Characterizing the Structure of Preserved Information in Quantum Processes

Robin Blume-Kohout,[1,*] Hui Khoon Ng,[1] David Poulin,[2] and Lorenza Viola[3]

[1]*Institute for Quantum Information, Caltech, Pasadena, California 91125, USA*
[2]*Center for the Physics of Information, Caltech, Pasadena, California 91125, USA*
[3]*Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, New Hampshire 03755, USA*

(Received 29 May 2007; published 22 January 2008)

We introduce a general operational characterization of information-preserving structures—encompassing noiseless subsystems, decoherence-free subspaces, pointer bases, and error-correcting codes—by demonstrating that they are isometric to fixed points of unital quantum processes. Using this, we show that every information-preserving structure is a matrix algebra. We further establish a structure theorem for the fixed states and observables of an arbitrary process, which unifies the Schrödinger and Heisenberg pictures, places restrictions on physically allowed kinds of information, and provides an efficient algorithm for finding all noiseless and unitarily noiseless subsystems of the process.

DOI: 10.1103/PhysRevLett.100.030501 PACS numbers: 03.67.Pp, 03.65.Yz, 03.67.Lx, 89.70.−a

Quantum processes, also known as quantum channels, quantum operations, or completely positive (CP) maps [1,2], are central to the theory and practice of quantum information processing (QIP). They describe how quantum states evolve over a period of time in the presence of noise, or how a device's output depends on its input. They are also complex and unwieldy: to fully specify a quantum process on a d-dimensional system requires d^4 real numbers. Most of these data are irrelevant to what one really wants to know: What information can pass unharmed through the process? Besides being central to QIP, a general answer is broadly relevant to both fundamental physics and quantum technologies, for the information-preserving degrees of freedom are precisely those that may be reliably characterized and exploited. *Information-preserving structures* (IPS) in quantum processes—what they are and how to find them—are the subject of this Letter.

The quest for such structures has a long history in quantum physics. Pointer states (PS), defined in the context of quantum measurement theory, are "most classical" states that resist decoherence [3]. QIP science has spurred interest in the preservation of quantum information, leading to the notion of noiseless subsystems (NS) [4] as passive IPS that emerge from the existence of symmetries in the noise, and recover both decoherence-free subspaces (DFS) [5] and PS in special limits. Processes admitting no NS may still preserve information, which can be actively protected using quantum error correction (QEC) [6,7] to create an effective NS. Rapid experimental progress in implementing DFS [8], NS [9], and QEC [10] heightens the need for a complete and constructive characterization of preserved information.

In this Letter, we formulate a general operational theory of IPS. The key insight is to identify preserved information with sets of states (or codes) whose *mutual distinguishability* is left unchanged. We prove that every preserved code can, through error correction, be made noiseless, then show that every maximal noiseless code is isometric [11] to the fixed-point set of the dynamics. This set, in turn, is isometric to a matrix algebra; thus, we conclude that every IPS is an algebra. Finally, we provide an explicit structure for the fixed points of an arbitrary process and an efficient algorithm to determine its noiseless and unitarily noiseless IPS. Our results fill several gaps in existing work. Starting from basic operational definitions, our approach encompasses everything that could represent information perfectly preserved by a quantum process, and shows an explicit connection to fixed points. Our structure theorem is general, whereas previous results applied only to unital [12,13] maps, or ones with a full-rank fixed state [14]. While information preservation has been addressed in both the Schrödinger and Heisenberg [15] picture, we consistently unify them. Available algorithms to find IPS are either inefficient (e.g., Zurek's "predictability sieve" for PS [16] or Choi and Kribs's method for NS [17]) or restricted to purely noiseless information [18] or unital channels [19]. By explicitly shifting focus from the noise commutant to the fixed-point set (recent work, e.g., [15], has also moved in this direction), our approach paves the way to analyzing "approximate" IPS, beyond existing results on the stability of DFS/NS under symmetry-breaking perturbations [20].

