Separation of Heavy Particles in Turbulence

Itzhak Fouxon[1] and Péter Horvai[2]

[1]*Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel*
[2]*Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom*

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We study motion of small particles in turbulence when the particle relaxation time falls in the range of inertial time scales of the flow. Because of inertia, particles drift relative to the fluid. We demonstrate that the collective drift of two close particles makes them see local velocity increments fluctuate fast. This allows us to introduce Langevin description for separation dynamics. We describe the behavior of the Lyapunov exponent and give the analogue of Richardson's law for separation above viscous scale.

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Motion of small particles in a fluid, due to random molecular forces, is the subject of the classical theory of Brownian motion. Scale separation between the particle relaxation time and time scales of the forces allows to introduce an effective Langevin description of the driving force as white noise in time [1]. In contrast, here we consider the situation where the random driving force originates not from the microscopic motions, but rather from the macroscopic turbulent motion of the surrounding fluid [2]. The limit of the particle relaxation time much larger than the characteristic time scales of turbulence (very heavy particles) can be described as in the Brownian motion case [3]. In the opposite limit, when particle relaxation time is much smaller than the characteristic time scales of turbulence (very large friction), particles follow the flow closely, and the two-particle dispersion—of interest to us here—is approximately the same as for fluid particles. In this Letter we study the intermediate case of heavy particles, where the relaxation time falls in the range of flow time scales corresponding to the inertial interval of turbulence. This precludes Langevin description for the single-particle motion. However, for *two* particles, because of their collective drift relative to the fluid, velocity increments determining the separation do vary fast. This allows to introduce effective Langevin description for the dynamics of separations. The description enables us to establish several results, not employing a particular model of turbulence.

Behavior of small inertial particles in turbulence has received much attention lately [4–19]. This problem has many applications including rain formation in clouds [4,5], ocean physics [6], and engineering [7]. Theoretical study of the problem mostly involved modeling turbulence by a white noise in time, Gaussian velocity field, the so-called Kraichnan model. Even in that case theoretical study is rather difficult; analytic results were mainly obtained for the one-dimensional case [8–10]. The limit of heavy particles, considered here for turbulence, was studied numerically for the Kraichnan model in [3,11]. For turbulence, numerical studies of intermediate regime of moderately heavy particles were performed in [12–14].

We consider the motion of a small spherical particle in an incompressible, statistically steady, turbulent flow $\boldsymbol{u}(\boldsymbol{x}, t)$. We assume that the drag force acting on the particle obeys Stokes' law. Designating the particle position and velocity by $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t)$, Newton's law reads

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = -\frac{\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{x}(t), t)}{\tau}. \tag{1}$$

Here $\tau = (2/9)(\rho_0/\rho)(a^2/\nu)$, where ρ_0 and a are the particle density and radius, while ρ and ν are the fluid's density and kinematic viscosity. We briefly review relevant properties of $\boldsymbol{u}(\boldsymbol{x}, t)$, see, e.g., [2] for details. Velocity field, excited at the integral scale L, fluctuates in a wide (inertial) range of spatial scales $\eta \ll l \ll L$. The characteristic velocity u_l of fluctuations at a scale l is related to the temporal scale t_l by $u_l t_l / l \sim 1$. At the viscous scale η we have $t_\eta \sim \eta^2/\nu$. For moderate Reynolds numbers Re one can use Kolmogorov theory (below K41) that gives $u_l \sim (\epsilon l)^{1/3}$, $t_l \sim \epsilon^{-1/3} l^{2/3}$, and $\eta \sim (\nu^3/\epsilon)^{1/4}$, where ϵ is the mean energy injection rate. Equation (1) is valid if $\eta \gg a$ and inertia-induced particle drift relative to the flow, described by $\boldsymbol{w}(t) \equiv \boldsymbol{v}(t) - \boldsymbol{u}[\boldsymbol{x}(t), t]$, has small Reynolds number wa/ν [15]. We shall consider $t_\eta \ll \tau \ll t_L$, which implies that particles are heavy, $\rho_0/\rho \sim (\eta^2/a^2)(\tau/t_\eta) \gg 1$, justifying the neglect of such effects as added mass in Eq. (1) [15]. Beyond K41, quantities like η and w have strong spatiotemporal fluctuations, and in that setup we will refer to their local (in space and time) values on statistically relevant events.

