Mach-Zehnder Interferometry at the Heisenberg Limit with Coherent and Squeezed-Vacuum Light

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We show that the phase sensitivity $\Delta\theta$ of a Mach-Zehnder interferometer illuminated by a coherent state in one input port and a squeezed-vacuum state in the other port is (i) independent of the true value of the phase shift and (ii) can reach the Heisenberg limit $\Delta\theta \sim 1/N_T$, where N_T is the average number of input particles. We also demonstrate that the Cramer-Rao lower bound of phase sensitivity, $\Delta\theta \sim 1/\sqrt{|\alpha|^2 e^{2r} + \sinh^2 r}$, can be saturated for arbitrary values of the squeezing parameter r and the amplitude of the coherent mode α by using a Bayesian phase inference protocol.

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Introduction.—The goal of quantum interferometry is to estimate phases beyond the shot-noise (or "standard quantum") limit. This field was initiated by Caves in 1981 [1], who first suggested how to reach a sub–shot-noise sensitivity by using coherent ⊗ squeezed-vacuum light as input of a Mach-Zehnder (MZ) interferometer. A large body of theoretical [2–5] and experimental [6,7] studies have followed this work. The scheme proposed by Caves is sketched in Fig. 1. One of the input states of the linear lossless MZ is the coherent field $|\alpha\rangle_a \equiv \sum_{m=0}^{+\infty} C_m |m\rangle_a$, with $\alpha \equiv e^{i\theta_c}|\alpha|$ and $C_m \equiv \frac{\alpha^m e^{-|\alpha|^2/2}}{\sqrt{m!}}$. The second input is the squeezed-vacuum $|\zeta\rangle_b \equiv \sum_{m=0}^{+\infty} S_m |m\rangle_b$, where $\zeta \equiv re^{i\theta_s}$, $S_m \equiv \frac{(e^{i\theta_s}\tanh r)^{m/2}}{2^{m/2}\sqrt{m!}\cosh r} H_m(0)$ [2] and $H_m(x)$ are the Hermite polynomials. According to the current literature, based on the original works of the 1980s, the unknown value of the phase shift between the arms of the interferometer can be estimated from the measurement of the relative number of particles at the output ports, $\hat{M}_{\rm out} = \hat{N}_c - \hat{N}_d$. Fluctuations on the results obtained in p independent measurements propagate to the estimated value of the phase shift θ [8], which can therefore be determined with uncertainty [9]:

$$\Delta\theta = \frac{1}{\sqrt{p}} \sqrt{\frac{|\alpha|^2 e^{-2r} + \sinh^2 r}{(|\alpha|^2 - \sinh^2 r)^2} + \frac{|\alpha|^2 + 2\sinh^2 r \cosh^2 r}{(|\alpha|^2 - \sinh^2 r)^2 \tan^2\theta}}. \tag{1}$$

According to Eq. (1), we can obtain an increase in phase sensitivity with respect to the shot noise only when the true value of the phase shift is sufficiently close to the optimal point $\theta = \pi/2$ [1,5] (dark fringe), where $\langle \hat{M}_{\rm out}\rangle = \langle \hat{N}_c - \hat{N}_d\rangle = 0$. On the other hand, $\langle \hat{M}_{\rm out}\rangle$ depends weakly on the phase shift when $\theta \approx 0$, π and the error propagation formula Eq. (1) predicts large phase fluctuations around these points. Asymptotically in the amplitude of the coherent state, $|\alpha|^2 \gg \sinh^2 r$, Eq. (1) predicts, at the optimal point, a sub–shot-noise sensitivity [1,10,11]

$$\Delta\theta = \frac{1}{\sqrt{p}} \frac{e^{-r}}{\sqrt{\bar{n}}}, \qquad (\theta = \pi/2), \tag{2}$$

with the average number of photons input to the MZ, $\bar{n} = |\alpha|^2 + \sinh^2 r \simeq |\alpha|^2$.

