Energy Decay Laws in Strongly Anisotropic Magnetohydrodynamic Turbulence

Barbara Bigot,[1,2] Sébastien Galtier,[1] and Hélène Politano[2]
[1]Institut d'Astrophysique Spatiale (IAS), Bâtiment 121, F-91405 Orsay (France); Université Paris-Sud XI and CNRS (UMR 8617), France
[2]Laboratoire Cassiopée, UMR 6202, OCA, BP 42229, 06304 Nice Cedex 4, France
(Received 21 June 2007; published 19 February 2008)

We investigate the influence of a uniform magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_\parallel$ on energy decay laws in incompressible magnetohydrodynamic (MHD) turbulence. The nonlinear transfer reduction along $\mathbf{B}_0$ is included in a model that distinguishes parallel and perpendicular directions, following a phenomenology of Kraichnan. We predict a slowing down of the energy decay due to anisotropy in the limit of strong B_0, with distinct power laws for energy decay of shear- and pseudo-Alfvén waves. Numerical results from the kinetic equations of Alfvén wave turbulence recover these predictions, and MHD numerical results clearly tend to follow them in the lowest perpendicular planes.

DOI: 10.1103/PhysRevLett.100.074502 PACS numbers: 47.27.Jv, 47.65.−d, 52.30.Cv, 95.30.Qd

Predicting the evolution of freely decaying turbulence remains one of the most difficult problems in turbulence. For the sake of simplicity, this problem is often tackled under the assumption of homogeneity and isotropy. In the case of 3D Navier-Stokes flows, the Kolmogorov prediction for the kinetic energy decay law thus reads $E_v(t) \sim (t - t_*)^{-10/7}$, where t_* is the time origin for power law decay [1]. Close power law indices are measured in grid turbulence experiments although boundary effects may alter the result [2]. The generalization of decay laws to other situations is still currently under debate. For example in rotating turbulence, a recent experiment shows that neutral fluids behave differently from pure (non rotating) flows with a slowing down of the kinetic energy decay due to the presence of anisotropy [3]. In this case, the original isotropic Kolmogorov description has to be modified to include rotation effects that may lead to strong spectral anisotropy (see, e.g., Ref. [4]).

In this Letter, we investigate the influence of an external uniform magnetic field on the energy decay laws in freely incompressible magnetohydrodynamic (MHD) turbulence. The MHD approximation has proved to be quite successful in the study of a variety of astrophysical plasmas like those found in the solar corona, the interplanetary medium or in the interstellar clouds. These media are characterized by extremely large Reynolds numbers (up to 10^{13}) [5] with a range of available scales from 10^{18} m to few meters. The isotropy assumption is particularly difficult to justify when dealing with astrophysical flows since a large-scale magnetic field is almost always present like in the inner interplanetary medium where the magnetic field lines form an Archimedean spiral near the equatorial plane (see, e.g., Ref. [6]). The present study, although theoretical, appears therefore particularly important to extract some universal features of turbulent plasmas. The MHD equations in presence of an external uniform magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_\parallel$ read

$$\partial_t \mathbf{v} - B_0\partial_\parallel \mathbf{b} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P_* + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\Delta\mathbf{v}, \quad (1)$$

$$\partial_t \mathbf{b} - B_0\partial_\parallel \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{v} + \eta\Delta\mathbf{b}, \quad (2)$$

with $\nabla\cdot\mathbf{v} = 0$ and $\nabla\cdot\mathbf{b} = 0$. The magnetic field $\mathbf{b}$ is normalized to a velocity ($\mathbf{b} \rightarrow \sqrt{\mu_0 n m_i}\mathbf{b}$, with m_i the ion mass and n the electron density), $\mathbf{v}$ is the plasma flow velocity, P_* the total (magnetic plus kinetic) pressure, ν the viscosity and η the magnetic diffusivity. The role of the $\mathbf{B}_0$ field on the flow behavior has been widely discussed in the community (see, e.g., Ref. [7–14]). One of the most clearly established results is the bidimensionalization of an initial isotropic energy spectrum with a strong reduction of nonlinear transfers along $\mathbf{B}_0$.