Quantum states and processes.—We consider an open quantum system with a (finite) d-dimensional Hilbert space $\mathcal{H}$. Its state is described by a non-negative, trace-1, $d \times d$ density matrix ρ, which is also a vector in the system's Hilbert-Schmidt space $\mathcal{B}(\mathcal{H})$ (the space of bounded operators on $\mathcal{H}$). The system's dynamical evolution over a time t is described by a quantum process $\mathcal{E}:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. $\mathcal{E}$ is linear, trace preserving (TP), and CP, which ensures that $\mathcal{E}$ does not produce negative probabilities when acting on arbitrary states. $\mathcal{E}$ is CP if and only if $\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$ for some set of Kraus operators $\{K_i\}$ and TP if and only if $\sum_i K_i^\dagger K_i = \mathbb{1}$. $\mathcal{E}$ is unital if and only if, in addition, $\mathcal{E}(\mathbb{1}) = \sum_i K_i K_i^\dagger = \mathbb{1}$ (see [1,2] for further details).

Preserved information and distinguishability.—To encode information, we prepare the system in a state ρ, chosen from a convex set $\mathcal{C}$ of possible states, the *code*. The code defines the kind of information encoded. In particular, our definition includes all the familiar examples: e.g., a QEC code contains all the states in a subspace $\mathcal{P} \subseteq \mathcal{H}$; a classical code comprises a discrete set of orthogonal states. Many other kinds of codes are possible, and our first goal is to classify them.

To access the information, we must distinguish between states $\rho, \rho' \in \mathcal{C}$. If we assign prior probabilities $\{q, 1-q\}$ to ρ and ρ', and make the optimal measurement to distinguish them, we guess correctly with probability $p = \frac{1}{2} \times (1 + \|q\rho - (1-q)\rho'\|_1)$ (see Ref. [21]). Clearly, if $\mathcal{E}$ makes the states in $\mathcal{C}$ less distinguishable, then information was not perfectly preserved. We therefore propose the following operational criterion: A code $\mathcal{C}$ is *preserved* by a process $\mathcal{E}$ if and only if each pair of states $\rho, \rho' \in \mathcal{C}$ is just as distinguishable after $\mathcal{E}$ as before it. More technically, $\mathcal{C}$ is preserved by $\mathcal{E}$ if and only if, for every $\rho, \rho' \in \mathcal{C}$ and $x \in \mathbb{R}_+$, $\|\mathcal{E}(\rho - x\rho')\|_1 = \|\rho - x\rho'\|_1$.

Such a code can be extended by (real) linear combination, so we can think of a preserved code $\mathcal{C}$ as comprising all the states in an operator subspace of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{C}$ is preserved if it is isometric to $\mathcal{E}(\mathcal{C})$; that is, if $\mathcal{E}$ acts as a 1:1 trace-distance-preserving map on $\mathcal{C}$. Several other operational notions of "preserved" will be relevant:

(1) $\mathcal{C}$ is *noiseless* for $\mathcal{E}$ if and only if it is preserved by any convex mixture $\sum_n p_n \mathcal{E}^n$, with $p_n \geq 0$ and $\sum_n p_n = 1$;

(2) $\mathcal{C}$ is *unitarily noiseless* [22] for $\mathcal{E}$ if and only if it is preserved by $\mathcal{E}^n$, for every fixed $n \in \mathbb{N}$;

(3) $\mathcal{C}$ is *correctable* for $\mathcal{E}$ if and only if there exists a correction process $\mathcal{R}$ such that $\mathcal{C}$ is noiseless for $\mathcal{R} \circ \mathcal{E}$.

Both noiseless and unitarily noiseless codes preserve information forever, without intervention, but in unitarily noiseless codes, the preserved information can "move around." The optimal measurement to distinguish two states in a noiseless code is independent of the number n of applications of $\mathcal{E}$ (and can be derived from $\mathcal{E}_\infty$; see proof of Theorem 2). In contrast, for a unitarily noiseless code [e.g., a system evolving unitarily, as $\mathcal{E}(\rho) = U\rho U^\dagger$] this measurement may depend on n, so we must keep track of how many times $\mathcal{E}$ has occurred. Correctable codes are not inherently stable, but they can be stabilized indefinitely by applying $\mathcal{R}$. We can collapse the lowest levels of this hierarchy:

Theorem 1 *A code $\mathcal{C}$ is preserved by the process $\mathcal{E}$ if and only if it is correctable for $\mathcal{E}$.*

