Coulomb Interaction, Ripples, and the Minimal Conductivity of Graphene

Igor F. Herbut,[1] Vladimir Juričić,[1] and Oskar Vafek[2]

[1]*Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6*
[2]*National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306, USA*

We argue that the unscreened Coulomb interaction in graphene provides a positive, universal, and logarithmic correction to scaling of zero-temperature conductivity with frequency. The combined effect of the disorder due to wrinkling of the graphene sheet and the long-range electron-electron interactions is a finite positive contribution to the dc conductivity. This contribution is disorder strength dependent and thus nonuniversal. The low-energy behavior of such a system is governed by the line of fixed points at which both the interaction and disorder are finite, and the density of states is exactly linear. An estimate of the typical random vector potential representing ripples in graphene brings the theoretical value of the minimal conductivity into the vicinity of $4e^2/h$.

PACS numbers: 71.10.Pm, 71.10.Hf, 71.23.−k, 73.61.Wp

Graphene, an atom-thick layer of graphite, has recently defined a new frontier of condensed matter physics. Its essential electronic property, inherent to the honeycomb lattice formed by the carbon atoms, is that the low-energy quasiparticle excitations can be thought of as being massless Dirac fermions, which propagate with a Fermi velocity of around three thousandths of the velocity of light. This pseudorelativistic nature of the quasiparticle excitations makes the electronic properties of graphene fundamentally new in many respects [1]. When the chemical potential is tuned to the Dirac point graphene provides a rare example of a critical two-dimensional fermionic system [2]. In contrast to its textbook bosonic equivalent [3], all sufficiently weak interactions between electrons, including the long-ranged Coulomb, are then irrelevant perturbations [4,5]. The effects of electron interactions thus become progressively less important as the system is probed at lower frequencies and temperatures. One important consequence of this "infrared freedom" is that the zero-temperature dc conductivity of clean graphene is finite and universal, and simply determined by its Gaussian value of $\sigma = (\pi/2)e^2/h$ [6]. Including scattering of impurities in a self-consistent Born approach yields another, similar in magnitude and still universal, value of $(4/\pi)e^2/h$ [2,6,7]. Localization corrections are also expected to set in at very low temperatures [8] and thus further diminish the conductivity. The actual measurements of graphene's conductivity, however, are in significant discord with these results: experimentally, $\sigma \approx 4e^2/h$, and thus significantly larger than all the theoretical values [1]. The origin of this discrepancy is unclear at the moment, with several recent works focusing on the role played by the extrinsic charged impurities [9].

In this Letter we show that the long-range Coulomb interaction between electrons in graphene provides the leading correction to the Gaussian value of conductivity, which is positive and by itself only logarithmically slowly vanishing when frequency approaches zero. This suggests

that the origin of the observed unusually large conductivity at the Dirac point may be intrinsic, and originate from the Coulomb correction which is then effectively cut off by a finite temperature, disorder, or size effects.

We consider a specific mechanism of such cutoff which invokes a nontrivial interplay between long-range interactions and disorder [10], the latter being presently provided by the apparently unavoidable wrinkling of the graphene sheet. The combination of the Coulomb interaction and the random vector potential that may be used to represent such ripples in graphene leads to a line of stable fixed points (see Fig. 1). The finite zero-temperature dc conductivity, obtained here from the Kubo formula, varies continuously along the line and, most importantly, increases with the increasing disorder strength. We provide the symmetry arguments for the existence of the line of stable fixed points, and extract the density of states at low energies. A crude estimate of the typical parameters in graphene gives a sizable correction to the Gaussian value already to the lowest order in our calculation and significantly nar-
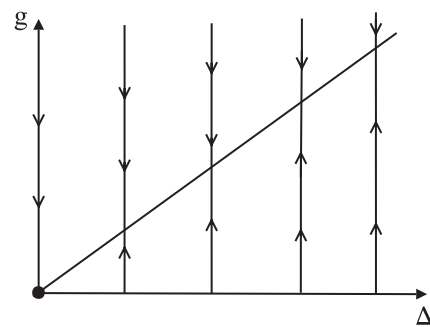

FIG. 1. The flow of the disorder strength Δ characterizing ripples of the graphene sheet and the Coulomb coupling g at weak couplings. Two effects are balanced at the attractive line of fixed points that emanates from the Gaussian fixed point at the origin. At the line the system exhibits linear density of states and a finite nonuniversal dc conductivity.

rows the gap that presently exists between the theory and the experiment. Further implications of our theory and the connections with the critical bosonic theories and related theoretical results in literature are discussed.