In this Letter we show that the choice of the average relative number of photons as a phase estimator is not optimal. Further information about the true value of the phase shift is contained in the quantum fluctuations of the number of particles measured at the output ports. As a consequence, we show that the ultimate phase sensitivity of a Mach-Zehnder fed by coherent ⊗ squeezed-vacuum light is not given by Eqs. (1) and (2), but is

$$\Delta\theta = \frac{1}{\sqrt{p}} \frac{1}{\sqrt{|\alpha|^2 e^{2r} + \sinh^2 r}} \qquad (0 \le \theta \le \pi). \tag{3}$$

The phase sensitivity (i) is independent from the true value of the phase shift over the whole interval $0 \le \theta \le \pi$ [12] and (ii) it reaches the Heisenberg scaling

$$\Delta\theta = \frac{1}{\sqrt{p}} \frac{1}{\bar{n}}, \qquad (0 \le \theta \le \pi), \tag{4}$$

when $|\alpha|^2 \simeq \sinh^2 r \simeq \bar{n}/2$ and $\bar{n}$, $p \gg 1$.

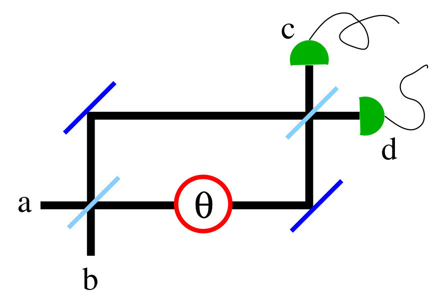

FIG. 1 (color online). Schematic representation of the Mach-Zehnder interferometer. The input modes a, b are a coherent and a squeezed-vacuum field, respectively.

In the following, we will first analytically calculate the Cramer-Rao lower bound (CRLB) [13], Eq. (3), and then demonstrate that it is saturated by a Bayesian phase inference approach.

The Cramer-Rao lower bound.—The output state of a lossless Mach-Zehnder interferometer is given by $|\psi_{\rm out}\rangle = e^{-i\theta\hat{J}_y}|\psi_{in}\rangle$ [14], where, in our case, $|\psi_{\rm in}\rangle = |\alpha\rangle_a|\zeta\rangle_b$. The conditional probability to measure N_c and N_d particles at the output ports, given an unknown phase shift θ, is

$$P(N_c, N_d|\theta) = \left|\sum_{n=0}^{N} C_{N-n}S_n d^{N/2}_{\mu,N/2-n}(\theta)\right|^2, \qquad (5)$$

where $\mu = (N_c - N_d)/2$, $N = N_c + N_d$, and $d^j_{\mu,\nu}(\theta)$ are rotation matrix elements. According to the Cramer-Rao theorem, the phase sensitivity of an unbiased estimator is bounded by $\Delta\theta = \frac{1}{\sqrt{pF(\theta)}}$, where the Fisher information is $F(\theta) = \sum_{N_c,N_d=0}^{+\infty}\frac{1}{P(N_c,N_d|\theta)}(\frac{\partial P(N_c,N_d|\theta)}{\partial\theta})^2 = |\alpha|^2e^{2r} + \sinh^2 r$. By replacing this expression in the CRLB we retrieve Eq. (3). There are a few important regimes which deserve to be outlined. When $r = 0$ *or* $\alpha = 0$ we obtain the (θ-independent) shot-noise limit $\Delta\theta = 1/\sqrt{p\bar{n}}$. The phase independence of the case $r = 0$ has been studied and experimentally demonstrated in [16]. When $\sinh^2 r \ll |\alpha|^2$ we obtain the sub–shot-noise limit discussed by Caves, $\Delta\theta = e^{-r}/\sqrt{p\bar{n}}$, Eq. (2), with, again, the important difference that, here, the phase sensitivity is independent of the true value of the phase shift. Notice that, in the limit of very high squeezing, $\sinh^2 r \gg |\alpha|^2$, Eq. (1) predicts $\Delta\theta = 1/\sqrt{p\bar{n}}$ (at $\theta = \pi/2$), while Eq. (3) gives a sub–shot-noise scaling $\Delta\theta = 1/(\sqrt{p\bar{n}}\sqrt{4|\alpha|^2+1})$.

The most important regime predicted by Eq. (3) is obtained when $|\alpha|^2 \simeq \sinh^2 r \simeq e^{2r}/4 \sim \bar{n}/2$ (i.e., with half of the input intensity provided by the coherent state and half by the squeezed light). This gives the Heisenberg scaling $\Delta\theta = 1/(\bar{n}\sqrt{p})$ when $\bar{n}$, $p \gg 1$. It is interesting to note that, for these optimal values of the parameters α and r, the error propagation formula Eq. (1) diverges. In Fig. 2(a) we compare, as a function of r, the quantity $\sqrt{\bar{n}p}\Delta\theta$ calculated with Eq. (1) (dashed line) and with Eq. (3) (solid line), for $\theta = \pi/2$.