While we defer a full proof to Ref. [23], the central idea is simple: If $\mathcal{C}$ is preserved, then we can correct it with the *transpose channel* [24],

$$\hat{\mathcal{E}}_{\mathcal{P}}(\rho) = \sum_i (PK_i^\dagger P \mathcal{E}(P)^{-1/2}) \rho (\mathcal{E}(P)^{-1/2} P K_i P),$$

where $\mathcal{P}$ is the joint support of every $\rho \in \mathcal{C}$, and P projects onto it. Notice that $\hat{\mathcal{E}}_{\mathcal{P}} \circ \mathcal{E}(P) = \mathcal{E}^\dagger(\mathcal{E}(P)^{-1/2} \times \mathcal{E}(P)\mathcal{E}(P)^{-1/2}) = P$; thus, the corrected map is not only TP but also unital on the code's support.

Because a correctable code for $\mathcal{E}$ is a noiseless code for some other channel $\mathcal{R} \circ \mathcal{E}$, we can characterize all preserved codes by characterizing noiseless codes. The first step is to relate $\mathcal{E}$'s noiseless codes to its fixed points:

Theorem 2 *If $\mathcal{C}$ is a noiseless code for $\mathcal{E}$, then $\mathcal{C}$ is isometric to a subset of the fixed states of $\mathcal{E}$.*

Proof: $\mathcal{C}$ is preserved by any channel of the form $\sum_n p_n \mathcal{E}^n$ ($\sum p_n = 1$), including $\mathcal{E}_N = \frac{1}{N+1}\sum_{n=0}^N \mathcal{E}^n$, and therefore also by $\mathcal{E}_\infty = \lim_{N\to\infty}\mathcal{E}_N$ [25] (the limit is well defined for finite-dimensional $\mathcal{H}$). Thus, $\mathcal{C}$ is isometric to $\mathcal{E}_\infty(\mathcal{C})$. But $\mathcal{E} \circ \mathcal{E}_\infty = \mathcal{E}_\infty$, so if $\sigma = \mathcal{E}_\infty(\rho)$, then $\mathcal{E}(\sigma) = \sigma$. Therefore, $\mathcal{E}_\infty$ projects onto the fixed points of $\mathcal{E}$, so $\mathcal{E}_\infty(\mathcal{C})$ is a subset of $\mathcal{E}$'s fixed states. □

Theorem 2 has important consequences for maximal codes—ones that encode as many states as possible. First, every maximal noiseless code for $\mathcal{E}$ is isometric to the set of all fixed states of $\mathcal{E}$. The fixed states are themselves a noiseless code $\mathcal{C}_0$, so if $\mathcal{C}$ is not isometric to $\mathcal{C}_0$, then it is isometric to a proper subset, and cannot be maximal. Next, every maximal preserved code for $\mathcal{E}$ is isometric to the set of all fixed states of a unital, TP map. This follows from Theorem 1. If $\mathcal{C}$ is preserved, $\hat{\mathcal{E}}_{\mathcal{P}}$ corrects it, so $\mathcal{C}$ is noiseless for $\hat{\mathcal{E}}_{\mathcal{P}} \circ \mathcal{E}$ and (by Theorem 2) isometric to its fixed points. Optimal preserved codes come in equivalence classes characterized by a fixed geometry (as defined by the pairwise distances between elements): $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if and only if they are isometric. Equivalent codes use different states to encode the same information—they are manifestations of the same IPS:

Definition 1 *An IPS of a process $\mathcal{E}$ is the geometric structure common to an equivalence class of maximal preserved codes.*

A maximal preserved code is isometric to the fixed-point set of $\hat{\mathcal{E}}_{\mathcal{P}} \circ \mathcal{E}$. Because this set (and its geometry) depend on $\mathcal{P}$, $\mathcal{E}$ may have several distinct IPS. However, all its maximal *noiseless* codes belong to a single class as they all share the geometry of $\mathcal{E}$'s fixed-point set. They are manifestations of a unique noiseless IPS:

Definition 2 *The noiseless IPS of a process $\mathcal{E}$ is the unique geometric structure common to all of its maximal noiseless codes.*

Structure of codes.—The next step toward characterizing the possible IPS is to determine the structure of fixed states for arbitrary $\mathcal{E}$. Because $\mathcal{E}$ is linear, its fixed points are closed under linear combination, hence form an operator subspace of $\mathcal{B}(\mathcal{H})$. For the special case where $\mathcal{E}$ is unital, several authors have shown [12,13] that (a) the fixed points of $\mathcal{E}$ form a complex matrix algebra $\mathcal{A}$, (b) $\mathcal{A}$ is the commutant of $\mathcal{E}$' s Kraus operators, and (c) $\mathcal{E}$ and $\mathcal{E}^\dagger$ have the same fixed points.