The low-energy excitations in the vicinity of the Dirac points at $\sigma \vec{K}$, $\sigma = \pm 1$ may be represented by two four-component Dirac spinors $\Psi_\sigma^\dagger = (\Psi_{\sigma\uparrow}^\dagger(\mathbf{x}, \tau), \Psi_{\sigma\downarrow}^\dagger(\mathbf{x}, \tau))$. $\Psi_{\sigma\uparrow}$ is a two-component Grassmann field representing the components of the electron with the third component of its (real) spin up on the two sublattices of the honeycomb lattice and with wave vectors near $\sigma \vec{K}$ [11]. The imaginary-time Lagrangian density for the interacting system of quasiparticles in the presence of a random vector potential representing ripples of the graphene sheet [6] is then $L = L_0 + L_C + L_D$, where

$$L_0 = \bar{\Psi}_\sigma^\alpha \gamma_\mu \partial_\mu \Psi_\sigma^\alpha, \tag{1}$$

$$L_C = -ia_0^\alpha \bar{\Psi}_\sigma^\alpha \gamma_0 \Psi_\sigma^\alpha + a_0^\alpha \frac{|\vec{\nabla}|}{2g} a_0^\alpha, \tag{2}$$

and

$$L_D = -i\sigma \bar{\Psi}_\sigma^\alpha \gamma_n \Psi_\sigma^\alpha A_n(\mathbf{x}) + \frac{1}{2\Delta} A_n^2(\mathbf{x}). \tag{3}$$

Here the index $\alpha = 1, 2, \ldots, N$ labels replicas introduced to average over disorder, $\mu = 0, 1, 2$, and $n = 1, 2$, and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The summation over repeated indices is assumed and the limit $N \to 0$ is to be taken at the end. The integration over $a_0(\mathbf{x}, \tau)$ reproduces the standard $\sim g/|\mathbf{x}|$ electron-electron interaction, whereas integrating out the static vector potential $A_n(\mathbf{x})$ yields an alternative form of L_D [6]:

$$\tilde{L}_D = \frac{\Delta}{2} \int d\tau' (\sigma \bar{\Psi}_\sigma^\alpha \gamma_n \Psi_\sigma^\alpha)(\mathbf{x}, \tau)(\sigma' \bar{\Psi}_{\sigma'}^\beta \gamma_n \Psi_{\sigma'}^\beta)(\mathbf{x}, \tau'), \tag{4}$$

which will be also used. For convenience, we have set $\hbar = e/c = v_F = 1$, where $v_F \approx 10^6$ m/s is the Fermi velocity. In this convention there are two dimensionless coupling constants in the theory: the Coulomb interaction $g = 2\pi e^2/\epsilon\hbar v_F$, and the strength of disorder Δ.

In analogy with the two-dimensional bosonic critical systems [3], at $T = 0$ the conductivity at frequencies well below the microscopic energy scale Ω/b in the units of e^2/h can be written in the scaling form as

$$\sigma(\omega) = F(b\omega, g(b), \Delta(b)), \tag{5}$$

where $F(x, y, z)$ is a universal scaling function. The functional dependence of $g(b)$ and $\Delta(b)$ ensures the ultimate independence of $\sigma(\omega)$ on the arbitrary factor b. We have set all the irrelevant couplings to zero. Let us choose then $b\omega = \Lambda \ll \Omega$, with Λ as an arbitrary scale. Then

$$\sigma(\omega) = F(\Lambda, g(\Lambda/\omega), \Delta(\Lambda/\omega)). \tag{6}$$

If the couplings $g(\Lambda/\omega)$ and $\Delta(\Lambda/\omega)$ flow to small values as $\omega \to 0$, one can expand the function $F(\Lambda, g, \Delta)$ as

$$\sigma(\omega) = F(\Lambda, 0, 0) + ug(\Lambda/\omega) + w\Delta(\Lambda/\omega), \tag{7}$$

to the first order in the two coupling constants. u and w are constants. Kubo formula yields the value of the Gaussian term $F(\Lambda, 0, 0) = \pi/2$ in the continuum limit [6] [see also Eq. (16) below]. In what follows we compute the remaining two terms in the last expression.