In the past, several papers have been devoted to predictions of energy decay laws in isotropic MHD turbulence [15–19]. We first review one of them [19] which follows the Kolmogorov-Kraichnan spirit that we then adapt to strongly anisotropic flows. In the simplest case (balance turbulence), the derivation of self-similar decay laws in MHD relies on few basic assumptions which are (i) a weak correlation between velocity and magnetic fields (which allows to use below the variable z instead of $z^\pm$, where $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$), (ii) a power law spectrum in $E(k) \sim k^s$ (with $s = D + 1$, D being the space dimension) for the low wave number energy, i.e., at scales larger than the integral scale ℓ from which the inertial range begins, and, obviously (iii) a power law time dependence for $E(t) \sim (t - t_*)^{-\alpha}$ and $\ell(t) \sim (t - t_*)^{\beta}$, where α and β are two unknown indices. Another hypothesis, directly related to the invariance of the Loitsianskii integral [20], tells that the modal spectrum scales like k^{s-2} at low wave numbers and, with sufficient scale separation, this dominates the total energy so that $E \sim \ell^{-(s+1)}$, hence the first relation $\alpha = \beta(s + 1)$. A second relation may be obtained from the energy transfer equation $\epsilon = -dE/dt \sim E/\tau_{\rm tr} \sim (t - t_*)^{-\alpha-1}$, where $\tau_{\rm tr}$ is the transfer time and ϵ is the transfer rate. According to the Iroshnikov-Kraichnan (IK) phenomenology [21],

$\tau_{tr} = \tau_{NL}^2/\tau_A$, with $\tau_{NL} = \ell/z_\ell$, the eddy turnover time, and $\tau_A = \ell/B_0$, the Alfvén time. Substituting previous times into the energy transfer equation leads to $1 = \alpha + \beta$. Finally one obtains $\alpha = (s+1)/(s+2)$ and $\beta = 1/(s+2)$. The predictions for 3D MHD turbulence ($s = 4$) are then $E(t) \sim (t - t_*)^{-5/6}$ and $\ell(t) \sim (t - t_*)^{1/6}$. The energy decay law for isotropic MHD turbulence has been favorably compared with direct numerical simulations, in particular, in the 2D case for which higher Reynolds numbers can be reached than in the 3D case [18,19]. Note that other phenomenologies exist for explaining, in particular, the influence of magnetic helicity on the decay laws [22], a situation different from here where it is negligible.

We now derive an anisotropic version of the previous heuristic description which is well adapted to MHD flows permeated by a strong uniform magnetic field. The main characteristic of such flows is that nonlinear transfers are strongly damped along the mean field $\mathbf{B}_0$ direction, leading preferentially to perpendicular ($\perp$) transfers. In Fourier space, this means that the modal energy spectrum mainly develops a power law scaling in $k_\perp$-wave numbers whereas the scaling along parallel wave numbers, $k_\parallel$, does not change very much. In practice, we assume that the intensity B_0 is strong enough to ignore parallel transfers and we make the approximation $k \sim k_\perp \gg k_\parallel$ which means, in particular, that the initial condition of the system is confined to the largest parallel scales. In this situation, it is straightforward to show that shear-Alfvén wave energy corresponds to perpendicular fields ($\mathbf{z}_\perp$) whereas pseudo-Alfvén wave energy mainly comes from parallel fields ($\mathbf{z}_\parallel$). Our heuristic description is focused on shear-Alfvén waves, and not pseudo-Alfvén waves, since it is well known that under such anisotropic configuration, the latter are slaved to the former [9,12].

We make three equivalent hypotheses as in the isotropic model and we note $E_\perp$, $\ell_\perp$ and $\ell_\parallel$ the shear-Alfvén wave energy, the perpendicular and parallel integral scales, respectively. The $\ell_\parallel$ length scale, although appearing sometimes in our derivation, is assumed time independent because of the strong nonlinear anisotropy. In that case, we assume (a) $v_\perp \sim b_\perp \sim z_\perp$, (b) for $k_\parallel$ fixed, $E_\perp(k_\perp, k_\parallel) \sim k_\perp^D$ (where D is the space dimension) at low perpendicular wave numbers, i.e., at scale larger than $\ell_\perp$, and (c) $E_\perp(t) \sim (t - t_*)^{-\bar{\alpha}}$ and $\ell_\perp \sim (t - t_*)^{\bar{\beta}}$, where $\bar{\alpha}$ and $\bar{\beta}$ are the new unknown indices. The second assumption means a $D - 1$ power law index for the modal spectrum before the integral $k_\perp$-wave number; furthermore it leads to estimate $E_\perp(t) \sim \ell_\perp^{-(D+1)}\ell_\parallel^{-1}$, and hence the (same) first relation

