Extreme Statistics of Complex Random and Quantum Chaotic States

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Complex random states have the statistical properties of the Gaussian and circular unitary ensemble eigenstates of random matrix theory. Even though their components are correlated by the normalization constraint, it is nevertheless possible to derive compact analytic formulas for their extreme values' statistical properties for all dimensionalities. The maximum intensity result slowly approaches the Gumbel distribution even though the variables are bounded, whereas the minimum intensity result rapidly approaches the Weibull distribution. Since random matrix theory is conjectured to be applicable to chaotic quantum systems, we calculate the extreme eigenfunction statistics for the standard map with parameters at which its classical map is fully chaotic. The statistical behaviors are consistent with the finite-N formulas.

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The study of the statistics of extreme values [1] has found many applications in diverse areas such as geophysics, meteorology, economics, structural engineering, ocean waves, and dynamical systems. The subject is currently undergoing a resurgence of interest due to recent catastrophic events such as hurricanes, floods, and a particularly deadly tsunami as well as a number of research advances [2]. The questions being asked are, for example, what are the distributions for extreme events, or what are the interevent gap distributions? It has long been known that if the underlying events are independent and identically distributed, then for appropriately rescaled variables there are three possible limiting universal distributions for the extreme maximal events: the Fréchet, Gumbel, and Weibull distributions [1]. Respectively, they arise depending on whether the tail of the density is a power law, or faster than any power law, and unbounded or bounded. If there are correlations, then it is known that these universal distributions are reached for a sufficiently fast decay of autocorrelations [3].

In this Letter, these powerful methods are applied to the extreme properties of random vectors, or wave functions more generally, such as found in quantum mechanics, acoustics, and optics. Our motivation is that the eigenstate intensities in fully chaotic systems with no particular symmetries are conjectured to behave exactly as these random vectors subject only to a normalization constraint as in random matrix theory [4]. For chaotic systems, the Bohigas-Giannoni-Schmit conjecture states that the spectral fluctuations of quantized classically chaotic systems can be modeled by a suitable ensemble of random matrices [5]. In fact, a certain number of extreme spectral properties have already been derived [6–8] beginning with the well-known result for the distribution of the largest eigenvalue [9]. The corresponding treatment of random vectors or quantum eigenvectors has not yet been addressed. However, see [10] for an initial foray into random waves.

In fact, eigenstate intensities in strongly chaotic systems are known to follow an exponential density, which is consistent with states uniformly distributed over a standard simplex [11], as happens in the unitary ensembles (if an antiunitary symmetry is respected, the Porter-Thomas density, hypersphere, and orthogonal ensembles [12] are the relevant ones). A similar class of problems shows up in fragmentation, i.e., randomly cutting an object of fixed length into N pieces [13,14].

It is possible to give compact, exact formulas for all dimensionality N in spite of the correlations introduced by the normalization constraint. It turns out that the small-N distributions for the maxima differ considerably from their asymptotic limit (which turns out to be Gumbel, $\exp[-e^{-(t-a_N)/b_N}]$) thus giving the possibility of extracting system size from the distributions. In an N-dimensional complex Hilbert space a general state is represented in a fixed orthonormal basis $|i\rangle$ as $|\psi\rangle = \sum_{i=1}^{N} z_i|i\rangle$. If the z_i are complex components of a random state, then their joint probability distribution is

$$P(z_1, z_2, \ldots, z_N) = \frac{(N-1)!}{\pi^N}\delta\left(\sum_{j=1}^{N} |z_j|^2 - 1\right). \quad (1)$$

The real and imaginary parts of the components are spread in an unbiased, microcanonical, manner over the $2N$-dimensional unit sphere. The intensities $|z_i|^2$ are distributed uniformly on an $N-1$ simplex. Consider the probability distribution $\rho(t, N)$ of $t = \max\{|z_1|^2, |z_2|^2, \ldots, |z_N|^2\}$ and let $F(t, N)$ be the probability that all $|z_j|^2 \leq t$. This is called the distribution or cumulative density, i.e., $F'(t, N) = \rho(t, N)$ where the prime denotes differentiation with t:

$$F(t,N)=\Gamma(N)\left[\prod_{i=1}^{N}\int_0^t dt_i\right]\delta\left(\sum_{i=1}^{N}t_i-1\right), \quad (2)$$

where $t_j \equiv |z_j|^2$. Introduce an auxiliary function

$$G(t,N,u)=\Gamma(N)\left[\prod_{i=1}^{N}\int_0^t dt_i\right]\delta\left(\sum_{i=1}^{N}t_i-u\right) \quad (3)$$

such that $G(t,N,1)=F(t,N)$. Its Laplace transform is

$$\int_0^\infty e^{-us}G(t,N,u)\,du=\frac{\Gamma(N)}{s^N}\sum_{m=0}^{N}(-1)^m\binom{N}{m}e^{-stm}. \quad (4)$$

