha + 0.5 m_\alpha v_\alpha^2 + q_\alpha\phi)$ spells out the gradient forces; $p_\alpha(n_\alpha)$ is the pressure (density) and ϕ is the electrostatic potential.

Notice that in (11), I have separated the uniform static externally maintained field $B_0(\partial B_0/\partial t = 0)$ from the dynamic magnetic field $\mathbf{B}$. Equation (11) is tremendously simplified when one considers no variation along the confining field $(\partial/\partial z = 0)$ and deal with only compressible motions $(\nabla \cdot \mathbf{v}_\alpha = 0)$. In that case the velocity $\mathbf{v}_\alpha$ may be expressed as

$$\mathbf{v}_\alpha = v_{z\alpha}\hat{e}_z + \hat{e}_z \times \nabla\chi_\alpha, \quad (12)$$

implying

$$\mathbf{v}_\alpha \times \hat{e}_z = \nabla\chi_\alpha, \quad (13)$$

which converts (11) into

$$\frac{\partial \mathbf{P}_\alpha}{\partial t} = \mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha - \nabla\hat{\psi}_\alpha, \quad (14)$$

with $\hat{\psi}_\alpha = \psi_\alpha - B_0\chi$. We notice that Eq. (14), applicable to a plasma embedded in a strong magnetic field, is entirely equivalent to Eq. (7) with no confining field; the confining field has simply gone to modify the gradient force, which for the purposes of the present paper, is not pertinent. The only magnetic field of relevance in (14) is the dynamic field $\mathbf{B}$, and the dynamics is governed by Eq. (14), its curl

$$\frac{\partial \mathbf{\Omega}_\alpha}{\partial t} = \nabla \times (\mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha), \quad (15)$$

and the Ampere's law $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$ with the current $\mathbf{J} = \sum q_\alpha n_\alpha \mathbf{v}_\alpha$. Straightforward manipulations of (14) and (15) yield the following constants of motion: the total energy $[\langle\rangle = \int d^3x]$

$$E = \left\langle \frac{B^2}{8\pi} + \frac{1}{2}\sum_\alpha n_\alpha m_\alpha v_\alpha^2 \right\rangle, \quad (16)$$

and a generalized helicity,

$$h_\alpha = \frac{1}{8\pi}\langle \mathbf{P}_\alpha \cdot \mathbf{\Omega}_\alpha \rangle, \tag{17}$$

associated with each species. For a perfectly conducting system of n dynamical species, there are a total of $(n+1)$ bilinear invariants. It needs to be emphasized that in any magneto-fluid system, unless the fluid inertia is neglected, it is the generalized helicity h_α, and not the magnetic helicity h_m (6) that is conserved. The conservation of h_m in MHD occurs because the electron inertia is neglected.

One can extract through conventional techniques that such a system, in general, allows a variety of so-called relaxed states derived via the variational principle [9]

$$\delta(E - \mu_\alpha^{-1} h_\alpha) = 0, \tag{18}$$

minimizing the energy with the helicity constraints. The constant μ_α are Lagrange multipliers. The Euler–Lagrange equations that follow

$$\mathbf{\Omega}_\alpha = \mathbf{B} + \frac{m_\alpha c}{q_\alpha}\nabla \times \mathbf{v}_\alpha = (4\pi/c)\mu_\alpha q_\alpha n_\alpha \mathbf{v}_\alpha \tag{19}$$

align the generalized vorticities of each species along its velocity. All variations are incompressible (densities n_α are constant) and normal components of fields vanish at the boundaries. Equation (19) is an equilibrium solution of (11)–(15) provided the Bernoulli condition $(\nabla\psi) = 0$ is satisfied. The consequences of the latter conditions can be important [8,9], but for our current problem, the Bernoulli conditions are not directly relevant.

The structure of the magnetic and velocity fields can be obtained by solving (19) in conjunction with Ampere's law. The emerging field configurations belong to the class of minimum energy relaxed states widely investigated in plasma physics [5–9]. For this paper I will not pursue the general case, but limit further investigation to the simplest system most relevant to the study of classical superconductivity. I will assume (i) the plasma has two distinct components—a bulk plasma of essentially stationary (nondynamic) electrons and ions, and a dynamic fast component that could be of either electrons or ions, and (ii) the dynamic component (to be designated fast with a label f) carries all the current as well as the kinetic energy of the system, i.e., $|q_f n_f \mathbf{v}_f| \gg |q_e n_e \mathbf{v}_e|, |q_i n_i \mathbf{v}_i|$, and $n_f m_f v_f^2 \gg n_e m_e u_e^2, n_i m_i v_i^2$. The fast dynamical component is the one that will mimic the superelectrons; the bulk plasma is needed to insure charge neutrality, and does not contribute to the plasma current. The fast dynamical component due to its large speeds (relative to the ambient plasma) will undergo a negligible number of momentum changing collisions, and, thus, can be justifiably treated as ideal (infinite conductivity).