$$\bar{\alpha} = \bar{\beta}(D+1). \tag{3}$$

A second relation may be obtained by using the energy transfer equation

$$\epsilon_\perp = -dE_\perp/dt \sim E_\perp/\tau_{tr} \sim (t - t_*)^{-\bar{\alpha}-1}, \tag{4}$$

where $\tau_{tr} = \tau_{NL}^2/\tau_A$, with, for anisotropic transfers, $\tau_{NL} = 1/(k_\perp z_{\perp \ell_\perp})$ and $\tau_A = 1/(k_\parallel B_0)$. Substituting these times into (4) leads to the (new) relation

$$1 = \bar{\alpha} + 2\bar{\beta}. \tag{5}$$

Finally, (3) and (5) lead to two new scaling exponents:

$$\bar{\alpha} = \frac{D+1}{D+3}; \qquad \bar{\beta} = \frac{1}{D+3}. \tag{6}$$

Hence the predictions for 3D anisotropic MHD turbulence ($D = 3$) $E_\perp(t) \sim (t - t_*)^{-2/3}$ and $\ell_\perp(t) \sim (t - t_*)^{1/6}$. These results show, in particular, a slowing down of the energy decay for shear-Alfvén waves compared to the total energy in the isotropic case (where no distinction is made between shear- and pseudo-Alfvén waves).

The extension of our approach to pseudo-Alfvén waves is not direct since these waves are (mainly) slaved to shear-Alfvén waves. We remind that shear-Alfvén and pseudo-Alfvén waves are the two kinds of linear perturbations about the equilibrium, the latter being the incompressible limit of slow magnetosonic waves. For the pseudowaves, the heuristic description based on nonlinear transfers is misleading. Instead, it seems suitable to find a relationship between the different type of waves. The divergence free condition provides this relation which eventually leads to a prediction for the energy decay law. In Fourier space, the divergence free condition reads

$$\mathbf{k}_\perp \cdot \hat{\mathbf{z}}_\perp + \mathbf{k}_\parallel . \hat{\mathbf{z}}_\parallel = 0, \tag{7}$$

where $\hat{\mathbf{z}}_\perp$ and $\hat{\mathbf{z}}_\parallel$ are the Fourier transform of the Cartesian fields which may be associated, under strong anisotropy assumption, mainly to the shear- and pseudo-Alfvén waves, respectively. Since we assume a weak cross correlation (balance turbulence), it is not necessary to introduce the Elsässer variables. Simple manipulations lead to $k_\perp^2 E_\perp \sim k_\parallel^2 E_\parallel$, where $E_\parallel$ denotes the pseudo-Alfvén wave energy. Since nonlinear transfers along the $\mathbf{B}_0$ direction are negligible, $k_\parallel$ may be seen as a mute variable. Therefore, the energy decay law for pseudo-Alfvén waves should be

$$E_\parallel \sim (t - t_*)^{-1}. \tag{8}$$

Our anisotropic model thus predicts quite different energy decay laws for shear- and pseudo-Alfvén waves, the latter not depending on the system dimensions. An important issue concerns the balance $k_\parallel \sim k_\perp^{2/3}$ law found in many simulations (see, e.g., [23]). Our analysis is based on the permanence of big eddies [1] which is directly linked to the conservation of the spectral scaling law at the largest scales, i.e., at scales larger than the ones of the inertial range where the "critical balance" is well observed. The 2/3 law is *a priori* not included in the model (as well as the exact scaling law for the energy spectrum) because the assumption of the existence of an inertial range where energy is evacuated from the reservoir is enough. Therefore, the integral length scales $\ell_\perp$ and $\ell_\parallel$ are not linked by the 2/3 law even in the derivation of Eq. (5) since we are only dealing with scales before the inertial range. The possible persistence of the 2/3 law at the largest scales is nevertheless an important issue which, in principle, may be investigated from the derivation of the von Kármán–Howarth equation in anisotropic MHD turbulence, from which a Loitsianskii type invariant could be found. This is a huge challenge for the future which is, of course, out of the scope of this Letter.