Let us first obtain the cutoff-dependent couplings $g(b)$ and $\Delta(b)$. Nonanaliticity in momentum of the second term in L_C implies that the flow of g with the change of cutoff can be written exactly [10] as,

$$\frac{dg}{d\ln(b)} = (z - 1)g, \tag{8}$$

with the dynamical exponent z fixed by the requirement $b^{z-1} = Z_\omega/Z_k$. Z_ω and Z_k are the wave-function and the velocity renormalizations, respectively. To the first order in both couplings one finds

$$Z_\omega = 1 + \frac{\Delta}{\pi} \ln b, \qquad Z_k = 1 + \frac{g}{8\pi} \ln b, \tag{9}$$

in agreement with previous calculations [4–6,12,13]. At $g = 0$, the disorder strength Δ is exactly marginal coupling [6]. We have also confirmed this by an explicit calculation to order Δ^2 (Fig. 2). When $g \neq 0$, $\Delta(b) = \Delta Z_\Delta/Z_k^2$. Using the form in Eq. (4) and computing the first diagram in Fig. 2 then gives

$$Z_\Delta = 1 + \frac{g}{4\pi} \ln(b). \tag{10}$$

Thus, as the ultraviolet cutoff in the theory is changed from Ω to Ω/b, the Coulomb and the disorder couplings flow according to the differential equations [12]

$$\frac{dg}{d\ln(b)} = g\left(\frac{\Delta}{\pi} - \frac{g}{8\pi} + \mathcal{O}(g^2, \Delta^2, g\Delta)\right), \tag{11}$$

$$\frac{d\Delta}{d\ln(b)} = 0. \tag{12}$$

Under renormalization the electron interaction may thus both decrease or increase, depending on disorder.

Although the calculation has been performed here only to the leading order, we suspect that Eq. (12) may in fact be
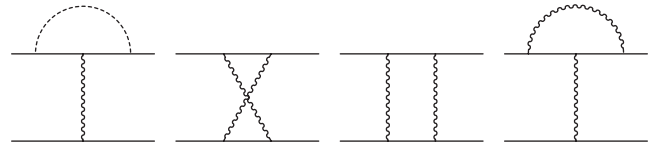

FIG. 2. The one-loop corrections to the disorder vertex in Eq. (4). The dashed and wiggly lines stand for the scalar and the vector field propagators, respectively. The last three diagrams sum up to zero.

exact. Without the last term in L_D the rest of the (interacting) Lagrangian L enjoys the symmetry under the time-independent gauge transformation

$$A_n(\mathbf{x}) \rightarrow A_n(\mathbf{x}) + \partial_n \chi(\mathbf{x}), \qquad \Psi_\sigma^\alpha \rightarrow e^{i\sigma\chi(\mathbf{x})}\Psi_\sigma^\alpha. \quad (13)$$

This implies that the usual Ward identities hold and that the polarization of the vector field $A_n(\mathbf{x})$ is transverse. The coupling Δ, which in L_D appears as the inverse mass for $A_n(\mathbf{x})$, should therefore not renormalize [14].

For completeness let us note that L is also symmetric under the purely time-dependent gauge transformation

$$a_0^\alpha(\mathbf{x}, \tau) \rightarrow a_0^\alpha(\mathbf{x}, \tau) + \partial_\tau f^\alpha(\tau), \qquad \Psi_\sigma^\alpha \rightarrow e^{if^\alpha(\tau)}\Psi_\sigma^\alpha, \quad (14)$$

which guarantees the preservation of the form of L_C and is ultimately responsible for Eq. (8) [3].

The conductivity to the first order in g and Δ may be computed next. We couple the external electromagnetic vector potential $\mathbf{a}$ minimally to Dirac fermions and choose $\mathbf{a} = a\hat{\mathbf{e}}_1$, for example. Using the Kubo formula for the replicated theory [15] and to the first order in the two couplings we find

$$F(-i\omega, g, \Delta) = I_G(\omega) + gI_C(\omega) + \Delta I_D(\omega), \quad (15)$$

where the Gaussian value is

$$I_G(\omega) = 16\pi \frac{d}{d\omega} \int \frac{dq_0 d^2\mathbf{q}}{(2\pi)^3} \frac{q_1^2 - q_2^2 - q_0(q_0 - \omega)}{q^2((q_0 - \omega)^2 + \mathbf{q}^2)}$$
$$= \frac{\pi}{2} + \mathcal{O}\left(\frac{\omega}{\Omega}\right). \quad (16)$$

To the first order in disorder only the self-energy diagram contributes and yields a negative contribution

$$I_D(\omega) = 64\pi \frac{d}{d\omega} \int \frac{d\nu d^2\mathbf{k} d^2\mathbf{p}}{(2\pi)^5} \frac{\nu(\nu^2 - \mathbf{p}^2)}{(\nu^2 + \mathbf{k}^2)(\nu^2 + \mathbf{p}^2)^2}$$
$$\times \frac{(\nu + \omega)}{(\nu + \omega)^2 + \mathbf{p}^2} = -\frac{1}{6} + \mathcal{O}\left(\frac{\omega}{\Omega}\right). \quad (17)$$