Convolution gives the inverse Laplace transform

$$\mathcal{L}_s^{-1}\left(\frac{e^{-smt}}{s^N}\right)=\frac{1}{\Gamma(N)}(u-mt)^{N-1}\Theta(u-mt), \quad (5)$$

where $\Theta(x)$ is the Heaviside function, and using this we can write the exact result for $F(t,N)$ as

$$F(t,N)=\sum_{m=0}^{N}\binom{N}{m}(-1)^m(1-mt)^{N-1}\Theta(1-mt). \quad (6)$$

More explicitly the expressions for $F(t,N)$ valid in the intervals $I_k=[1/(k+1),1/k]$, where $k=1,2,\ldots,N-1$ are

$$F(t\in I_k,N)=\sum_{m=0}^{k}\binom{N}{m}(-1)^m(1-mt)^{N-1}, \quad (7)$$

and $F(t\le 1/N,N)=0$. Thus the cumulative density is a piecewise smooth function on the intervals I_k.

Given the simple form of the distribution above, it is useful to interpret them combinatorially and derive them from such an approach. First note that

$$P_l(z_1,\ldots,z_l)=\frac{\Gamma(N)}{\pi^l\Gamma(N-l)}\left(1-\sum_{j=1}^{l}|z_j|^2\right)^{N-l-1}, \quad (8)$$

where P_l is the reduced probability density for l complex components (valid for $1\le l<N$ and nonzero only in the relevant domain $\sum_{j=1}^{l}|z_j|^2<1$). If $t\in I_1=[1/2,1]$ there can be at most only one such component. Therefore, the fraction of states with a component larger than t is exactly the fraction of components larger than t. Since the desired quantity is the fraction of states such that all components are less than t, it is the simply the complement:

$$F(t,N)=1-N\int_{|z|^2\ge t}P_1(z)\,dz. \quad (9)$$

The factor N accounts for multiplicity of choice of this one component. This integral is elementary for the complex case and gives $F(t,N)=1-N(1-t)^{N-1}$, which agrees with the series in the right-hand side of Eq. (7) which terminates at $k=1$ for $t\in I_1$.

If $t\in I_2$, it is possible that there are at most two such components. The number of components $\ge t$ is no longer the number of sequences (states) with at least one component more than t as it double counts states which have two components larger than t, a contribution which must get subtracted. This same logic extends, and in the next interval I_3, the number of pairs over counts the contributions of triples. Similarly, this reasoning carries forward to any distribution with a unit norm constraint and gives for the cumulative density

$$F(t\in I_k,N)=\sum_{m=0}^{k}\binom{N}{m}(-1)^m \times\int_{|x_i|^2>t}P_m(x_1,\ldots,x_m)dx_1\cdots dx_m, \quad (10)$$

which generalizes Eq. (7) (the $m=0$ term is unity). This generalization, for instance, could be the starting point for an analysis of real random states as well as general density matrix eigenvalues whose sum is also constrained to be unity. It is interesting that Eq. (7) and other piecewise continuous extreme distributions have been found in fragmentation problems [13,14], and have been identified in dynamical systems [15].

From the distribution Eq. (6) above, it is possible to derive exact formulas for the moments. In particular, results for the first (mean) and the second moments of the maximum components are

$$\langle t\rangle=\frac{H(N,1)}{N}=\frac{\gamma+\ln N}{N}+\mathcal{O}\left(\frac{1}{N^2}\right), \quad (11)$$

$$\langle t^2\rangle=\frac{H^2(N,1)+H(N,2)}{N(N+1)}, \quad (12)$$

where $H(N,k)$ is the Harmonic number of order k defined by the finite sum $\sum_{m=1}^{N}m^{-k}$, and $\gamma=0.5772\ldots$ is the Euler-Mascheroni constant. The standard deviation can be calculated exactly and its large-N form is

$$\sigma(t)=\frac{\pi}{\sqrt{6}N}+\mathcal{O}\left(\frac{\ln(N)}{N}\right)^2. \quad (13)$$

It turns out that the asymptotic $N\to\infty$ limit here is also the result for uncorrelated exponentially distributed variables of mean $1/N$. This limit may be calculated by simply taking P_1 as the probability density of an independent process, which gives

$$F(t,N)=(1-e^{-Nt})^N\to\exp(-e^{-N[t-\ln N/N]}). \quad (14)$$

Expressed in terms of the standard linearly scaled variable $x=(t-a_N)/b_N$, this distribution is seen to coincide with the Gumbel distribution where the parameters are given by $a_N=\ln(N)/N$ and $b_N=1/N$. As should happen, the mean and the standard deviation calculated from the Gumbel distribution coincide with the leading order contributions derived in Eqs. (11) and (13).