In order to check the model validity, we perform numerical simulations of strongly anisotropic MHD flows. The first data set comes from integrations of the kinetic equations of Alfvén wave turbulence derived and analyzed in [12]. This regime describes the asymptotic limit of strong B_0. We do not rederive or even rewrite these equations that the reader can find in [12] [Eqs. (54) and (55)]. One of the main results found in this regime is a $k_\perp^{-2}$ scaling law for the energy spectrum (for both shear- and pseudo-Alfvén waves) which is an exact power law solution of the wave kinetic equations (in absence of cross-correlation). Another important result is the total absence of parallel transfer. For that reason, the numerical simulations are made at a fixed $k_\parallel$. A nonuniform grid in Fourier space (see for details [12]) is used to achieve a highly turbulent state at a Reynolds number of about 10^5, with $\nu = \eta = 2 \times 10^{-5}$. Figure 1 displays the time evolution of the shear-Alfvén, pseudo-Alfvén and total energies in absence of cross correlation.

We first note the existence of a transient period during which the energy is conserved. During this period the energy cascades towards smaller scales following a $k_\perp^{-7/3}$ scaling law rather than the $k_\perp^{-2}$ exact solution [12]. A clear power law behavior appears for both types of waves. These power laws are in agreement with the theoretical predictions made for 3D anisotropic flows (for comparison, the predicted slopes for the 3D and 2D isotropic cases are also given). Surprisingly, the total energy follows very precisely the theoretical prediction over more than two decades

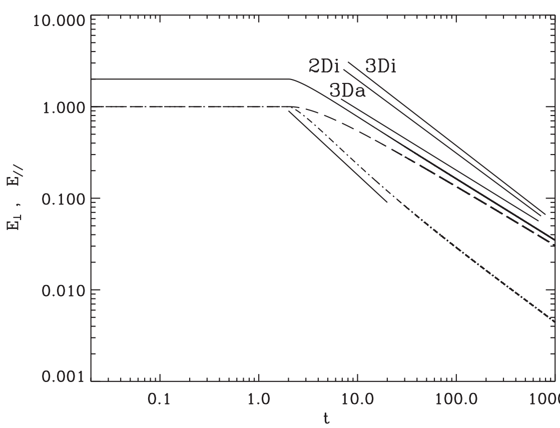

FIG. 1. Temporal decays of shear-Alfvén (long-dashed), pseudo-Alfvén (dash-dotted) and total (solid) energies, together with three theoretical slope predictions: 3D isotropic (3Di) in $t^{-5/6}$, 2D isotropic (2Di) in $t^{-4/5}$, and 3D anisotropic (3Da) in $t^{-2/3}$. A t^{-1} slope is also plotted for comparison with the pseudo-Alfvén energy decay.

whereas pseudo-Alfvén waves decay slower than t^{-1} at very large times. This discrepancy may be linked to the saturation of the integral length scale as it can be observed from the temporal evolution of the shear-Alfvén wave spectra (Fig. 2). Indeed, a self-similar energy decay is observed with a slow increase of the integral length scale that can be roughly estimated from the wave number at which the inertial range begins (i.e., the maxima of the spectra). At later times, this scale is close to the maximum size of the numerical box whereas the large-scale power law in $k_\perp^3$ is still preserved. The large scales are even more reduced for (slaved) pseudo-Alfvén waves since they decay faster. Actually, the finite-size box effect may explain the change of decay law at very large times for pseudo-Alfvén wave energy.