Finally, the Coulomb contribution is

$$I_C(\omega) = \frac{d}{d\omega} \int \frac{d^2\mathbf{k} d^2\mathbf{p}}{(2\pi)^3} \frac{4\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}}{|\mathbf{k} + \mathbf{p}|(\omega^2 + 4\mathbf{k}^2)}$$
$$\times \left\{ \frac{\omega^2 - 4\mathbf{k}^2}{\omega^2 + 4\mathbf{k}^2} - \frac{\omega^2 + 4\mathbf{k} \cdot \mathbf{p}}{\omega^2 + 4\mathbf{p}^2} \right\}$$
$$= \left(\frac{25}{48} - \frac{\pi}{8}\right) + \mathcal{O}\left(\frac{\omega}{\Omega}\right). \quad (18)$$

Few comments are in order at this point. The derivatives with respect to frequency in Eqs. (16)–(18), which are to be taken for strictly positive frequencies, serve to subtract the finite $\omega = 0$ contributions to the integrals. These are known to arise as the artifacts of the ultraviolet cutoff, which violates gauge invariance. Second, in contrast to the disorder contribution, the Coulomb term contains both the

self-energy and the vertex corrections, given by the first and the second term in the curly bracket in Eq. (18), respectively. While each of these two separately is logarithmically divergent in the continuum limit $\Omega \rightarrow \infty$, the divergences cancel out exactly in the full expression for the conductivity. The final result in Eq. (18) represents the finite remnant left after this cancellation. Indeed, the cancellation of logarithms in the conductivity is to be expected: without it the field-theoretic result would not be cutoff-independent, and would be physically meaningless. The results in Eqs. (16)–(18) are universal numbers characteristic of the continuum limit.

The last three expressions together with Eqs. (7), (11), and (12) at finite disorder then give the dc conductivity

$$\sigma(0) = \left[\frac{\pi}{2} + (4 - \pi)\Delta + \mathcal{O}(\Delta^2)\right] \frac{e^2}{h}, \quad (19)$$

and nonuniversal. The result is larger than the Gaussian value due to the positive Coulomb contribution at the line of fixed points. Note that Δ in the last expression is the same as the "bare" value of the disorder strength at the microscopic scale.

In the ideal sample with $\Delta = 0$, on the other hand, solving the Eq. (11) and inserting into Eq. (7) yields

$$\sigma(\omega) = \frac{\pi}{2} + \frac{\pi((25/6) - \pi)}{\ln(\Lambda/\omega)} + \mathcal{O}\left(\frac{1}{g\ln^2(\Lambda/\omega)}\right), \quad (20)$$

in the limit $\omega \rightarrow 0$. The bare value of g cancels out in the second term. Electron interactions provide therefore the leading universal logarithmic correction to scaling of conductivity at low frequencies in this case. Similar logarithmic corrections arise in perturbative quantum chromodynamics, for example.

To get an estimate of the size of the first-order correction to conductivity in Eq. (19) we restore all the constants we previously set to unity. For the dielectric constant of $\epsilon \approx 6$ we find $g \approx 2$ [16]. The conductivity is determined by the bare value of disorder, not the interaction, however. The disorder coupling can be written as $\Delta^{1/2} \approx \Phi_\xi/\Phi_0$, where $\Phi_\xi \approx h\xi^2$ is the flux of the effective random magnetic field $h = \partial_1 A_2 - \partial_2 A_1$ through the ripple of the average size ξ, and $\Phi_0 = \hbar c/e$ is the flux quantum. Effective magnetic field h can be estimated by assuming that the same random magnetic field is responsible for the observed suppression of weak localization in graphene [17]. Using $h \approx 1T$ and $\xi \approx 30$ nm [17] we find $\Delta \approx 2$, and $\sigma \approx 3e^2/h$, to the leading order. Clearly this is only a crude estimate and for Δ of order one the higher-order terms in Eq. (19) need to be included. It is encouraging, however, that the size of the lowest-order correction is significant and in the direction towards the experimental result. The latter result is a direct and a nontrivial consequence of the unscreened long-range Coulomb interaction in graphene. It may also be relevant that the experimental observation of normal localization properties in graphene seems to correlate with a lower

minimal conductivity [18]. The present theory would naturally account for this since both effects result from a low value of Δ.

