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Emergent Fermions and Anyons in the Kitaev Model

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We study the gapped phase of the Kitaev model on the honeycomb lattice using perturbative continuous unitary transformations. The effective low-energy Hamiltonian is found to be an extended toric code with interacting anyons. High-energy excitations are emerging free fermions which are composed of hard-core bosons with an attached string of spin operators. The excitation spectrum is mapped onto that of a single particle hopping on a square lattice in a magnetic field. We also illustrate how to compute correlation functions in this framework. The present approach yields analytical perturbative results in the thermodynamical limit without using the Majorana or the Jordan-Wigner fermionization initially proposed to solve this problem.

The study of elementary excitations in strongly correlated systems is a fascinating field of current research. As exemplified in the fractional quantum Hall effect, such excitations can be very different from the elementary constituents present in the system. In the same spirit, emergent fermions and gauge fields in purely boson or spin systems have attracted much attention recently [1]. The emergence of anyonic excitations in two-dimensional systems has also triggered a tremendous amount of interest, especially since its relevance for topological quantum computation (see Ref. [2] for a recent review) has been pointed out by Kitaev in a seminal paper [3] introducing the celebrated toric code.

More recently, Kitaev introduced a more realistic model [4] for which experiments using ultracold atoms or polar molecules have been proposed [5,6]. This model is a two-dimensional spin-1/2 system on the honeycomb or brick-wall lattice, as illustrated in Fig. 1. It consists solely of Ising-like interactions but in different quantization axis. More precisely, the Hamiltonian reads

$$H = - \sum_{\alpha=x,y,z} \sum_{\alpha\text{-links}} J_\alpha \sigma_i^\alpha \sigma_j^\alpha, \qquad (1)$$

where σ_i^α are the usual Pauli matrices at site i. Without loss of generality [4], in the following, we assume $J_\alpha \geq 0$ for all α and $J_z \geq J_x, J_y$.

Kitaev solved the model exactly by introducing Majorana fermions to represent the spin operators in an extended Hilbert space. In this way, the Hamiltonian is reduced to free fermions in a static $\mathbb{Z}_2$ gauge field on the honeycomb lattice. The physical states are selected by a projection step which simply amounts to a selection rule on the parity of the number of fermions [7], in agreement with the general conclusions of Ref. [8]. The system exhibits a gapless phase for $J_x + J_y > J_z$, with non-Abelian anyonic excitations arising when a magnetic field is switched on

[4]. A contrario, for $J_x + J_y < J_z$, the system is gapped and the low-energy effective Hamiltonian, at lowest nontrivial order in perturbation (J_x, $J_y \ll J_z$) and periodic boundary conditions, turns out to be exactly the toric code. Consequently, at this order, free Abelian anyons are present in the gapped phase.

Note also that an alternative treatment has been proposed [9–11] based on the Jordan-Wigner transformation which transforms the spin system into a system of fermions with *p*-wave BCS pairing and a site-dependent chemical potential.

In this Letter, we focus on the gapped phase, and we derive the low- and high-energy effective theory at high-order in perturbation. This allows us to show that (i) the low-energy effective Hamiltonian is an extended toric code Hamiltonian with still static but interacting Abelian anyons, and (ii) the high-energy excitations are free fermions, composed of a hard-core boson with an attached string of spin operators, hopping on a square lattice embedded in a magnetic field.

Our approach has some advantages which compensate for its perturbative nature and its restriction to the gapped
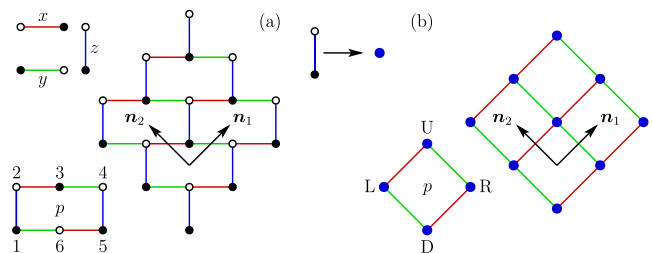

FIG. 1 (color online). Mapping of the brick-wall lattice (a) to an effective square lattice (b) with unit basis vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. The numbering of the sites of a plaquette p is shown in both cases.

phase. It provides a unified picture of the emergence of high- and low-energy excitations without introducing fermionic fields by hand as is done with Majorana or Jordan-Wigner fermionization. In addition, we work, from the beginning, in the thermodynamical limit, and derive analytical results for non-translational-invariant low-energy states.