We first consider the drift velocity $\boldsymbol{w}$. From Eq. (1), in the steady state, $\boldsymbol{w}(0) = \int_{-\infty}^{0} \delta_t^P \boldsymbol{u} \exp(t/\tau) dt/\tau$, where we set $\boldsymbol{x}(0) = 0$ and $\delta_t^P \boldsymbol{u} \equiv \boldsymbol{u}[\boldsymbol{x}(t), t] - \boldsymbol{u}(\boldsymbol{0}, 0)$ is the turbulent velocity difference in particle frame. The latter is similar to the Lagrangian difference $\delta_t^L \boldsymbol{u} \equiv \boldsymbol{u}[\boldsymbol{q}(t), t] - \boldsymbol{u}(\boldsymbol{0}, 0)$, where $\boldsymbol{q}(t)$ is a fluid particle trajectory obeying $\dot{\boldsymbol{q}} = \boldsymbol{u}[\boldsymbol{q}(t), t]$ and $\boldsymbol{q}(0) = \boldsymbol{0}$. As $\delta_t^L u$, the increment $\delta_t^P u$ should be, on a rough scale, a nondecreasing function of $|t|$, growing at most as a power law. Then the integral for $\boldsymbol{w}$ yields the order of magnitude estimate $w \sim \delta_\tau^P u$ and we also see that the characteristic time of variations of $\boldsymbol{w}$ is τ. Both $\boldsymbol{w}$ and particle acceleration $\boldsymbol{a} \equiv \dot{\boldsymbol{v}} = -\boldsymbol{w}/\tau$ characterize velocity fluctuations with time scale τ in the inertial range. Let us consider the time averages $\langle w^n\rangle$ along the particle trajectory $\boldsymbol{x}(t)$. The usual phenomenology suggests the existence of finite universal limits $\gamma_n = \lim_{\tau\to 0}\lim_{\nu\to 0}\ln\langle w^n\rangle/\ln(\tau/t_L)$, independent of the source of turbulence, where $\gamma_n/n \neq$ const due to intermittency [2,20]. Introducing dimensionless constants A_n which can depend on τ and ν only weakly, we have

$$\tau^n\langle a^n\rangle = \langle w^n\rangle = A_n u_L^n(\tau/t_L)^{\gamma_n}, \qquad \sqrt{\nu/\epsilon}\mathrm{Re}^{\alpha_n} \ll \tau \ll t_L, \tag{2}$$

where $\alpha_n \neq 0$ due to fluctuations of t_η [2]. Statistics of $\boldsymbol{a}$ was studied numerically for $\tau \lesssim t_\eta$ [12] and experimentally for $\tau < t_\eta$ [17]. The acceleration flatness $\langle a^4\rangle/\langle a^2\rangle^2$ was found to decrease with τ at $\tau \lesssim t_\eta$ [12]. This fits Eq. (2) giving $\langle a^4\rangle/\langle a^2\rangle^2 = (A_4/A_2)(\tau/t_L)^{\gamma_4-2\gamma_2}$, where $\gamma_4 < 2\gamma_2$ due to $\gamma_0 = 0$ and convexity of γ_n [2].