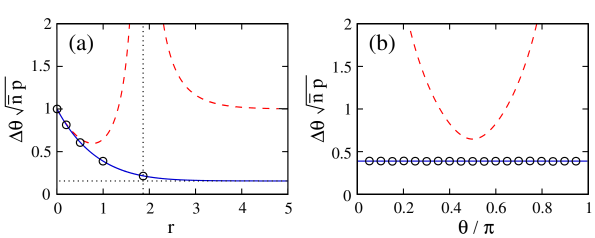

FIG. 2 (color online). Comparison between Eq. (1) (dashed line) and Eq. (3) (solid line). Circles are the results of a Bayesian analysis. (a) Phase sensitivity $\Delta\theta\sqrt{\bar{n}p}$ as a function of the squeezing parameter r, for $\theta = \pi/2$ and $|\alpha|^2 = 10$. Notice that Eq. (1) diverges at $\sinh^2 r = |\alpha|^2$ (dotted vertical line). For $r \gg 1$, Eq. (3) gives $\Delta\theta\sqrt{\bar{n}p} \to 1/\sqrt{4|\alpha|^2+1}$ (dotted horizontal line). (b) $\Delta\theta\sqrt{\bar{n}p}$ in the limit $p \to \infty$ as a function of the true value of the phase shift. Here $|\alpha|^2 = 10$ and $r = 1$.

Why does the error propagation formula Eq. (1) provide such poor phase sensitivity with respect to the CRLB? The answer is that a phase estimate based only on the analysis of the average relative number of particles does not exploit all available information. In particular, it does not consider information contained in the fluctuations of the number of particles and (since the relevant probability distributions are not Gaussian) in the higher moments [17]. In Fig. 2(b) we plot $\sqrt{\bar{n}p}\Delta\theta$ as a function of the true value of the phase shift. The dashed line is Eq. (1) and the solid line Eq. (3).

Bayesian analysis.—Is it possible to saturate the CRLB and, furthermore, to demonstrate a phase estimation sensitivity at the Heisenberg limit $\Delta\theta \sim 1/N_T$ [18]? (N_T is the average number of particles used in the process.) A possibility is to consider the maximum likelihood estimator which, according to the Fisher theorem, saturates the CRLB asymptotically in the number of measurements p. In the following, however, we consider a Bayesian protocol [19] and show that it also saturates the CRLB. To simulate a phase estimation experiment, we (i) randomly choose p values $N_c^{(i)}$, $N_d^{(i)}$ at the output ports distributed according to $P(N_c, N_d|\theta)$ with an unknown θ; (ii) invert the distribution Eq. (5) using the Bayes theorem and associate to the measured values $\{N_c^{(i)}, N_d^{(i)}\}_{i=1\ldots p}$ the probability distribution $P(\phi|\{N_c^{(i)}, N_d^{(i)}\}_{i=1\ldots p}) \sim \prod_{i=1}^{p} P(\phi|N_c^{(i)}, N_d^{(i)})$; (iii) calculate the phase sensitivity as the 68% confidence interval around the maximum of the phase distribution. In Figs. 2(a) and 2(b), the circles, obtained with the Bayesian probabilities asymptotically in the number of independent measurements p, coincide with the analytical expression of the CRLB, Eq. (3).

Yet, in order to demonstrate the possibility to reach the Heisenberg limit, $\Delta\theta \sim 1/N_T$, we have to carefully analyze the role of p [20]. Within the optimal choice of parameters, $|\alpha|^2 \simeq \sinh^2 r \simeq \bar{n}/2 \gg 1$, we fix a total number of particles, $N_T = p\bar{n}$, distributed in ensembles of p independent measurements. There are two concurring behaviors contributing, in average, to $\Delta\theta$. For small p, we are in a preasymptotic regime characterized by large oscillations of $\sum_{i=1}^{p}(N_c^{(i)} + N_d^{(i)})$, which still provides sub–shot noise but not the Heisenberg limit. For larger values of p, we saturate the Fisher information and obtain $\Delta\theta \sim \sqrt{p}/N_T$. The prefactor $\sqrt{p}$ is due to the statistics of independent measurements. As shown in Fig. 3(a), the optimal value is $p_{\rm opt} \simeq 30$. The crucial point to notice is that $p_{\rm opt}$ does not depend on N_T. If it would, we could not claim the Heisenberg limit. The phase sensitivity calculated at $p_{\rm opt}$ is plotted in Fig. 3(b) as a function of N_T (circles). The dashed line is $\Delta\theta = 7.12/N_T$, while the solid line is $\Delta\theta \sim 1/(\sqrt{p_{\rm opt}}\sqrt{|\alpha|^2e^{2r} + \sinh^2 r})$. For comparison, we include in the figure the shot-noise limit (dot-dashed line).