A limitation of our result should also be pointed out. We have computed the $T = 0$, $\omega \rightarrow 0$ conductivity along the line of fixed points, whereas the measurements of the minimal dc conductivity in graphene typically correspond to the opposite, $\omega = 0$, $T \rightarrow 0$ limit. While both conductivities at a fixed disorder Δ are expected to be universal, the two numbers could in principle be different [19]. If the results from the related critical bosonic theories with [20] and without disorder [3,19] are of any guide, the latter universal number may be expected to be only larger. Its computation along the line of the fixed points at which both the disorder and interactions are finite may be a nontrivial task though.

Our result at $\Delta = 0$ is in stark contrast to the recent result in Ref. [21], where Coulomb interaction is claimed to suppress the conductivity at low frequencies. We note that this conclusion is in contradiction to the well-established infrared irrelevancy of the Coulomb coupling [4,5], which implies that the Coulomb interaction can only provide corrections to the Gaussian conductivity, which ultimately vanish in the dc limit. Furthermore, the result of Ref. [21] is explicitly dependent on the cutoff in the Dirac theory, which is completely arbitrary. In contrast, the cancellation of the two logarithmically divergent terms [22] in Eq. (18) guarantees the renormalizability of our result. The low-frequency conductivity we computed is consequently perfectly cutoff independent to the order of our calculation, as it has to be if the picture of Dirac quasiparticles is to remain physically meaningful in presence of the interactions.

Finally, let us address the density of single-particle states: $N(\omega) \propto \omega^{(2-z)/z}$, just as at the unstable line at $g = 0$. Since $g \neq 0$ however, Eq. (8) implies $z = 1$ at the stable line. The density of states is thus exactly linear, in contrast to the $g = 0$ line.

To summarize, our main finding is that the lowest-order combined effect of electron interactions and rippling in graphene is to increase its minimal dc conductivity in a nonuniversal, disorder-dependent fashion. A testable prediction of our theory would be a decrease of minimal conductivity in graphene with the suppression of wrinkling, which, incidentally, should also bring back the usual localization behavior at finite density.

[1] For a review, see A. K. Geim and K. S. Novoselov, Nat. Mater. **6**, 183 (2007).
[2] E. Fradkin, Phys. Rev. B **33**, 3263 (1986).
[3] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, Cambridge, U.K., 2007), Chap. 8.
[4] J. Gonzales *et al.*, Nucl. Phys. **B424**, 595 (1994).
[5] I. F. Herbut, Phys. Rev. Lett. **97**, 146401 (2006).
[6] A. W. W. Ludwig *et al.*, Phys. Rev. B **50**, 7526 (1994).
[7] N. H. Shon and T. Ando, J. Phys. Soc. Jpn. **67**, 2421 (1998); T. Ando, *ibid.* **75**, 074716 (2006).
[8] I. L. Aleiner and K. B. Efetov, Phys. Rev. Lett. **97**, 236801 (2006); A. Altland *ibid.* **97**, 236802 (2006).
[9] K. Nomura and A. H. MacDonald, Phys. Rev. Lett. **98**, 076602 (2007); S. Adam *et al.*, Proc. Natl. Acad. Sci. U.S.A. **104**, 18 392 (2007).
[10] I. F. Herbut, Phys. Rev. Lett. **87**, 137004 (2001).
[11] See, V. P. Gusynin, S. G. Sharapov, and J. P. Carbotte, Int. J. Mod. Phys. B **21**, 4611 (2007).
[12] T. Stauber *et al.*, Phys. Rev. B **71**, 041406(R) (2005).
[13] O. Vafek, Phys. Rev. Lett. **97**, 266406 (2006); **98**, 216401 (2007).
[14] See, J. Collins, *Renormalization* (Cambridge University Press, Cambridge, U.K., 1984), Chap. 12.9.
[15] I. F. Herbut, Phys. Rev. Lett. **79**, 3502 (1997); Phys. Rev. B **57**, 13 729 (1998).
[16] See, however, F. Schedin *et al.*, Nat. Mater. **6**, 652 (2007), about a possibly larger values of ϵ.
[17] S. V. Morozov *et al.*, Phys. Rev. Lett. **97**, 016801 (2006); D. A. Abanin *et al.*, *ibid.* **98**, 156801 (2007).
[18] Note the lowest data point in Fig. 5 in Ref. [1].
[19] K. Damle and S. Sachdev, Phys. Rev. B **56**, 8714 (1997).
[20] I. F. Herbut, Phys. Rev. Lett. **81**, 3916 (1998).
[21] E. G. Mishchenko, Phys. Rev. Lett. **98**, 216801 (2007).
[22] See also, D. E. Sheehy and J. Schmalian, Phys. Rev. Lett. **99**, 226803 (2007).