Equation (2) is analogous to $\langle[\delta_\tau^L u]^n\rangle_q = B_n u_L^n(\tau/t_L)^{\mu_n}$ for $\sqrt{\nu/\epsilon}\mathrm{Re}^{\beta_n} \ll \tau \ll t_L$ [2,20], where the time average is along the fluid particle trajectory $\boldsymbol{q}(t)$, rather than $\boldsymbol{x}(t)$. Let us show that $\gamma_n = \mu_n$ and $\alpha_n = \beta_n$ is expected (but not $A_n = B_n$). In K41 this is automatic: both $\langle w^n\rangle$ and $\langle[\delta_\tau^L u]^n\rangle_q$ are determined by ϵ and τ only, and proportional to $(\epsilon\tau)^{n/2}$ by dimensional analysis, while $\alpha_n = \beta_n = 0$ (thus in K41 $w \sim \sqrt{\epsilon\tau}$). For general case, let us consider the separation of inertial and fluid particles. The relative velocity $\delta\boldsymbol{v} \equiv \boldsymbol{v}(t) - \boldsymbol{u}[\boldsymbol{q}(t), t]$ can be written as a sum of the stationary process $\boldsymbol{w}(t)$ and $\delta\boldsymbol{u} \equiv \boldsymbol{u}[\boldsymbol{x}(t), t] - \boldsymbol{u}[\boldsymbol{q}(t), t]$, roughly growing with time: $\delta\boldsymbol{v} = \boldsymbol{w} + \delta\boldsymbol{u}$. Then, as shown by Olla in [16], there is a crossover: $\delta\boldsymbol{v} \approx \boldsymbol{w}$ at $|t| \ll \tau$ and $\delta\boldsymbol{v} \approx \delta\boldsymbol{u}$ at $|t| \gg \tau$ (this result can be shown based on explosive separation [18,21]). In the latter regime the separation is explosive and $\delta u(t) \sim \delta_t^L u$ [2,21], implying $\delta_t^P u = |\delta_t^L \boldsymbol{u} + \delta\boldsymbol{u}| \sim \delta_t^L u$ at $|t| \gg \tau$. This gives $\delta_\tau^P u \sim \delta_\tau^L u$ which by $w \sim \delta_\tau^P u$ produces $w \sim \delta_\tau^L u$. Since $\delta_\tau^L u$ is the characteristic velocity of turbulent fluctuations with time scale τ, the estimate just means that particles follow only the flow fluctuations with time scales larger than τ (in particular $\boldsymbol{v} \approx \boldsymbol{u}[\boldsymbol{x}(t), t]$).

The local equality $w \sim \delta_\tau^L u$ implies $\langle w^n\rangle \sim \langle[\delta_\tau^L u]^n\rangle$. To demonstrate $\gamma_n = \mu_n$ it remains to show $\langle[\delta_\tau^L u]^n\rangle \sim \langle[\delta_\tau^L u]^n\rangle_q$, not evident because of the particles' tendency to concentrate preferentially in specific regions of the flow [4], thus performing biased sampling of the latter [12,17]. Yet the numerical evidence indicates at $\tau \gg t_\eta$ the bias is of minor importance for the averaging [12]. Further, $\delta_\tau^L u$ is correlated in space over the scale l_τ, defined implicitly by $t_{l_\tau} \sim \tau$, beyond which the preferential concentration is expected to be small [14]. Then additional spatial averaging of $\langle[\delta_\tau^L u]^n\rangle$ over l_τ vicinity of $\boldsymbol{x}(t)$, which preserves the order of magnitude of $\langle[\delta_\tau^L u]^n\rangle$, washes out the effects of preferential concentration, giving $\langle[\delta_\tau^L u]^n\rangle \sim \langle[\delta_\tau^L u]^n\rangle_q$.

We now consider two-particle motion. We assume the particle separation $\boldsymbol{R} = \boldsymbol{x}' - \boldsymbol{x}$ much larger than radius a, so that hydrodynamic interactions between particles are negligible. Then each particle satisfies Eq. (1), producing

$$\tau\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}} = \delta\boldsymbol{u}(\boldsymbol{R}), \qquad \delta\boldsymbol{u}(\boldsymbol{R}) \equiv \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{R}) - \boldsymbol{u}(\boldsymbol{x}). \tag{3}$$

At $R \gg l_\tau$ particle dispersion, driven by $\delta\boldsymbol{u}(\boldsymbol{R}) \sim u_R$, is determined by turbulent fluctuations slower than τ, and the separation is like for fluid particles: $\dot{\boldsymbol{R}} = \boldsymbol{v}' - \boldsymbol{v} = \delta\boldsymbol{u}(\boldsymbol{R}) + \boldsymbol{w}' - \boldsymbol{w} \approx \delta\boldsymbol{u}(\boldsymbol{R})$, by $w \sim u_{l_\tau} \ll u_R$. In contrast, at $R \ll l_\tau$ dispersion laws peculiar for inertial particles hold [note that in K41 $l_\tau \sim \epsilon^{1/2}\tau^{3/2} \sim \eta(\tau/t_\eta)^{3/2}$]. We first treat $R(t) \ll \eta$ where $\delta u_i(\boldsymbol{R}) \approx R_j\nabla_j u_i[\boldsymbol{x}(t), t]$ and

$$\tau\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}} = \boldsymbol{R}\cdot\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]. \tag{4}$$

