Synchronization of Moving Chaotic Agents

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We consider a set of mobile agents in a two dimensional space, each one of them carrying a chaotic oscillator, and discuss the related synchronization issues under the framework of time-variant networks. In particular, we show that, as far as the time scale for the motion of the agents is much shorter than that of the associated dynamical systems, the global behavior can be characterized by a scaled all-to-all Laplacian matrix, and the synchronization conditions depend on the agent density on the plane.

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Synchronization of chaotic oscillators, i.e., the regime in which two or more chaotic oscillators evolve following the same chaotic trajectory despite their different initial conditions, is a very interesting dynamical behavior studied under many points of view [1]. A case of particular relevance in biophysics, neuroscience, and technology is when the chaotic oscillators are the nodes of a complex network [2] and the edges represent the coupling among them. The conditions under which N identical oscillators (coupled by an arbitrary network configuration admitting an invariant synchronization manifold) can be synchronized have been unravelled in [3] by linearization of the network dynamics around the synchronization manifold.

In [3] the dynamics of each node is modeled as $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K\sum_j g_{ij}\mathbf{H}(\mathbf{x}^j)$ where $i = 1, \ldots, N$, $\mathbf{x}^i$ is a m-dimensional vector of dynamical variables of the ith node, $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i)$ represents the dynamics of each isolated node, K is the coupling strength, $\mathbf{H}$: $\mathbb{R}^m \rightarrow \mathbb{R}^m$ is the coupling function and $G = [g_{ij}]$ is a zero-row sum $N \times N$ matrix modeling network connections (i.e., the Laplacian of the network).

According to the analysis of Pecora and Carroll [3], a block diagonalized variational equation of the form $\dot{\xi}_h = [D\mathbf{F} - K\gamma_h D\mathbf{H}]\xi_h$ represents the dynamics of the system around the synchronization manifold; where γ_h is the hth eigenvalue of G, $h = 1, \ldots, N$. $D\mathbf{F}$ and $D\mathbf{H}$ are the Jacobian matrices of F and H computed around the synchronous state, and are the same for each block. Therefore, the blocks of the diagonalized variational equation differ from each other only for the term $K\gamma_h$. If one wants to study synchronization properties with respect to different topologies, the variational equation must be studied as a function of a generic (complex) eigenvalue $\alpha + i\beta$. This leads to the definition of the master stability equation (MSE): $\dot{\zeta} = [DF - (\alpha + i\beta)DH]\zeta$.

The maximum (conditional) Lyapunov exponent $\lambda_{\max}$ of the MSE is studied as a function of α and β, thus obtaining the master stability function (MSF), i.e., $\lambda_{\max} = \lambda_{\max}(\alpha + i\beta)$. Then, the stability of the synchronization manifold in a given network can be evaluated by computing the eigenvalues γ_h (with $h = 2, \ldots, N$) of the matrix G and studying the sign of $\lambda_{\max}$ at the points $\alpha + i\beta = K\gamma_h$. If all eigenmodes with $h = 2, \ldots, N$ are stable, then the synchronous state is stable at the given coupling strength. If G has real eigenvalues, the MSF can be computed only as function of α.

The MSF formalism allows us to study how the overall topology of networks influences the propensity to synchronization. Specifically, it gives a necessary condition (the negativity of all Lyapunov exponents transverse to the synchronization manifold) for the stability of a complete synchronization process. With this approach, both heterogeneous and homogeneous networks, scale-free and small-world topologies, weighted and unweighted networks have been studied [2]. Only recently, the case of complex networks with links which do evolve in time has been considered [4–6].

In this Letter we consider the case of mobile agents, each one associated with a chaotic oscillator coupled with those of the neighboring agents. This situation, indeed, can be considered as a good representation of problems like clock synchronization in mobile robots, in which the communication is limited by the range of the communication system [7], or task coordination of swarming animals which are not only able to coordinate their motion in the plane, but also to react collectively and synchronously when subjected to external threats or attacks, or the appearance of synchronized bulk oscillations in a suspension of yeast cells, which has been experimentally observed for sufficiently high cell density [8].