This is a powerful result because finite-dimensional matrix algebras share an elegant structure: Every such matrix algebra is a direct sum of the form

$$\mathcal{A} = \bigoplus_k \mathcal{M}_{d_k} \otimes \mathbb{1}_{n_k}, \qquad n_k, d_k \in \mathbb{N}, \tag{1}$$

where $\mathcal{M}_{d_k}$ is the algebra of all $d_k \times d_k$ matrices, and $\mathbb{1}_{n_k}$ is the trivial algebra containing the n_k-dimensional identity [26]. Thanks to this result, we have all the ingredients to describe the structure of preserved information for an arbitrary (not necessarily unital) $\mathcal{E}$: *Every* maximal preserved code is isometric to a matrix algebra. This follows from Theorem 1 (preserved codes are correctable, with $\mathcal{R} \circ \mathcal{E}$ unital) and Theorem 2 (maximal noiseless codes are isometric to fixed-point sets), together with the structure theorem cited above. We conclude that *any* IPS of a process on a d-dimensional system is a subalgebra of $\mathcal{M}_d$.

Fixed points of arbitrary maps.—While the above IPS characterization is fully general, it is nonconstructive as long as the projector P required to construct the transpose map is unknown. However, on one hand noiseless codes are isometric to the fixed states of $\mathcal{E}$ itself (rather than $\hat{\mathcal{E}}_{\mathcal{P}} \circ \mathcal{E}$); on the other hand, the set of all fixed states is a maximal noiseless code, whose unique isometric algebra $\mathcal{A}$ fully specifies $\mathcal{E}$'s noiseless IPS. To obtain a constructive characterization of this IPS, we need (1) a general description of the fixed states of $\mathcal{E}$ and (2) a way to extract the algebra to which they are isometric. Unfortunately, the structure theorem for unital maps does not extend to arbitrary processes. The following example violates every point listed earlier: $\mathcal{E}$ and $\mathcal{E}^\dagger$ have different fixed-point sets, which do not form algebras, and do not commute with the Kraus operators.

Example Let A be a qutrit and B a qubit, and $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_B$ be a process on $\mathcal{H}_A \otimes \mathcal{H}_B$, with Kraus operators

$$\mathcal{E} \sim \left\{|0\rangle\langle 0| + |1\rangle\langle 1|, \frac{1}{\sqrt{2}}|0\rangle\langle 2|, \frac{1}{\sqrt{2}}|1\rangle\langle 2|\right\}_A \otimes \left\{\frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|0\rangle\langle 1|, \frac{\sqrt{3}}{2}|1\rangle\langle 0|, \frac{\sqrt{3}}{2}|1\rangle\langle 1|\right\}_B.$$

$\mathcal{E}$ does nothing to the $\{|0\rangle, |1\rangle\}$ subspace of A, but maps $|2\rangle_A$ into an equal mixture of $|0\rangle\langle 0|_A$ and $|1\rangle\langle 1|_A$. At the same time, it forces B into $\tau_B = \frac{1}{4}|0\rangle\langle 0|_B + \frac{3}{4}|1\rangle\langle 1|_B$. $\mathcal{E}$'s fixed states are $\sigma_A \otimes \tau_B$ (for any 2×2 matrix σ_A), and the fixed observables of $\mathcal{E}^\dagger$ are $(\sigma_A + \frac{1}{2}\mathrm{Tr}(\sigma_A)|2\rangle\langle 2|_A) \otimes \mathbb{1}_B$. The commutant of the Kraus operators is nothing but $\mathbb{1}$.