Equation (4) describes the exponential growth of the distance $\boldsymbol{p} \equiv (\boldsymbol{R}, \tau\dot{\boldsymbol{R}})$ between two infinitesimally close trajectories in the *phase* space of Eq. (1) (where the relative particle velocity is also small cf. [10]). The main characteristic of the growth is the Lyapunov exponent $\lambda_1 = \lim_{t\to\infty}\ln[p(t)/p(0)]/t = \langle\dot{p}/p\rangle$. Below we study the dependence of λ_1 on the Stokes number $\mathrm{St} \equiv \lambda_1^{\mathrm{turb}}\tau$, where $\lambda_1^{\mathrm{turb}}$ is the Lyapunov exponent of fluid particles, $\lambda_1^{\mathrm{turb}} = \lambda_1|_{\tau=0}$. At physically relevant Reynolds numbers, K41 estimate $\lambda_1^{\mathrm{turb}} \sim \sqrt{\epsilon/\nu}$ is valid [13], which means that the events determining $\lambda_1|_{\tau=0}$ are but weakly intermittent. Below we assume that λ_1 is determined by weakly intermittent events also at finite τ, and we use K41 for order of magnitude estimates, justifying this later.

The main observation enabling us to analyze λ_1 is that $\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]$ in Eq. (4) is short correlated in time at $\mathrm{St} \gg 1$. To see this note that the time derivative of the velocity gradient in the *particle* frame, $\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]$, can be written as the time derivative along the fluid particle trajectory plus a spatial derivative due to the inertial drift:

$$\frac{d}{dt}\nabla\boldsymbol{u}[\boldsymbol{x}(t), t] = [(\partial_t + \boldsymbol{u}\cdot\nabla) + \boldsymbol{w}\cdot\nabla]\nabla\boldsymbol{u}. \tag{5}$$

The term $(\boldsymbol{w}\cdot\nabla)\nabla u \sim \nabla u(w/\eta)$ produces characteristic time of variations $\eta/w \sim t_\eta/\sqrt{\mathrm{St}}$. During this time the particle drifts away from the carrying flow by the spatial scale of variations of the velocity gradient η. At $\mathrm{St} \gg 1$ this drift time scale is smaller than the time scale t_η given by the substantial derivative in Eq. (5). It follows that the correlation time τ_c of $\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]$ is $t_\eta/\sqrt{\mathrm{St}}$ and it decreases with St. On the other hand, the characteristic value of $\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]$, given by $u_\eta/\eta \sim 1/t_\eta$, is independent of St. Thus at large St it should be possible to describe $\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]$ by a white noise. Such description, however, has a subtlety—properties of local correlations of $\nabla\boldsymbol{u}[\boldsymbol{x}(t), t]$ [such as the correlation time $\eta/w(t)$] depend on the slowly changing parameter $\boldsymbol{w}(t)$, and thus change at the time scale τ (notice that $\tau \gg \tau_c$ at $\mathrm{St} \gg 1$). To incorporate this feature we first consider the evolution described by Eq. (4) at time intervals shorter than τ, where one can write $\boldsymbol{x}(t) = \boldsymbol{q}(t) + \boldsymbol{w}t$. Here $\boldsymbol{w}(t) = \boldsymbol{w}$ and we set the observation moment at $t = 0$. We make a natural assumption (verified below) that $\lambda_1^{\mathrm{turb}} \gg \lambda_1$ at $\mathrm{St} \gg 1$. Since $\lambda_1 \sim t_\eta \gtrsim \tau_c$, we find that the characteristic time of variations of $\boldsymbol{R}$, given by λ_1^{-1}, obeys $\lambda_1^{-1} \gg \tau_c$. Then averaging Eq. (4) over time Δt satisfying $\lambda_1^{-1} \gg \Delta t \gg \tau_c$, we obtain $\ddot{R}_i + \dot{R}_i/\tau = R_j \bar{\nabla}_j u_i/\tau$, where $\bar{\nabla}_j u_i \equiv \int_t^{t+\Delta t} \nabla_j u_i[\boldsymbol{q}(t') + \boldsymbol{w}t', t']dt'/\Delta t$. We observe that $\bar{\nabla}_j u_i$ is a sum of a large number $\sim\Delta t/\tau_c$ of independent random variables and thus is Gaussian. It is fully fixed by its mean (equal to zero) and the pair correlation, which—due to stationarity, spatial homogeneity of small-scale turbulence and incompressibility—is determined by $F_{ijmn}(\boldsymbol{w}) \equiv \int dt \langle \nabla_j u_i(\boldsymbol{0}, 0)\nabla_n u_m[\boldsymbol{q}(t) + \boldsymbol{w}t, t]\rangle_u$. Here the averaging is performed over the statistics of the turbulent velocity $\boldsymbol{u}$, assuming that $\boldsymbol{w}$ is just a constant vector: $\boldsymbol{w}(t)$ is determined by velocity fluctuations with $t_l \gtrsim \tau$ so that the statistics of the velocity gradients, determined by fluctuations with time scale $t_\eta \ll \tau$, are approximately independent of the local value of $\boldsymbol{w}(t)$. Then $\bar{\nabla}_j u_i(t)$ is statistically equivalent to $\int_t^{t+\Delta t} \sigma_{ij}(\boldsymbol{w}, t')dt'/\Delta t$, where