In this Letter we adopt the constraint of fast switching [4,5] to derive synchronization conditions which relate synchronization to a scaled all-to-all Laplacian matrix and to the parameters describing the agent motion (for instance, density of agents). We also study how the system scales as the number of agents is varied. The results obtained do not rely on the quite restrictive hypothesis of commutative graphs [6].

We consider N moving individuals distributed in a planar space of size L with periodic boundary conditions. Each individual moves with velocity $\mathbf{v}_i(t)$ and direction

of motion $\theta_i(t)$ [v being the modulus of the agent velocity, which is the same for all individuals]. In our model, the agents are random walkers whose position and orientation are updated according to

$$\begin{aligned} \mathbf{y}_i(t + \Delta t_M) &= \mathbf{y}_i(t) + \mathbf{v}_i(t)\Delta t_M; \\ \theta_i(t + \Delta t_M) &= \eta_i(t + \Delta t_M), \end{aligned} \tag{1}$$

where $\mathbf{y}_i(t)$ is the position of the ith agent in the plane at time t, $\eta_i(t)$ are N independent random variables chosen at each time with uniform probability in the interval $[-\pi, \pi]$, and Δt_M is the motion integration step size. The choice of random walk is motivated by the numerous examples of real systems which can be modeled through random walk (for example, living cells in a fluid [9], mobile robots performing exploration tasks [7], and social systems [10]).

In addition, to include the possibility that individuals can move through the bidimensional world with shorter time constants (as in some social system models [10]), we consider the case that individuals may perform long-distance jumps. This is accounted for by defining a parameter p_j that quantifies the probability for an individual to perform a jump into a completely random position. In summary, at each time step, each agent evolves following Eqs. (1) [with $\mathbf{v}_i(t) = (v\cos\theta_i(t), v\sin\theta_i(t))$] with probability $1 - p_j$ or performs a jump with probability p_j. In the latter case the agent position is updated into a new position chosen at random in the plane in which agents move.

Furthermore, a dynamical system, and, in particular, a chaotic one, is associated to each agent. Each agent is then characterized by a state variable vector $\mathbf{x}^i(t) \in \mathbb{R}^n$ which evolves according to a given chaotic law. In the following, without lack of generality, we consider the case of Rössler oscillators, where the state dynamics of each agent is described by $\dot{x}_1^i = -(x_2^i + x_3^i)$; $\dot{x}_2^i = x_1^i + ax_2^i$; $\dot{x}_3^i = b + x_3^i(x_1^i - c)$ with $\mathbf{x}^i(t) = [x_1^i(t)\ x_2^i(t)\ x_3^i(t)]^T$. The following parameters have been used: $a = 0.2$; $b = 0.2$; $c = 7$.

Each agent interacts at a given time with only those agents located within a neighborhood of an interaction radius, defined as r [7,11,12]. When two agents interact, the state equations of each agent are changed to include diffusive coupling with the neighboring agent, acting on the state variable x_1. Based on these assumptions, the state dynamics of each agent can be described in terms of the following equations:

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K\sum_{j=1}^{N} g_{ij}(t)E\mathbf{x}^j \tag{2}$$

for $i = 1, \ldots, N$ with $\mathbf{F}$: $\mathbb{R}^3 \to \mathbb{R}^3$ given by the Rössler dynamics,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $g_{ij}(t)$ are the elements of a time-varying matrix $G(t)$ which defines the neighborhood of each agent at a given time t and depends on the trajectory of each agent. More in detail, $g_{ij}(t) = g_{ji}(t) = -1$ if the ith agent and the jth agent are neighbor at time t; and $g_{ii}(t) = h$ where h is the number of neighbors of the ith agent at time t. Equations (2) are integrated with a fixed time step defined as Δt_s (in all the simulations $\Delta t_s = 0.001$).