Finding a Gumbel distribution is interesting because the intensities have finite support due to the δ-function constraint of Eq. (1). In fact, the Weibull distribution ($1 - \exp[-(t-a)^\gamma/b]$) would be expected, but is not found due to the induced correlations. In Fig. 1 we compare the exact probability density with that of the Gumbel density ($\exp[-x - e^{-x}]$) after appropriately rescaling. It is clear that the approach of the exact density to the Gumbel density is rather slow and at N around 100 there are still significant differences. It is also instructive to note that without the rescaling, the densities actually diverge as N increases.

The distribution of the minimum intensity $s = \min\{|z_1|^2, |z_2|^2, \ldots, |z_N|^2\}$ on the other hand is a much simpler quantity and is not asymptotically a Gumbel distribution; interestingly, it appears in the decay rates of lasing modes in chaotic cavities [16]. The fraction of states such that the minimum is larger than some s is the fraction of states such that all the components are larger than s. Thus if $F(s, N)$ is the cumulative distribution of the minimum, it is given by

$$F(s, N) = 1 - \Gamma(N)\left[\prod_{i=1}^{N} \int_s^1 dt_i\right] \delta\left(\sum_{i=1}^{N} t_i - 1\right), \quad (15)$$

which is very similar to the integral in Eq. (2). Its evaluation proceeds similarly to the maximum, and gives

$$F(s, N) = \begin{cases} 1 - (1 - Ns)^{N-1} & 0 \le s \le 1/N, \\ 1 & 1/N \le s \le 1. \end{cases} \quad (16)$$

It is clear that the minimum cannot exceed $1/N$, just as the maximum cannot be less than this. The average minimum component is easily calculated and is exactly equal to $\langle s \rangle = 1/N^2$. The distribution for the minimum does not have the piecewise continuous character observed for the maximum. This has a geometrical interpretation in terms of the standard simplex. In the case of the maximum, the

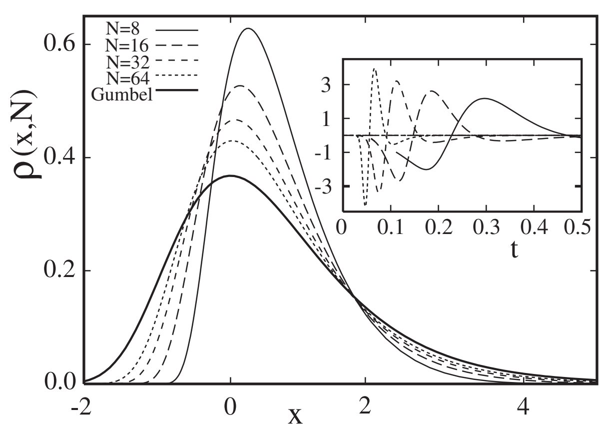

FIG. 1. The convergence of the exact probability density to the asymptotic Gumbel distribution using the scaled variable $x = N(t - \ln(N)/N)$ with increasing N. The inset shows the difference between the exact and the Gumbel distribution for the same values of N, but in the unscaled variable.

integral in Eq. (2) can be interpreted in terms of volumes of subsets contained in the region bounded by the standard $N - 1$ simplex. These volumes enclose more complex shapes for increasing t, as they pierce the simplex boundary. On the other hand, for the integral in Eq. (15) the volumes involved are those of the entire simplex and the volume of a subset that never pierces the $N - 1$ simplex.

For large N, the distribution of the minimum approaches the exponential:

$$F(s, N \to \infty) = 1 - \exp(-N^2 s). \quad (17)$$

This being a special case of the Weibull distribution, it is indeed what one would expect of uncorrelated variables with a compact support. That the minimum cannot be less than zero presents a strong constraint and for small components the normalization correlation is not so important. Thus, the distribution of the maximum and minimum of the complex random vector intensities follow different universal distributions asymptotically. It is noteworthy, however, that the limiting large deviations of the maximum component toward a small value which occur in the interval $I_{N-1} = 1/N \le t \le 1/(N-1)$ is distributed as $F(t, N) = (Nt - 1)^{N-1}$ which is an exact reflection about the value $1/N$ of the behavior of the minimum component.