$$\langle \sigma_{ij}(\boldsymbol{w}, t)\sigma_{mn}(\boldsymbol{w}, t')\rangle = \delta(t' - t)F_{ijmn}(\boldsymbol{w}). \tag{6}$$

Dropping the auxiliary time averaging we find that the evolution over times shorter than τ can be described by the anisotropic white-noise model $\tau\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}} = \sigma(\boldsymbol{w})\boldsymbol{R}$.

To use the above reduction we need some results on the Kraichnan model in which $\nabla \boldsymbol{u}$ in Eq. (4) is modeled by the white noise $\hat{\sigma}_{ij}$ obeying $\langle \hat{\sigma}_{ij}(t)\hat{\sigma}_{mn}(t')\rangle = D\delta(t - t')[(d + 1)\delta_{im}\delta_{jn} - \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}]$, where d is the space dimension [6]. Here, passing to dimensionless time $s = D^{1/3}t/\tau^{2/3}$, one finds $\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/(D\tau)^{1/3} = \sigma'(s)\boldsymbol{R}$, where $\langle \sigma'_{ij}(s_1)\sigma'_{mn}(s_2)\rangle = \delta(s_1 - s_2)[(d + 1)\delta_{im}\delta_{jn} - \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}]$ cf. [11]. At $(D\tau)^{1/3} \gg 1$ one can drop $\dot{\boldsymbol{R}}/(D\tau)^{1/3}$ obtaining the scaling law $\lambda_1 = \lambda^0 D^{1/3}/\tau^{2/3}$ where λ_0 is the dimensionless growth exponent of $\ddot{\boldsymbol{R}} = \sigma'(s)\boldsymbol{R}$. At $(D\tau)^{1/3} \sim 1$ all terms in the equation are of order unity and though there is no scaling, the estimate $\lambda_1 \sim D^{1/3}/\tau^{2/3}$ still holds (this explains the numerical results of [3,11]). Thus $\lambda_1 \sim D^{1/3}/\tau^{2/3}$ at $D\tau \gtrsim 1$. For example, in $d = 2$, $\tilde{\lambda}_1[(D\tau)^{-1/3}] \equiv \lambda_1\tau^{2/3}/D^{1/3}$ slowly varies from $\tilde{\lambda}_1(1) \approx 0.5$ to $\lambda^0 \equiv \tilde{\lambda}_1(0) \approx 2$ [3], giving $\lambda_1 \approx \lambda^0 D^{1/3}/\tau^{2/3}$ at $(D\tau)^{1/3} \gg 1$. We make an important remark that the time scale beyond which $\langle \dot{p}(t)/p(t)\rangle$ relaxes to its steady-state value λ_1, forgetting the initial conditions, can be estimated as λ_1^{-1}. Indeed, λ_1^{-1} is the only time scale at $(D\tau)^{1/3} \gg 1$, while at $(D\tau)^{1/3} \sim 1$ all coefficients in $\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}}/(D\tau)^{1/3} = \sigma'(s')\boldsymbol{R}$ are of order unity so again $\tau^{2/3}/D^{1/3} \sim \lambda_1^{-1}$ is the only possible time scale.