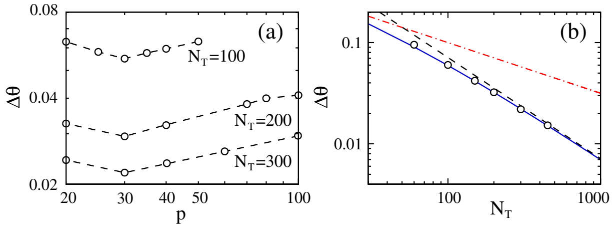

FIG. 3 (color online). Demonstration of the Heisenberg limit $\Delta\theta \sim 1/N_T$. (a) Circles are the phase sensitivity obtained with the Bayesian analysis as a function of the number of measurements p with fixed total number of particles $N_T = p\bar{n}$. The optimal value, $p_{\rm opt} = 30$, corresponds to the minimum of $\Delta\theta$ and does not depend on $\bar{n}$. Dashed lines are guides to the eye. (b) Phase sensitivity as a function of N_T calculated with the optimal number of measurements $p_{\rm opt} = 30$. The dashed line is the asymptotic limit $\Delta\theta = 7.12/N_T$, the solid line is $\Delta\theta \sim 1/(\sqrt{p_{\rm opt}}\sqrt{|\alpha|^2 e^{2r} + \sinh^2 r})$. Shot noise has been included for comparison (dot-dashed line).

We emphasize that the CRLB predicts a phase sensitivity at the Heisenberg limit also when monitoring a single output port (reduced MZ configuration). In this case, a numerical calculation of the Fisher information at $|\alpha|^2 \simeq \sinh^2 r$ shows a strong dependence on θ, the optimal working point being close to 0 or π (depending on which output port is monitored).

Discussion.—What is the physics underlying the increase in phase sensitivity using squeezed-vacuum light? In [1] Caves associated sub–shot-noise sensitivity to quadrature squeezing. Indeed, under the conditions $\theta = \pi/2$, and $|\alpha|^2 \gg \sinh^2 r$, Eq. (1) reduces to $\Delta\theta = \frac{2\Delta\hat{X}_1}{\sqrt{p\bar{n}}}$, with the quadrature $\hat{X}_1 = (\hat{b}^\dagger + \hat{b})/2$. With squeezed-vacuum light $\Delta\hat{X}_1 = e^{-r}/2$ and we recover Eq. (2). Yet, the saturation at the Heisenberg limit is the result of the large entanglement among the two modes created by the beam splitter. In the following, for the sake of simplicity, we fix (postselect) a total number of particles $N = \bar{n}$. The input state $|\psi_N\rangle \equiv \sum_{\mu=-N/2}^{N/2} A_\mu |N/2-\mu\rangle_a |N/2+\mu\rangle_b$ is characterized by a relative number of particles distribution $P(\mu) = |A_\mu|^2$, where $A_\mu = 0$ for odd values of $N/2 - \mu$, see Fig. 4(a). After the first beam splitter, the distribution has the largest peaks centered at $\mu = \pm N/2$, see Fig. 4(b), which indicates that the corresponding quantum state $|\psi_N^{\rm BS}\rangle$ contains a large "NOON" component, $|{\rm NOON}\rangle \simeq |N, 0\rangle + |0, N\rangle$. This is characterized by a mean-square fluctuation of the order of total number of particles N, which is typical of states attaining the Heisenberg limit [21]. This can be intuitively understood by considering the phase distribution obtained by projecting $|\psi_N^{\rm BS}\rangle$ over the phase states $|\phi\rangle = \sum_{\nu=-N/2}^{N/2} e^{-i\nu\phi}|N/2-\nu\rangle_a|N/2+\nu\rangle_b$, see Fig. 4(c). The frequency of the oscillations is $\sim N$, so that each peak has a width $\sim 2\pi/N$. It is also interesting to notice that the highest NOON component is obtained