Direct numerical simulations of 3D incompressible MHD Eqs. (1) and (2) are also performed with a pseudo-spectral code including dealiasing. A high resolution with 512^2 in the $\mathbf{B}_0$-transverse planes whereas only 64 grid points are taken in the longitudinal direction. Such a situation was analyzed to explore the self-consistency of the reduced MHD model [24] with the conclusion that small values of viscosities, adjusted according to the transverse dynamics, are not incompatible with the smaller spatial resolution in the longitudinal direction since the transfer towards small scales is also reduced along the uniform magnetic field. We have checked that viscosity values, $\nu = \eta = 5 \times 10^{-4}$, are indeed well adjusted [25]. The initial condition corresponds to a modal energy spectrum in agreement with the phenomenology described above, with a $D = 3$ power law at largest scales: $E^\pm(k_\perp, k_\parallel) = C(k_\parallel)k_\perp^3$ for $k_\perp$ and $k_\parallel \in [0, 4]$, the value of $C(k_\parallel)$ increasing with $k_\parallel$ to reach a maximum at $k_\parallel = 4$. This initial spectrum allows a transient period of cascade towards smaller scales during which energy is mainly conserved. The uniform magnetic field is fixed to $B_0 = 15$. Initially, the ratio between kinetic and magnetic energies is one, whereas the correlation $2\langle \mathbf{v} \cdot \mathbf{b} \rangle / \langle |\mathbf{v}|^2 + |\mathbf{b}^2| \rangle$ is zero (remaining less than 4% up to $t = 40$). In Fig. 3, time evolutions of shear- and pseudo-Alfvén wave energies are

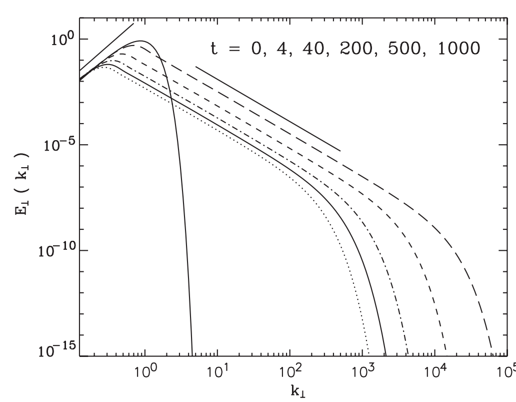

FIG. 2. Temporal evolution of shear-Alfvén energy spectra. A $k_\perp^3$ and $k_\perp^{-2}$ power laws are plotted for comparison.

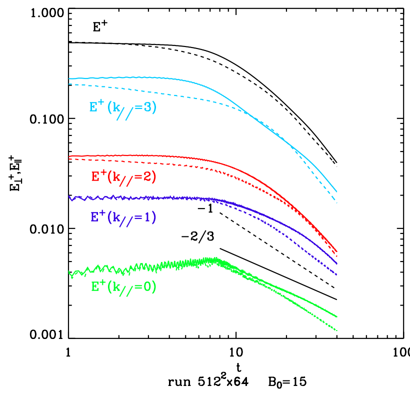

FIG. 3 (color online). Evolution of shear- (solid lines) and pseudo-Alfvén (dash lines) wave spectral energies at fixed $k_\parallel$ wave number, together with the total energies. Note that curves at $k_\parallel = 0$ and $k_\parallel = 1$ are shifted by a factor of 2 for clarity.

plotted at fixed parallel wave numbers, namely $k_\parallel = 0$, 1, 2, 3, together with total energies (as obtained by integrating $E_\parallel$ over all $k_\parallel$). Note that we chose to show the behavior of one type of waves, i.e. $E^+_{\perp,\parallel}$ (with by definition $E^\pm = \langle |\mathbf{z}^\pm|^2 \rangle /2$), as $E^-_{\perp,\parallel}$ energies behave the same way. We observe that systematically $E^+_\perp > E^+_\parallel$ at the final time which confirms the previous results. A clear tendency towards power law behavior is also found, in particular, at low $k_\parallel$, with exponents close to the heuristic predictions. This was already observed at smaller resolution (not shown) and seems to be a general tendency of anisotropic MHD flows. However, this is less clear in planes at $k_\parallel = 2$ and 3, probably due to enhanced dissipations there (for example, at $t = 20$, $\sim$60% of the shear wave energy is lost at $k_\parallel = 2$, and $\sim$73% at $k_\parallel = 3$) leading thus to shortened self-similar decay ranges. Actually, this remark may explain the power law steepening of the energy decay, when integrated over all $k_\parallel$, since the energy loss is even more pronounced in higher $k_\parallel$ planes. This average effect has never been emphasized in the literature but seems to be the most important obstacle to see the decay laws as well as the spectral laws predicted by anisotropic theories like wave turbulence [26]. Note that other initial conditions with an extended energy spectrum in $k_\parallel$ might reduce the discrepancy found between the model and the simulation at high $k_\parallel$, a situation not investigated here because of the constraint due to the reduced parallel resolution.