We now return to Eq. (6) and introduce $\lambda_1(w)$ as the Lyapunov exponent for $\tau\ddot{\boldsymbol{R}} + \dot{\boldsymbol{R}} = \sigma(\boldsymbol{w})\boldsymbol{R}$. At $\sqrt{\mathrm{St}} \gg 1$ the latter model simplifies to the ordinary $2d$ Kraichnan model. Here the drift time scale $\tau_c \sim t_\eta/\sqrt{\mathrm{St}}$ is much less than t_η and the right-hand side of Eq. (5) is dominated by the last term. As a result, the change in $\nabla \boldsymbol{u}[\boldsymbol{x}(t), t]$ is determined by the spatial variations of turbulence only and one finds $F_{ijmn}(\boldsymbol{w}) \approx \int dt \langle \nabla_j u_i(\boldsymbol{0})\nabla_n u_m(\boldsymbol{w}t)\rangle$. A degeneracy $w_n F_{ijmn} = \int dt \partial_t \langle \nabla_j u_i(0)u_m(\boldsymbol{w}t)\rangle = 0$ appears, allowing to set $\sigma_{i3} \equiv 0$, where we chose z axis parallel to $\boldsymbol{w}$. As a result, $\boldsymbol{r} \equiv (R_1, R_2)$ satisfies closed equation $\ddot{\boldsymbol{r}} + \dot{\boldsymbol{r}}/\tau = \tilde{\sigma}\boldsymbol{r}/\tau$, where $\tilde{\sigma}$ is a 2×2 matrix with $\tilde{\sigma}_{ij} = \sigma_{ij}$. Using $\langle \nabla_j u_i(0)\nabla_n u_m(\boldsymbol{r})\rangle = \nabla_j\nabla_n S_{im}(\boldsymbol{r})/2$, where $S_{ij}(\boldsymbol{r}) = \langle [u_i(\boldsymbol{r}) - u_i(\boldsymbol{0})][u_j(\boldsymbol{r}) - u_j(\boldsymbol{0})]\rangle$, one can express F_{ijmn} with the help of second order structure function of turbulence $S_2(r) = \langle ([\boldsymbol{u}(\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{0})] \cdot \boldsymbol{r}/r)^2\rangle$. A straightforward calculation [18] shows that $\tilde{\sigma}$ has the statistics of $\hat{\sigma}$ in $2d$ with $D = D(w) = w^{-1}\int_0^\infty S_2(r)/(2r^2)dr$ [note $D(w) \sim \lambda_1^{\mathrm{turb}}/\sqrt{\mathrm{St}}$ by $w \sim \sqrt{\epsilon\tau}$]. It follows that at $\sqrt{\mathrm{St}} \gg 1$ one has $\lambda_1(w) = D(w)^{1/3}\tilde{\lambda}_1\{[D(w)\tau]^{-1/3}\}/\tau^{2/3}$. For any $\mathrm{St} \gtrsim 1$, like in the Kraichnan model, one finds $\lambda_1(w) \sim D^{1/3}(w)/\tau^{2/3} \sim \mathrm{St}^{1/6}/\tau$.

Now we can analyze $\lambda_1 = \lim_{t\to\infty}\langle \dot{p}(t)/p(t)\rangle_u$. We average $\langle \dot{p}/p\rangle_u$ over the statistics of turbulence in two steps: first averaging over the statistics of the velocity gradients at fixed $\boldsymbol{w}(t)$ and then averaging over $\boldsymbol{w}(t)$. We note from the analysis of the Kraichnan model that if $\lambda_1^{-1}(w) \ll \tau$ then the average $\langle \dot{p}(t)/p(t)\rangle_{\nabla u}$ over the statistics of the gradients is determined by the time interval $[t - \lambda_1^{-1}(w), t]$ shorter than τ, for which Eq. (6) applies with $\boldsymbol{w} = \boldsymbol{w}(t)$. Thus $\langle \dot{p}(t)/p(t)\rangle_{\nabla u} \approx \lambda_1[w(t)] = \lambda^0 D^{1/3}[w(t)]/\tau^{2/3}$, where we use for $\lambda_1(w)$ the $2d$ Kraichnan model expression at $\lambda_1(w)\tau \gg 1$. Using the expression for $D(w)$ and averaging over $\boldsymbol{w}$, we obtain

$$\lambda_1 \approx G, \quad \text{for} \quad G \equiv \lambda^0\langle w^{-1/3}\rangle\left(\int_0^\infty \frac{S_2(r)dr}{2\tau^2 r^2}\right)^{1/3} \gg \frac{1}{\tau}. \tag{7}$$