We first study the behavior of the system under the constraint of fast switching as defined in [13]. Stilwell *et al.* consider a time-varying network topology of coupled chaotic oscillators and describe the synchronization properties of the system by means of the time-average of the coupling matrix $G(t)$. The main result discussed in [13] can be expressed as follows: if the set of coupled oscillators defined by $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K\sum_{j=1}^{N} \bar{g}_{ij}E\mathbf{x}^j$ (with fixed topology $\bar{G} = [\bar{g}_{ij}]$) admits a stable synchronization manifold and if there exists a constant T such that $\frac{1}{T}\int_t^{t+T} G(\tau)d\tau = \bar{G}$, then there exists ε^* such that for all fixed $0 < \varepsilon < \varepsilon^*$ the set of coupled oscillators defined by $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K\sum_{j=1}^{N} g_{ij}(t/\varepsilon)E\mathbf{x}^j$ [i.e., coupled through a time-variant network $G(t/\varepsilon)$] also admits a stable synchronization manifold. According to [13] this implies that, if the time average of the coupling matrix $G(t)$, defined as $\bar{G} = \frac{1}{T}\int_t^{t+T} G(\tau)d\tau$, supports synchronization of the whole system and if the switching between all the possible network configurations is sufficiently fast, then the time-varying network will synchronize.

For the sake of simplicity, let us first consider the case of two agents, i.e., $N = 2$. In this case, two network configurations are possible only at each time t, since the two agents can be neighbor or not. Under the constraint of fast switching, we can make the approximation of analyzing the synchronization properties inherited by the time-average matrix $\bar{G}$ given by $\bar{G} = p_A G_A + p_0 G_0$, where p_A is the probability that the two agents are neighbor (all-to-all coupling), while p_0 is the probability that the two agents are not neighbor, and

$$G_A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$G_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are the corresponding $N \times N$ coupling matrices. Since $G_0 = 0$, $\bar{G}$ is given by $\bar{G} = p_A G_A$.

Thus, p_A plays the role of a coupling parameter. Since the agents have random starting positions, under static conditions the probability that the two agents are neighbor (i.e., agent 2 is within the interaction radius r of agent 1) is given by $p_A = \pi r^2/L^2$. Under dynamic conditions [i.e., when the two agents move following Eqs. (1)] there are fluctuations around this value whose amplitude depends on v. Under the hypothesis of large v very small fluctuations are observed.

It can be shown that a similar result, namely $\bar{G} = pG_A$, also holds for $N > 2$, where G_A is the $N \times N$ all-to-all coupling matrix. In fact, in general, there will be more than

two cases, depending on the possible configurations of the neighborhood of N agents. For instance, for $N = 3$ there will be eight possible configurations (since the agents are numbered) which as shown in Fig. 1 can be classified into four different cases. Therefore, we have $\bar{G} = p_A G_A + p_0 G_0 + p_{12} G_{12} + p_{23} G_{23} + p_{13} G_{13} + p_1 G_1 + p_2 G_2 + p_3 G_3$ where all the matrices in the sum are $N \times N$ matrices ($N = 3$), p_A is the probability that all the agents are neighbor (and G_A the all-to-all coupling matrix), p_{12} is the probability that 1 is close to 2, but 3 is not close to either 1 or 2 (and G_{12} is the matrix which corresponds to this case), p_1 is the probability that 2 is close to 1 and 3 is close to 1, but 2 and 3 are not neighbors (and G_1 is the matrix which corresponds to this case), and so on. By direct calculation it can be observed that $G_{12} + G_{23} + G_{13} = G_A$ and $G_1 + G_2 + G_3 = 2G_A$. Since $p_{12} = p_{13} = p_{23}$ and $p_1 = p_2 = p_3$, $\bar{G} = p_A G_A + p_{12}(G_{12} + G_{23} + G_{13}) + p_1(G_1 + G_2 + G_3) = (p_A + p_{12} + 2p_1)G_A$ and thus $\bar{G} = pG_A$. The same considerations can be repeated for $N \neq 3$. In a certain sense $\bar{G}$ is a kind of rescaled all-to-all matrix. In analogy with the case of blinking networks [4], it can be shown that p is the probability that a link is activated (i.e., that two agents are neighbor) and thus p is given by $p = \pi r^2/L^2$.

By taking into account that $\rho = N/L^2$, it can be derived that p depends on the density ρ as $p = \pi r^2 \rho/N$. Furthermore, $\bar{G} = \frac{\pi r^2 \rho}{N} G_A$. This relationship and the MSF allow us to derive the conditions under which the agents can be synchronized. To do this, the N eigenvalues of the coupling matrix $\bar{G}$ are calculated. They are $\lambda_1 = 0$ (since $\bar{G}$ is zero-row sum) and $\lambda_i = pN$ for $i = 2, \ldots, N$. Let us define λ as $\lambda = pN$.