Let us consider the model in the limit J_x, $J_y \ll J_z$. In the limiting case $J_x = J_y = 0$, the model is a collection of isolated z-dimers. Each dimer has four possible configurations : two low-energy states $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ with energy $-J_z$ and two high-energy states $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$ with energy J_z. It is thus natural to interpret the change from a ferromagnetic to an antiferromagnetic dimer configuration as the creation of a particle, with an energy cost that we set equal to 1 by choosing $J_z = 1/2$. By construction, these particles are hard-core bosons hopping on the sites of an effective square lattice (see Fig. 1), together with an effective spin-1/2 indicating which kind of (anti-)ferro dimer configuration is realized. Among the four possible mappings, we choose the following:

$$|\uparrow\uparrow\rangle = |\Uparrow 0\rangle, |\downarrow\downarrow\rangle = |\Downarrow 0\rangle, |\uparrow\downarrow\rangle = |\Uparrow 1\rangle, |\downarrow\uparrow\rangle = |\Downarrow 1\rangle, \quad (2)$$

where the left (right) spin is the one of the black (white) site of the dimer, and double arrows represent the state of the effective spin. Let us denote by $b_i^\dagger$ (b_i) the creation (annihilation) operator of a hard-core boson at site i (bold letters denote effective sites), and τ_i^α the Pauli matrices of the effective spin at the same site. With this notation, the number of bosons in the system is $Q = \sum_i b_i^\dagger b_i$, and the Hamiltonian (1) can be rewritten as [7]

$$H = -\frac{N}{2} + Q + T_0 + T_{+2} + T_{-2}, \quad (3)$$

where N is the number of z-dimers,

$$T_0 = -\sum_i (J_x t_i^{i+n_1} + J_y t_i^{i+n_2} + \text{h.c.}), \quad (4)$$

$$T_{+2} = -\sum_i (J_x v_i^{i+n_1} + J_y v_i^{i+n_2}) = T_{-2}^\dagger, \quad (5)$$

with hopping and pair creation operators

$$t_i^{i+n_1} = b_{i+n_1}^\dagger b_i \tau_{i+n_1}^x, \qquad t_i^{i+n_2} = -i b_{i+n_2}^\dagger b_i \tau_{i+n_2}^y \tau_i^z, \quad (6)$$

$$v_i^{i+n_1} = b_{i+n_1}^\dagger b_i^\dagger \tau_{i+n_1}^x, \qquad v_i^{i+n_2} = i b_{i+n_2}^\dagger b_i^\dagger \tau_{i+n_2}^y \tau_i^z. \quad (7)$$

The vectors n_1 and n_2 are shown in Fig. 1(b). Let us underline that $[W_p, t_i^j] = [W_p, v_i^j] = 0$ for all (p, i, j) where the conserved plaquette operators [4] read (with notation given in Fig. 1)

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = (-1)^{b_L^\dagger b_L + b_D^\dagger b_D} \tau_L^y \tau_U^z \tau_R^y \tau_D^z. \quad (8)$$

At this stage, note that both the mapping (2) and the form of the Hamiltonian (3) are simply an alternative description of the problem which is always valid, even in a nonperturbative regime. The main difficulty resides in the fact that, now, one has to deal with hard-core bosons coupled to effective spin degrees of freedom. Of course, one could use a fermionization trick and solve the model directly as done by Kitaev [4]. However, the procedure used in the following can be applied to nonexactly solvable models and further allows, in the present case, for a clear identification of the excitations.

In general, a Hamiltonian of the form (3) cannot be diagonalized exactly. Here, following Kitaev, we choose to treat it perturbatively in the limit J_x, $J_y \ll J_z$. Nevertheless, the Green's functions method used in Ref. [4] turns out to be rather hard to implement at high order and/or high energy. Instead, we use an alternative approach based on continuous unitary transformations (CUTs) [12] whose perturbative version [13,14] is especially well-suited to the problem at hand. Technical details will be given in a forthcoming publication [7].