Since $G \sim \mathrm{St}^{1/6}/\tau$ the limit above is $\mathrm{St}^{1/6} \gg 1$. On the other hand, at $\mathrm{St}^{1/6} \sim 1$ one has $\lambda_1^{-1}(w) \sim \tau/\mathrm{St}^{1/6} \sim \tau$ and the time interval $[t - \lambda_1^{-1}(w), t]$ is of order τ, which is the boundary of the applicability of the averaging over the gradients using Eq. (6). Here we find only the order of magnitude estimate $\langle \dot{p}(t)/p(t)\rangle_{\nabla u} \sim \lambda_1[w(t)]$ giving $\lambda_1 \sim \langle \lambda_1(w)\rangle \sim \mathrm{St}^{1/6}/\tau$. One can combine the results as

$$\lambda_1/\lambda_1^{\mathrm{turb}} \sim \mathrm{St}^{-5/6} \quad \text{for} \quad \mathrm{St} \gtrsim 1. \tag{8}$$

The above says λ_1 can be estimated from the naive white-noise model with $D \sim \int \langle \nabla u(0)\nabla u(t)\rangle dt \sim (\lambda_1^{\mathrm{turb}})^2\tau_c$. The dependence of D on τ brings faster decay of $\lambda_1/\lambda_1^{\mathrm{turb}}$ than in the ordinary Kraichnan model. The use of K41 for estimates above is self-consistent: at realistic Re (within multifractal model [2] $\mathrm{Re} \gtrsim 10^{15}$) both $\langle w^{-1/3}\rangle$ and $\int_0^\infty S_2(r)dr/r^2$ entering G are well described by K41.

We now consider the growth of $\boldsymbol{R}$ in the inertial range, at $\eta \lesssim R \ll l_\tau$. In contrast to Richardson's law for fluid particles $R(t) \sim \epsilon^{1/2}t^{3/2}$ [2,21], K41 dimensional analysis does not fix the separation law for inertial particles, due to the additional time scale τ. We shall assume moderate $\dot{\boldsymbol{R}}(0)$ not to have mere ballistic motion, e.g., the analysis below applies to $R(0) \sim \eta$, $\dot{R}(0) \sim \lambda_1\eta$, holding after separation at $R \ll \eta$. As we will see, $R(t)$ reaches l_τ within $t \sim \tau$, so to study separation at $R(t) \ll l_\tau$ we assume $t \ll \tau$. The "friction" term $\dot{\boldsymbol{R}}/\tau$ in Eq. (3) produces negligible effect over $t \ll \tau$ and it can be omitted giving $\tau\ddot{\boldsymbol{R}} = \delta\boldsymbol{u}(\boldsymbol{R}) \approx \boldsymbol{u}[\boldsymbol{q}(t) + \boldsymbol{w}t + \boldsymbol{R}(t), t] - \boldsymbol{u}[\boldsymbol{q}(t) + \boldsymbol{w}t, t]$, where $\boldsymbol{w} \equiv \boldsymbol{w}(0)$. The correlation time $t_c(R)$ of $\delta\boldsymbol{u}(R)$ is due to the drift, $t_c(R) \sim R/w \lesssim t_R$. As we verify later, the time scale $\tau_c(R)$ of variations of R obeys $\tau_c(R) \gg t_c(R)$. Then, like in the viscous range, we may introduce Langevin description of $\delta u_i(\boldsymbol{R})$, substituting it by white noise $D_{ij}(\boldsymbol{R})\gamma_j$, where $\langle\gamma_i(t)\gamma_j(t')\rangle = \delta_{ij}\delta(t - t')$. Here $D_{ik}(\boldsymbol{R})D_{jk}(\boldsymbol{R}) = \int dt\langle[u_i(\boldsymbol{R}) - u_i(0)][u_j(\boldsymbol{w}t + \boldsymbol{R}) - u_j(\boldsymbol{w}t)]\rangle$ to provide the correct dispersion of the time averaged $\delta\boldsymbol{u}(\boldsymbol{R})$ [18]. We assumed for simplicity $t_R \gg R/w$ or $(l_\tau/R)^{1/3} \gg 1$ (we use K41 as at $R \ll \eta$), so that the time correlations of $\delta\boldsymbol{u}(\boldsymbol{R})$ are determined by the drift (cf. to $t_\eta \gg \eta/w \sim \tau_c$ at $R \ll \eta$). The above Kraichnan model for particles is not the same as used usually to model turbulence in the inertial range [21]: $t_c(R)$ depends on R differently than t_R. Noting $D_{ik}(\boldsymbol{R})D_{jk}(\boldsymbol{R}) \sim S_2(R)t_c(R) \sim S_2(R)R/w$, we conclude that the dependence on ϵ and τ in $\ddot{R}_i = D_{ij}(\boldsymbol{R}, \boldsymbol{w})\gamma_j/\tau$ is via a single parameter $\epsilon^{2/3}/w\tau^2 \sim l_\tau^{1/3}/\tau^3$. Now dimensional analysis is enough to fix the answer. We find $\tau_c(R) \sim \tau(R/l_\tau)^{1/9}$, so the applicability condition $\tau_c(R) \gg R/w$ gives $(R/l_\tau)^{8/9} \ll 1$, close to just $R \ll l_\tau$. At $t \gg \tau_c[R(0)]$ the initial condition is forgotten (we assume explosive separation characteristic of the inertial range [21]) and $R(t)$ depends only on t and $l_\tau^{1/3}/\tau^3$ giving