Let us consider a typical type III MSF (i.e., a system for which stability of the synchronization manifold is guaranteed only in a given interval $[\alpha_1, \alpha_2]$ [2]) and let us suppose that $K\lambda \in [\alpha_1, \alpha_2]$. Since $0 < p < 1$, this means that there is a critical value of p ($p_c = \frac{\alpha_1}{KN}$) so that if $p < p_c$ the two agents will not synchronize and, on the opposite, if $p > p_c$ the two agents will synchronize. In terms of the density of agents, we can conclude that there exists a critical threshold such that for $\rho > \rho_c = \frac{\alpha_1}{\pi r^2 K}$ agents do synchronize.

In the opposite case, i.e., if $K\lambda > \alpha_2$, the system will behave in a different way: in this case, the agents will synchronize at densities such that $\frac{\alpha_1}{\pi r^2 K} = \rho_{c1} < \rho < \rho_{c2} = \frac{\alpha_2}{\pi r^2 K}$. As it can be noticed p_c depends on the number of agents N, but ρ_c, ρ_{c1}, and ρ_{c2} do not depend on N.

To verify that the critical value of the density does not depend on the number of agents, and to support the validity of the approximations made in the above analytical treatment, we carried on simulations of the full system at different densities and with different values of N ($N = 2$, $N = 10$, and $N = 100$). The other parameters have been fixed so that fast switching is achieved ($\upsilon = 1$, $p_j = 0$, $\Delta t_M = \Delta t_s = 10^{-3}$). We considered both identical and nonidentical agents and we averaged the results of the simulations over 50 different realizations. In the case of nonidentical agents, for each agent we fixed a different value of the parameter c in the interval [6.9, 7.1], so that each system has a chaotic behavior.

For the analysis of the simulation results, we defined the following synchronization error: $\delta(t) = \sum_{i=2}^{N} \frac{(|x_1^i - x_1^1| + |x_2^i - x_2^1| + |x_3^i - x_3^1|)}{3(N-1)}$, and we defined $\langle\delta\rangle = \langle\delta(t)\rangle$ as synchronization index, where the average is performed on the interval $[4T/5, T]$ ($T = 500$ s is the total length of the simulation).

Figure 2 reports the results for $K = 10$ (which, for Rössler oscillators, is such that $K\lambda > \alpha_2$). In the case of identical systems, as expected, for $\frac{\alpha_1}{\pi r^2 K} = \rho_{c1} < \rho < \rho_{c2} = \frac{\alpha_2}{\pi r^2 K}$ the index $\langle\delta\rangle$ is zero; this corresponds to complete synchronization on a synchronization manifold. In the case of nonidentical systems, a synchronization manifold cannot be formally defined, and therefore the MSF approach does not rigorously apply. We expect, however, that, for $\rho_{c1} < \rho < \rho_{c2}$, a synchronization motion will be established where the difference between the states of the systems will only slightly oscillate around zero. This is

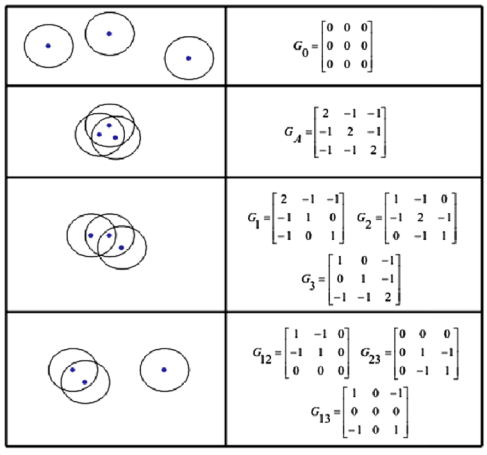

FIG. 1 (color online). Possible neighborhood configurations for $N = 3$ agents and corresponding Laplacian matrices.