The main idea is to transform the Hamiltonian (3) which does not conserve the number of bosons into an effective Hamiltonian H_eff which satisfies $[H_\text{eff}, Q] = 0$. As explained in Ref. [15], H_eff is a sum of k-quasi-particle (QP) operators with $k \in \mathbb{N}$. The $k = 0, 1$ contributions can be written as

$$H_\text{eff}^{0\text{qp}} = E_0 - \sum_{\{p_1,...,p_n\}} C_{p_1,...,p_n} W_{p_1} W_{p_2} \ldots W_{p_n}, \quad (9)$$

$$H_\text{eff}^{1\text{qp}} = \mu Q - \sum_{\{j_1,...,j_n\}} D_{j_1,...,j_n} t_{j_{n-1}}^{j_n} \ldots t_{j_2}^{j_3} t_{j_1}^{j_2}. \quad (10)$$

Here, $\{p_1, p_2, \ldots, p_n\}$ denotes a set of n plaquettes, and the sum, $\{j_1, \ldots, j_n\}$ represents a sequence of n connected sites. The perturbative aspect comes from the fact that the coefficients appearing in H_eff are obtained as series expansions in J_x and J_y. We have computed the 0-QP amplitudes E_0 and $C_{p_1,...,p_n}$ up to order 10 and the 1-QP amplitudes μ and $D_{j_1,...,j_n}$ up to order 4. These expressions, being quite lengthy, will be given in a longer paper [7], as well as a discussion of $(k \geq 2)$-QP sectors.

Let us first discuss the low-energy physics, i.e., the spectrum of $H_\text{eff}^{0\text{qp}}$. The most striking result is that the latter is only expressed in terms of the conserved quantities $W_p|_{0\text{qp}} = \tau_L^y \tau_U^z \tau_R^y \tau_D^z$. Thus, for any (vortex) configuration of the W_p's which can take two values ± 1, one readily gets the ground state energy of the corresponding sector. We wish to emphasize that at order 4 (lowest nontrivial order), as already discussed by Kitaev [4], the only nonvanishing contribution (apart from E_0) involves only single plaquette terms, and one recovers the toric code Hamiltonian with its Abelian anyons [3]. At order 6, one gets the following corrections to these terms:

$$\frac{E_0}{N} = -\frac{1}{2} - \frac{J_x^2 + J_y^2}{2} - \frac{J_x^4 + J_y^4}{8} - \frac{J_x^6 + J_y^6}{8}, \quad (11)$$

$$C_p = \frac{1}{2}J_x^2 J_y^2 + \frac{J_x^4 J_y^2 + J_x^2 J_y^4}{4}, \quad (12)$$

but, more interestingly, one also has some two-plaquette terms:

$$C_{p,p+n_1} = \frac{7}{8}J_x^4 J_y^2, \qquad C_{p,p+n_2} = \frac{7}{8}J_x^2 J_y^4. \quad (13)$$

Actually, when increasing the perturbation theory order, one generates higher and higher multi-plaquette terms. Thus, the low-energy effective theory of the Kitaev model turns out to be an interacting anyon theory whose eigenstates are those of the toric code. The single-vortex energy and the two vortices interaction energies of three configurations are given in Fig. 2 at order 10, for $J_x = J_y = J$. It should finally be noticed that although the vortices interact, they remain static since W_p's are conserved quantities.

In each sector defined by a configuration of the W_p's, we shall now see that the excitation spectrum is of fermionic nature. As proposed by Levin and Wen [8], the statistics can be directly determined from the hopping operators algebra. Let us consider the exchange process depicted in Fig. 3. The corresponding operator sequence is

$$t_i^j t_k^i t_i^l t_j^i t_i^k t_l^i = -1, \quad (14)$$

or, equivalently, $t_j^l t_i^k t_l^i = -t_l^i t_i^k t_j^i$. This latter identity shows that the quasiparticles made of a hard-core boson and an effective spin-$1/2$ obey fermionic statistics.

We shall now see that solving the excitation spectrum is completely equivalent to solving a problem of free fermions in a magnetic field on the square lattice. Therefore, let us focus on the first-order perturbation theory for which $H_{\text{eff}}^{1\text{qp}} = Q + T_0$ and consider a single quasiparticle. Then, one can easily see that the momenta $(H_{\text{eff}}^{1\text{qp}} - 1)^m$ for all m are strictly equal to those of a pure hopping fermion Hamiltonian in a magnetic field provided the magnetic flux per plaquette mimics the configuration of the W_p's. This is easily seen by noting that the product of the hopping operator around a plaquette p is given by $t_l^i t_k^l t_j^k t_i^j = W_p$, with $\{i, j, k, l\}$ being any sequence of connected sites chosen among L, U, R, and D (see Fig. 1).