In order to compare these extreme statistics to the statistical properties of the eigenfunctions of a Hamiltonian system, consider a quantum kicked rotor on the torus [17]. This is a stroboscopic mapping of a kicked one-dimensional particle of unit mass moving on a circle of unit perimeter with the Hamiltonian $H(q, p) = p^2/2 - (K/4\pi^2)\cos(2\pi q)\sum_{n=-\infty}^{\infty}\delta(t - n)$. The resultant mapping is the well-known standard map. For $K \gg 5$ the system is highly chaotic, although the phase space almost always contains some tiny proportion of regular motion mixed in.

In a position basis, the quantum evolution operator is

$$U_{nn'} = \frac{1}{N}\exp\left(\frac{iNK}{2\pi}\cos\left(\frac{2\pi(n' + \alpha)}{N}\right)\right) \times \sum_{m=0}^{N-1}\exp\left(-\pi i\frac{(m+\beta)^2}{N} + \frac{2\pi i(m+\beta)(n-n')}{N}\right). \quad (18)$$

The two phases $0 \le \alpha, \beta \le 1$ can be used for controlling parity and time-reversal symmetry breaking ($\alpha = 1/2$, $\beta = 0$ preserves both symmetries). Choosing β well away from 0 or $1/2$ breaks time-reversal invariance, which for $K \gg 5$ leads to quantum chaotic states that are complex and whose extremes should follow the distributions of the unitary ensembles.

The dimensionality of the Hilbert space N is the inverse (scaled) Planck constant.

An ensemble of roughly 30 000 quantum chaotic eigenstates is created from the N orthonormal states, as well as

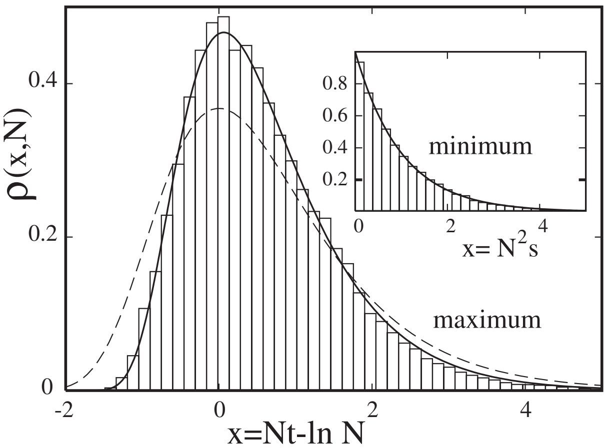

FIG. 2. The probability densities (histograms) of the scaled maximum and minimum (inset) intensity of eigenfunctions in the position basis of the quantum kicked rotor for $N = 32$ in the parameter range $13.8 < K < 14.8$. Shown as a continuous line is the exact density for random states while the dotted ones are the respective Gumbel and Weibull densities.

from those obtained by a variation of parameters K, α, and β such that while the quantum spectrum is significantly changed, the classical dynamics is not. Figure 2 shows the density of the maximal and minimal intensities in position bases for a range of K values where the classical map has no significant islands and is highly chaotic [18]. The derivatives of the exact results in Eqs. (7) and (8) fit the quantum system histograms very well. The deviations are roughly of the scale of the expected sample size errors. Note that the dynamical system results require the exact density for the maximum as the asymptotic approach is too slow. This is in contrast to the exact finite-N Dyson-Mehta fluctuation measures [4], which approach their asymptotic limits so quickly that finite-N results are seldom used. The inset shows that the fit to Weibull is excellent even for the small value of N used.

Given the excellent agreement between the analytic forms and the standard map on the torus for this strongly chaotic parameter range, deviations may in turn be used to investigate the important issue of eigenstate localization. For example, extreme statistical measures could be used to detect a variety of subtle effects such as significant eigenstate scarring in simple chaotic systems [19], other forms of localization due to dynamical effects [20], or detecting distinctions between simple chaotic and disordered systems, where, for example, the tails of eigenfunction statistics are predicted to be lognormal [21].

In conclusion, this work initiates the application of extreme statistics methods to random complex vectors, which appear naturally in a broad range of wave mechanical contexts as well as problems involving an isotropic norm constraint. The combinatorial approach, Eq. (10), is especially well suited to such problems. In the context of quantum eigenvectors, this offers a new approach to the study of nonergodic or localization behaviors, and of extracting system size information. Finally, it is worth noting that exact results for extreme value distributions of correlated variables such as those given here are quite rare. These show a number of surprising features related to slow convergence to asymptotic results and unexpected limiting forms.

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