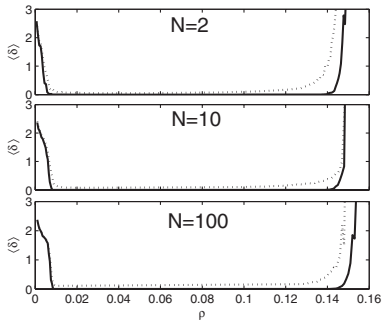

FIG. 2. Synchronization index $\langle\delta\rangle$ vs density ρ for identical (continuous line) and nonidentical (dotted line) systems with $N = 2$, $N = 10$, and $N = 100$. The coupling is fixed to $K = 10$. The other parameters have been chosen as follows: $\Delta t_M = \Delta t_s = 10^{-3}$, $\upsilon = 1$, $r = 1$. Results are averaged over 50 realizations.

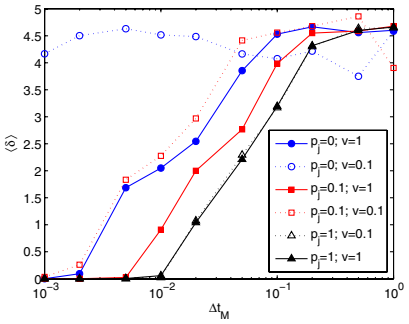

FIG. 3 (color online). Synchronization index $\langle\delta\rangle$ vs Δt_M for different values of p_j and v (identical systems). The other parameters have been fixed as $\Delta t_s = 10^{-3}$, $N = 100$, $\rho = 0.009$, $K = 10$. Results are averaged over 50 realizations.

confirmed in Fig. 2 where the corresponding values of $\langle\delta\rangle$ are very close to zero. The transition from a synchronized behavior to a nonsynchronized behavior is sharper in the case of identical systems. It can be noticed that the critical values of the density are quite independent of the number of agents. This suggests that the real size of the system is measured not by N but by the density of the agents. Furthermore, it is interesting to note that the effect of increasing the density is such that first the system synchronizes and then, for larger values of density, the synchronization is lost. This is related to the assumption of a type III MSF, whereas for type II MSF a single threshold is expected. The latter case may be pertinent to the experimental evidence in yeast cell populations where sustained oscillations depend on a sufficiently high cell density [8].

We then analyzed how the other parameters of the system, namely v, Δt_M, and p_j, influence synchronization. The system has two time scales: the motion integration step size Δt_M and that associated to the dynamics of the chaotic system. If we fix Δt_M to the smallest admissible value, i.e., $\Delta t_M = \Delta t_s$, then at each time step Eqs. (2) will be integrated and a motion step will be performed according to Eqs. (1). Otherwise, if, for instance, Δt_M is chosen to be $\Delta t_M = \tau \Delta t_s$, a motion step will be performed each τ integration step sizes of the dynamics equations (2). In this latter case, the neighborhood of each agent will be updated each τ integration step sizes Δt_s, and thus the switching between the possible network configurations will be slower than in the case $\Delta t_M = \Delta t_s$. Thus, decreasing Δt_M makes the switching between neighborhood configurations faster.

Both increasing v and p_j go in the direction of fast switching. As v increases, the distance covered in each motion step by the agents increases, making changes in the coupling matrix $G(t)$ more likely and the switching between possible network topologies faster. Similarly to the effect due to increasing values of v, the presence of long-distance jumps (i.e., $p_j \neq 0$) also makes the switching between the possible configurations faster and thus can help synchronization when Δt_M is low and $\rho_{c1} < \rho < \rho_{c2}$. This is shown in Fig. 3 which reports the synchronization index $\langle\delta\rangle$ vs Δt_M for different values of p_j and v with $\rho = 0.009$. The density ρ has been fixed so that the system is close to the transition for $v = 1$ and $p_j = 0$. As can be noticed for $v = 0.1$ and $p_j = 0$ agents are not synchronized, while increasing p_j they do synchronize at low values of Δt_M. For increasing values of p_j synchronization may be achieved at higher values of Δt_M with respect to the case of $p_j = 0$. This is also more evident for $v = 0.1$ than for $v = 1$, since the switching rate also depends on v. Furthermore, we notice that obviously when $p_j = 1$ the behavior is independent of v.

In conclusion, we have studied theoretically and numerically synchronization in a group of mobile agents, showing that a critical parameter is the density of the agents. When the density is kept constant, the behavior seems independent of the number of agents N. This study relates the system behavior to the properties of the dynamical network underlying the system. We have also shown that the velocity of the agents and the presence of long-distance jumps may help the onset of synchronization by increasing the switching rate between possible network configurations.

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