The W_p's being conserved quantities, from a purely spectral point of view one can replace the Pauli matrices in the hopping operators by pure numbers ± 1 constraining the fluxes per plaquette (in unit of the flux quantum) to be $\phi_p = 0$ for $W_p = +1$ or $\phi_p = 1/2$ for $W_p = -1$. At higher order, this mapping remains true (with hoppings from one site to any other site), but one needs to go beyond this simple argument to prove the correspondence between both spectra [7]. Let us simply mention that this is done by building a basis of the 1-QP subspace, which turns out to be made of states with one string of spin flips attached to one hard-core boson (yielding a fermion), as can be inferred from the form (6) of the hopping operators, and illustrated in Fig. 4. Note that the string fluctuates, as in the construction of anyons in [3].

To illustrate the mapping, let us consider the 1-QP vortex free state ($W_p = +1, \forall p$). The spectrum of $H_{\text{eff}}^{1\text{qp}}$ is made of a single band whose dispersion relation is given, at second order, by

$$E_{\text{free}}(k_x, k_y) = 1 - 2[J_x \cos(k_x) + J_y \cos(k_y)] + 2[J_x \sin(k_x) + J_y \sin(k_y)]^2, \quad (15)$$

$$\Delta E_{1\text{v}} = J^4 + 8J^6 + 75J^8 + 784J^{10}$$

$$\Delta E_{2\text{v}} = 2\Delta E_{1\text{v}} - \frac{7}{2}J^6 - \frac{177}{4}J^8 - \frac{17831}{32}J^{10}$$

$$\Delta E_{2\text{v}} = 2\Delta E_{1\text{v}} - \frac{165}{4}J^8 - 572J^{10}$$

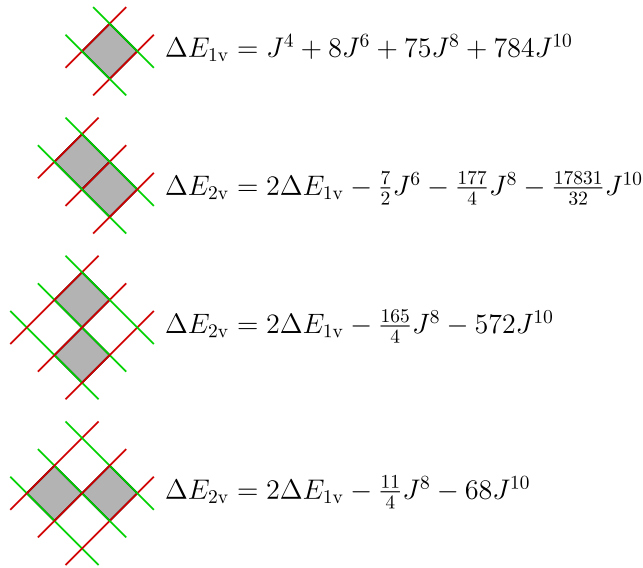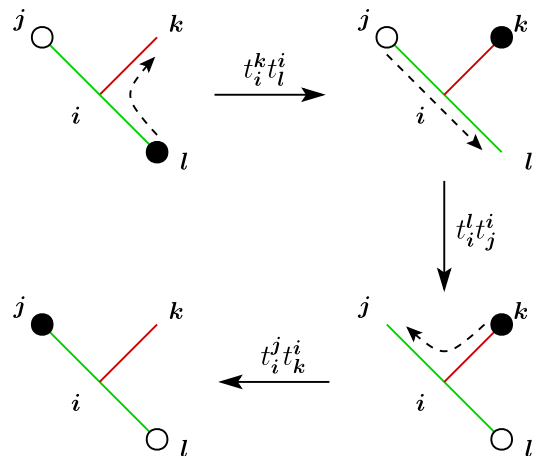$$\Delta E_{2\text{v}} = 2\Delta E_{1\text{v}} - \frac{11}{4}J^8 - 68J^{10}$$

FIG. 2 (color online).   One-anyon and some two-anyon configurations (gray plaquettes) on a vortex-free background. ΔE_{1v} (ΔE_{2v}) is the energy cost for adding one vortex (two vortices) to the vortex-free state. We have set $J_x = J_y = J$.