In this scenario, our system is fully defined by a single generalized helicity h_f, and the equilibrium fields can be calculated by solving the set

$$\mathbf{\Omega}_f \equiv \mathbf{B} + \left(\frac{m_f c}{q_f}\right)\nabla \times \mathbf{v}_f = \mu_f\left(\frac{4\pi}{c}\right) q_f n_f \mathbf{v}_f, \tag{20}$$

$$\nabla \times \mathbf{B} = \left(\frac{4\pi}{c}\right) q_f n_f \mathbf{v}_f. \tag{21}$$

Normalizing $|\nabla|$ to λ_f^{-1} [λ_f is the skin depth associated with the fast component, $\lambda_f^2 = c^2/\omega_{\rm pf}^2$, $\omega_{\rm pf}^2 = (4\pi q_f^2 n_f/m_f)$], Equations (20) and (21) yield after simple algebra ($\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B}$):

$$\nabla \times \nabla \times \mathbf{B} + \mathbf{B} = \frac{\mu_f}{\lambda_f}\nabla \times \mathbf{B}, \tag{22}$$

which, for the Lagrange multiplier $\mu_f = 0$ is nothing but the London equation with fields restricted to a skin depth λ_f associated with the fast dynamic component. Surely $\mu_f = 0$ (which for the relaxed state insures $\mathbf{\Omega}_f = 0$) is just the necessary and sufficient condition for perfect diamagnetism that we set out to derive.

Before we find a physical interpretation of μ_f, we note that (14) has a general solution ($A_\pm$ are constants) [9]

$$\mathbf{B} = A_+ \mathbf{G}_+ + A\mathbf{G}_- \tag{23}$$

where $\mathbf{G}_\pm$, known as the Beltrami fields, are the solutions of

$$\nabla \times \mathbf{G}_\pm = \lambda_\pm \mathbf{G}_\pm \tag{24}$$

with

$$\lambda_\pm = 0.5\left\{\frac{\mu_f}{\lambda_f} \pm \left[\left(\frac{\mu_f}{\lambda_f}\right)^2 - 4\right]^{1/2}\right\}. \tag{25}$$

The roots $\lambda_\pm$ are real for $(\mu_f/\lambda_f)^2 > 4$ but form a complex conjugate pair for $(\mu_f/\lambda_f)^2 < 4$. In the latter case $\mathbf{G}_+^* = \mathbf{G}_-$ and A_+ must be A_-^* in order for the solution [$\mathbf{B} = 2\mathrm{Re}(A_+\mathbf{G}_+)$] to be real. As $|\mu_f/\lambda_f|$ goes from zero to larger values, the system begins with perfect diamagnetism ($\lambda_\pm = \pm i$), switches to partial diamagnetism ($\pm$ complex), and finally succumbs to the behavior exhibited by Eq. (4) ($\lambda_\pm$ real), which we may just label "zero diamagnetism"; the transition from complex to real roots happens at the critical value $|\mu_f/\lambda_f| = 2$. Evidently the amount of diamagnetism displayed by this relaxed state is controlled by the parameter $|\mu_f/\lambda_f|$.

An expression for μ_f for the relaxed equilibrium state (20)–(22) is readily derived through the following steps:

$$\begin{aligned} h_f &= \frac{1}{8\pi}\langle \mathbf{P}_f \cdot \mathbf{\Omega}_f \rangle = \frac{\mu_f}{2c}\left\langle\left(\mathbf{A} + \frac{m_f c}{q_f}\mathbf{v}_f\right)\cdot q_f n_f \mathbf{v}_f\right\rangle \\ &= \mu_f\left\langle \frac{1}{2} m_f n_f v_f^2 + \frac{1}{8\pi}\mathbf{A}\cdot\nabla\times\mathbf{B}\right\rangle \\ &= \mu_f\left\langle \frac{1}{2} m_f n_f v_f^2 + \frac{B^2}{8\pi}\right\rangle, \end{aligned}$$

yielding the revealing identification

$$\mu_f = \frac{h_f}{E}; \tag{26}$$

the Lagrange multiplier (dimensions of a length) is nothing but a measure of the generalized helicity as a fraction of the total energy. A ratio of two constants of motion, the control parameter μ_f is an invariant of the system and is fully determined by the initial "preparation" of the system.