In this Letter, we show the influence of a uniform magnetic field on energetic decays in MHD turbulence. Modified self-similar laws are derived, with a $t^{-2/3}$ and a t^{-1} decay, respectively, for shear- and pseudo-Alfvén waves, under strong anisotropy. To our knowledge, this decay analysis is the first in the context of wave turbulence and could be extended to many other problems. Integrations of kinetic equations of Alfvén wave turbulence recover the predicted laws, while our MHD numerical flows follow them in planes at lowest parallel wave numbers.

Financial support from PNST/INSU/CNRS are gratefully acknowledged. Computations were performed at IDRIS (CNRS) Grant No. 0597.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **31**, 538 (1941); **32**, 19 (1941).
[2] G. Comte-Bellot and S. Corrsin, J. Fluid Mech. **25**, 657 (1966).
[3] C. Morize and F. Moisy, Phys. Fluids **18**, 065107 (2006).
[4] S. Galtier, Phys. Rev. E **68**, 015301 (2003).
[5] T. Tajima and K. Shibata, *Plasma Astrophysics* (Westview, Boulder, USA 2002).
[6] S. Galtier, J. Low Temp. Phys. **145**, 59 (2006).
[7] H. R. Strauss, Phys. Fluids **19**, 134 (1976); D. Montgomery and L. Turner, Phys. Fluids **24**, 825 (1981).
[8] J. V. Shebalin, W. H. Matthaeus, and D. Montgomery, J. Plasma Phys. **29**, 525 (1983); S. Oughton, E. R. Priest, and W. H. Matthaeus, J. Fluid Mech. **280**, 95 (1994).
[9] P. Goldreich and S. Sridhar, Astrophys. J. **438**, 763 (1995).
[10] C. S. Ng and A. Bhattacharjee, Astrophys. J. **465**, 845 (1996).
[11] W. H. Matthaeus, S. Oughton, S. Ghosh, and M. Hossain, Phys. Rev. Lett. **81**, 2056 (1998).
[12] S. Galtier, S. Nazarenko, A. C. Newell, and A. Pouquet, J. Plasma Phys. **63**, 447 (2000); S. Nazarenko, A. C. Newell, and S. Galtier, Physica (Amsterdam) **152–153D**, 646 (2001).
[13] L. J. Milano, W. H. Matthaeus, P. Dmitruk, and D. C. Montgomery, Phys. Plasmas **8**, 2673 (2001).
[14] B. D. G. Chandran, Phys. Rev. Lett. **95**, 265004 (2005).
[15] T. Hatori, J. Phys. Soc. Jpn. **53**, 2539 (1984);
[16] H. Politano, A. Pouquet, and P.-L. Sulem, Phys. Fluids B **1**, 2330 (1989).
[17] D. Biskamp, *Nonlinear Magnetohydrodynamics* (Cambridge University Press, Cambridge, England, 1994); M. Hossain, P. Gray, D. Pontius, W. Matthaeus, and S. Oughton, Phys. Fluids **7**, 2886 (1995).
[18] R. Kinney, J. C. McWilliams, and T. Tajima, Phys. Plasmas **2**, 3623 (1995).
[19] S. Galtier, H. Politano, and A. Pouquet, Phys. Rev. Lett. **79**, 2807 (1997).
[20] P. A. Davidson, *Turbulence* (Oxford University Press, New York, 2004).
[21] P. S. Iroshnikov, Sov. Astron. **7**, 566 (1964); R. H. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[22] D. Biskamp and W.-C. Müller, Phys. Rev. Lett. **83**, 2195 (1999).
[23] J. Cho, A. Lazarian, and E. T. Vishniac, Astrophys. J. **564**, 291 (2002).
[24] S. Oughton, P. Dmitruk, and W. H. Matthaeus, Phys. Plasmas **11**, 2214 (2004).
[25] Such a small aspect ratio may reduce the number of resonant wave interactions which in turn may affect the dynamics (see, e.g., L. Smith and F. Waleffe, Phys. Fluids **11**, 1608 (1999)). However, in Alfvén wave turbulence, the resonant manifolds foliate wave vector space [12] which, in principle, prevent such a problem.
[26] B. Bigot, S. Galtier, and H. Politano (unpublished).