FIG. 3 (color online).   Illustration of the exchange of two particles discussed in the text, for $j = i + n_2$, $k = i + n_1$, and $l = i - n_2$.
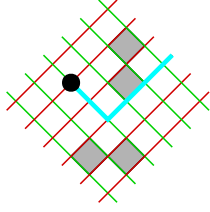
FIG. 4 (color online).   Pictorial representation of an emergent fermion as a composite object made of a hard-core boson (dot) and an attached string of spin-flips (thick line). Gray plaquettes are anyons, as in Fig. 2.

$$C_i^{zz} = 1 - (J_x^2 + J_y^2) - \tfrac{3}{4}(J_x^4 + J_y^4)$$
$$-\tfrac{1}{4}J_x^2 J_y^2 (W_r + W_l + 5W_u + 5W_d)$$

FIG. 5 (color online).   Perturbative expansion at order 4, of the correlation function C_i^{zz} defined in the text, in the 0-QP subspace.

where the wave vector $(k_x, k_y) \in [-\pi, \pi] \times [-\pi, \pi]$. The gap in this sector is then obtained by minimizing E_{free}, and one gets $\Delta_{\text{free}} = 1 - 2(J_x + J_y)$. Note that, in this case, the perturbative result at order 1 coincides with the nonperturbative result obtained by Kitaev [4] and one recovers the transition point at $J_x + J_y = 1/2 = J_z$.

Let us complete our analysis with a comparison to the Majorana (or Jordan-Wigner) fermion formalism. In the latter, one can compute the full spectrum exactly, by diagonalizing, for each configuration of the W_p's with $2N$ spins, a $2N \times 2N$ matrix. Within our approach, we have a $N \times N$ matrix to consider. Although we have seen that analytical (perturbative) results can be obtained for $H_{\text{eff}}^{0\text{qp}}$, even for non-translational-invariant W_p's, for $H_{\text{eff}}^{1\text{qp}}$, the problem is thus essentially as complicated as in the Majorana (or Jordan-Wigner) fermion formalism. In this case, the main interest of our method is thus to give a physical picture of the high-energy excitations as emerging fermions.

Furthermore, the continuous unitary transformations allow for the calculation of correlation functions. In the 0-QP subspace, as we show below, some of these can again be expressed as a series in W_p operators, as was the case for the eigenvalues of $H_{\text{eff}}^{0\text{qp}}$. Here, we focus on the correlation function $C_i^{zz} = \langle \sigma_{i,\bullet}^z \sigma_{i,\circ}^z \rangle$ where $i, \bullet$ and $i, \circ$ are the black and white sites of a given z-dimer i (see Fig. 1). Such a correlation function can be computed exactly for a translationally invariant configuration of W_p's, either by using the technique developed in [16] or, in a much simpler way, thanks to the Hellman-Feynman theorem [7]. However, none of these two methods can be used efficiently for a non-translational-invariant configuration of the W_p's.

As a first step, one has to translate C_i^{zz} into the effective spin and boson degrees of freedom, which yields $C_i^{zz} = \langle (-1)^{b_i^\dagger b_i} \rangle$. Then, the observable appearing in the angular brackets has to be transformed with the same unitary transformation as the one applied to the Hamiltonian [7,15]. As an illustration, we give the result in Fig. 5 at order 4 which is the lowest nontrivial order where the W_p's appear in the expansion. Note that we have used the same notation W for the operator and its expectation value on a 0-QP state. This result shows how the correlation function is modified by
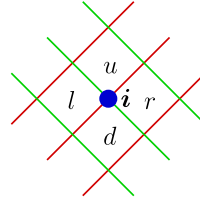
the presence of surrounding anyons. Of course, at higher order, anyons further apart would also contribute [7] as was already the case for the spectrum of $H_{\text{eff}}^{0\text{qp}}$.

To conclude, we wish to emphasize that the method we have used here, based on a perturbative approach of the CUTs, can be applied to nonexactly solvable models as well. In addition, as we have shown, it is especially efficient to compute the spectrum but also the expectation value of observables, even at high order in perturbation. Thus, we strongly hope that it constitutes a powerful tool to investigate, for instance, the non-Abelian anyons recently proposed by Yao and Kivelson for a time-reversal symmetry breaking version of the Kitaev model [7,17].

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