$$R(t) \sim l_\tau(t/\tau)^9, \qquad \tau_c[R(0)] \ll t \ll \tau. \tag{9}$$

The above law is closer than Richardson's law to the exponential separation holding for smooth flows (cf. the Kraichnan model for fluid particles where Richardson's law exponent grows indefinitely as the flow becomes less rough [21]). The smoothing is due to the effective time averaging of the turbulent velocity difference $\delta\boldsymbol{u}(\boldsymbol{R})$ performed by separating particles. As $R(0) \gtrsim \eta$, the observability of the power law entails $\tau \gg \tau_c[R(0)] \gtrsim \tau_c(\eta)$. This gives $(l_\tau/\eta)^{1/9} \sim \mathrm{St}^{1/6} \gg 1$, equivalent to the natural "forgetting" condition $\lambda_1^{-1} \ll \tau$. At $\mathrm{St}^{1/6} \sim 1$, the time of forgetting of the initial condition obeys $\tau_c[R(0)] \sim \tau$ so $R(t)$ at $t \ll \tau$ depends on the details of initial conditions. Equation (9) then can be used as order of magnitude estimate at $t \sim \tau$ giving $R(\tau) \sim l_\tau$. This is expectable—for *fluid* particles the time of separation to l_τ is of order τ and determined by the stage of evolution with $R(t) \sim l_\tau$ where fluid and inertial particles behave similarly. Note that $\mathrm{St}^{1/6} \gg 1$ at $\tau \ll t_L$ demands a very large inertial interval so the limiting case solution given by Eq. (9) is mainly a theoretical device to understand the generic features of the separation.

In summary, we studied the motion of inertial particles in turbulence at $\mathrm{St} \gg 1$. We showed that the particle drift velocity grows with inertia as the Lagrangian velocity increment of turbulence at time τ. The Lyapunov exponent $\lambda_1 \sim \lambda_1^{\mathrm{turb}}/\mathrm{St}^{5/6}$ can be estimated from the Kraichnan model with τ-dependent statistics. The analogue of Richardson's law in the inertial range $\eta \lesssim R \ll l_\tau$ is $R(t) \sim l_\tau(t/\tau)^9$. The law, observable only at $\mathrm{St}^{1/6} \gg 1$, shows property expected at any $\mathrm{St} \gg 1$: explosive separation to l_τ in $t \sim \tau$, closer than Richardson's law to the exponential separation holding for the smooth flows. The analysis can be generalized to include gravitational acceleration $\boldsymbol{g}$ where one finds $\lambda_1 = \lambda_1(w = g\tau)$ [18,19]. The implications of the results on separation for the two-particle distribution function (determining the preferential concentration) are an interesting subject for future work. Our main qualitative result is that collective drift of inertial particles through the flow makes their relative motion subject to Langevin description.

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