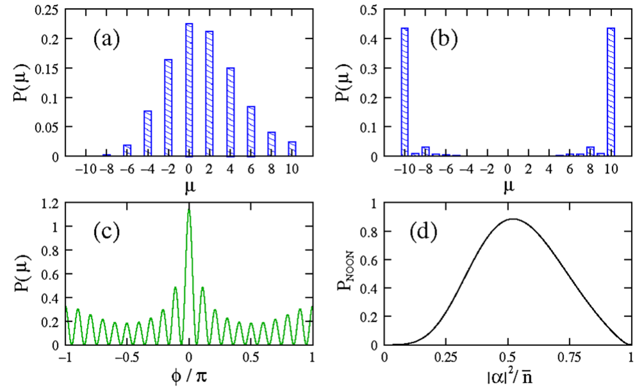

FIG. 4 (color online). Relative number of particles distribution $P(\mu)$ for (a) the input state $|\psi_N\rangle$ with a postselected number of particles $N = \bar{n}$ and $|\alpha|^2 = \sinh^2 r$ and (b) the state created by the first beam splitter, $|\psi_N^{\rm BS}\rangle$. (c) Phase distribution $P(\phi)$ obtained by projecting $|\psi_N^{\rm BS}\rangle$ over phase states. (d) Probability to obtain a NOON state after the first beam splitter, $P_{\rm NOON}$, as a function of $|\alpha|^2/\bar{n}$. The maximum is reached at $|\alpha|^2 \simeq \bar{n}/2$. Here $\bar{n} = 20$.

when $|\alpha|^2 = \bar{n}/2$ and $\bar{n} \gg 1$, which precisely correspond to the optimal conditions to reach the Heisenberg limit Eq. (4). This is illustrated in Fig. 4(d), where $P_{\rm NOON} \equiv |\langle {\rm NOON}|\psi_N^{\rm BS}\rangle|^2$ is shown as a function of $|\alpha|^2/\bar{n}$ with $\bar{n} = 20$.

Conclusions.—The discovery that nonclassical states of light can dramatically improve the sensitivity of interferometric phase estimations has been crucial for the development of modern quantum optics [5]. Several states and strategies have been proposed in the literature to beat the standard quantum limit [1,6,15,19,22]. Here we have shown that the oldest of such proposals, a linear lossless Mach-Zehnder interferometer illuminated by a coherent ⊗ squeezed-vacuum light [1], can indeed reach the Heisenberg limit Eq. (4), but *only if* all information included in the measurement of the number of particles at the output ports is properly taken into account. Moreover, we have also shown that the phase sensitivity is independent of the true value of the phase shift for arbitrary values of squeezing, which can be proved within current technology. A proof of principle of Eq. (4) can also be obtained experimentally in the limit of small $\bar{n}$. High-efficiency number-resolving photo detectors have been recently applied to interferometry [16,23] and large squeezing has been created in [24,25]. A possible application is, for instance, the increase of sensitivity of the large scale interferometers dedicated to the detection of gravitational waves [26].

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[9] In this Letter we will always consider the optimal condition $\cos(\theta_s - 2\theta_c) = 1$.
[10] Asymptotically in $\bar{n}$, Eq. (1) predicts the highest sensitivity when $\sinh^2 r \sim \sqrt{\bar{n}}$: $\Delta\theta \approx 1/(\bar{n}^{3/4}\sqrt{p})$ for $\theta = \pi/2$ [4].
[11] In the limit $|\alpha|^2 \gg \sinh^2 r$, Eq. (1) gives $\Delta\theta = (1/\sqrt{p}) \times \sqrt{(e^{-2r}/\bar{n}) + \{[1 + (e^{4r}/8\bar{n})]/(\bar{n}\tan^2\theta)\}}$ which shows that the larger is r the faster is the increase of phase sensitivity around $\theta = \pi/2$.
[12] The Mach-Zehnder interferometer considered here cannot discriminate between positive and negative values of the phase shift. This is the reason for the reduced phase interval $[0, \pi]$. The extension of the phase estimation to the full $[0, 2\pi]$ interval requires two combined, out of phase, measurements.
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