In the light of preceding discussion and analysis, we are now in a position to make some definitive statements on our understanding of the electrodynamics of the superconducting state:

(i) The electrodynamics of a standard superconductor and that of a magnetically confined ideal plasma (consisting of a fast dynamic component and an essentially stationary ion-electron bulk plasma; the latter needed to provide overall charge neutrality) is almost identical as long as one concentrates on the dynamic magnetic field produced by the currents in the material. In either case, the generalized helicity $h = (8\pi)^{-1}\langle \mathbf{P} \cdot \mathbf{\Omega} \rangle$ (we will drop the subscript f in the remainder of the Letter), an integral measure of the "knottedness" of the field of generalized vorticity $\mathbf{\Omega}$, is a constant of the motion, and emerges as a fundamental determinant of the class of magnetic field configurations that the system can entertain.

(ii) The state of perfect diamagnetism corresponds to $h = 0$. For minimum energy relaxed states, this condition [via (20)] automatically leads to the constitutive relationship $\mathbf{\Omega} = 0$ [$\mathbf{\Omega} \propto \mu \mathbf{v}$, $\mu = h/E$] the very definition of superconductivity. Since h is a constant of the motion, its value is determined by the initial conditions. From this perspective, the electrodynamics of a superconductor is fully reproduced if the quantum correlations provided the correct initial condition, $h = 0$. It is quite remarkable that quantum correlations do produce the superelectrons precisely in this state.

(iii) Once the identification of helicity h as the fundamental determinant of the "diamagnetic content" is established, one can conceive of a whole series of experiments on classical systems. The classical plasma, as we note form (20)–(23), is not bound to be in a helicity-free state; it can, in principle, entertain (or be prepared in) configurations of arbitrary helicity (or helicity and energy). Its magnetic behavior, therefore, can vary over a broad range, from perfect or nearly perfect diamagnetism to no diamagnetism.

(iv) What must be emphasized, however, is that perfect diamagnetism is, by no means, denied to a classical system. As long as the length $\mu = h/E$ (may be termed the decorrelation length) is much smaller than the skin depth, the system approaches the state of perfect diamagnetism. A clever experimentalist can play, for example, with a beam-plasma system (by experimenting with how to inject an ion or an electron beam in an ambient plasma) to bring the generalized helicity to any arbitrary value including a vanishingly small one. In the process, she could accomplish the same feat for a classical plasma what quantum mechanics does in a conventional superconductor.

(v) Even in the zero or near zero helicity state, classical systems display immense variety in the degree of localization. With appropriate choices of fast electron and/or ion beams with a range of densities, one can create skin lengths which can vary over several orders of magnitude. Thus current channels of arbitrary extent could be experimentally created. Whenever one finds excessive localization of current in space, astrophysical or laboratory plasmas, one should look, it seems, for a classical superconducting explanation.

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[1] F. London, Physica (Amsterdam) **2**, 341 (1935); F. London, Physica (Amsterdam) **3**, 450 (1936); F. London, Nature (London) **137**, 991 (1936).
[2] See, for example, J. R. Schreiifer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964).
[3] There was an attempt to derive the London equation from classical dynamics by W. F. Edwards, Phys. Rev. Lett. **47**, 1863 (1981); and was shown to be erroneous by J. B. Taylor, Nature (London) **299**, 681 (1982).
[4] L. Woltjer, Proc. Natl. Acad. Sci. U.S.A. **44**, 489 (1958); S. Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A. **42**, 1 (1956); S. Chandrasekhar and L. Woltjer, Proc. Natl. Acad. Sci. U.S.A. **44**, 4 (1958).
[5] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).
[6] For an excellent introduction to magnetic helicity, see, H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*, (Cambridge University Press, Cambridge, England, 1978); For a very general relativistic formulation of generalized helicities, see, S. M. Mahajan, Phys. Rev. Lett. **90**, 035001 (2003).
[7] A. Hasegawa, Adv. Phys. **34**, 1 (1985). In this review, the use of the helicity invariant as an agent for inverse cascading is thoroughly described.
[8] S. M. Mahajan and Z. Yoshida, Phys. Plasmas **7**, 635 (2000).
[9] R. N. Sudan, Phys. Rev. Lett. **42**, 1277 (1979); J. M. Finn and T. M. Antonsen, Jr., Phys. Fluids **26**, 3540 (1983); L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997); S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).