Still, we can characterize fixed states and observables:

Theorem 3 *Let $\mathcal{E}$ be a quantum process on $\mathcal{B}(\mathcal{H})$, Σ the fixed points of $\mathcal{E}$, and $\mathcal{B}$ the fixed points of $\mathcal{E}^\dagger$. Then,*

(i) *Σ and $\mathcal{B}$ are each isometric to a matrix algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{P})$, where $\mathcal{P}$ is a subspace of $\mathcal{H}$;*

(ii) *Σ is supported on $\mathcal{P}$, and contains all operators $\sigma = \bigoplus_k M_{d_k} \otimes \tau_{n_k}$, where M_{d_k} is an arbitrary $d_k \times d_k$ operator, and τ_{n_k} is a fixed $n_k \times n_k$ state;*

(iii) *$\mathcal{B}$ contains all operators of the form $X = A_{\mathcal{P}} \oplus \mathcal{F}_{\mathcal{P}\to\overline{\mathcal{P}}}(A_{\mathcal{P}})$, where $A_{\mathcal{P}} \in \mathcal{A}$, $\overline{\mathcal{P}}$ is the complement of $\mathcal{P}$ in $\mathcal{H}$, and $\mathcal{F}_{\mathcal{P}\to\overline{\mathcal{P}}}$ is a fixed linear map from $\mathcal{B}(\mathcal{P})$ to $\mathcal{B}(\overline{\mathcal{P}})$;*

(iv) *Projecting $\mathcal{B}$ onto the support $\mathcal{P}$ of Σ yields a representation of $\mathcal{A}$.*

The proof is deferred to [23]. The central result—that the fixed states are isometric to a matrix algebra—is already implied by the fact they form a preserved code. Notice that if $\mathcal{E}$ is unital, Σ *coincides* with $\mathcal{A}$—the non-negative, trace-1 operators in Σ directly determine the process's maximal noiseless codes, hence its noiseless IPS.

Familiar examples of noiseless IPS correspond to specific ways in which information is encoded in one or more blocks of $\mathcal{M}_d$ via Eq. (1). The simplest IPS corresponds to encoding purely classical information by a choice among multiple blocks. For a pointer basis, in particular, all blocks are one dimensional. Quantum information is preserved within a single higher-dimensional block. A DFS is represented by a single block with a trivial cofactor, and a NS by a single block tensored with an identity ("noise-full") subsystem. The most general IPS, a hybrid quantum memory [27], has n blocks of (possibly) different sizes d_k. It can be concisely described by its *shape*, the vector $\{d_1, d_2, \ldots, d_n\}$.

In each of the examples above, $\mathcal{E}$ must (by Theorem 2) have a set of fixed points. For a pointer basis, the projectors onto each PS are fixed. For a DFS, every state on the subspace is fixed. The fixed points associated with a NS are less obvious. If $\mathcal{E}$ has a NS, $\mathcal{H}$ may be decomposed as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \oplus \mathcal{H}_C$, and for all ρ_A and ρ_B, $\mathcal{E}(\rho_A \otimes \rho_B) = \rho_A \otimes \sigma_B$ [28]. That is, $\mathcal{E}$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ as $\mathcal{E} = \mathbb{1}_A \otimes \mathcal{E}_B$, and by Schauder's fixed-point theorem [29], $\mathcal{E}_B$ must have a fixed point τ_B. Thus, for any ρ_A, $\rho_A \otimes \tau_B$ is in Σ. Note how, for each σ_B, there is a distinct noiseless code $\mathcal{C}_\sigma = \{\rho_A \otimes \sigma_B \ \forall\ \rho_A\}$, which is isometric to the unique fixed code $\mathcal{C} = \{\rho_A \otimes \tau_B \ \forall\ \rho_A\}$.

In general, the explicit form of the fixed states given in Theorem 3(ii) illustrates what it means to be "isometric to a matrix algebra": The "noise-full" subsystems are represented, not by $\mathbb{1}_{n_k}$, but by a fixed state τ_{n_k}. Fixed observables have a different structure, also derived from that of $\mathcal{A}$. Their restriction to $\mathcal{P}$ coincides with $\mathcal{A}$, but each has an "echo" of itself on $\overline{\mathcal{P}}$. $\mathcal{E}^\dagger$ extends observables on $\mathcal{P}$ to $\overline{\mathcal{P}}$, so that they detect states initially outside of $\mathcal{P}$. This is the Heisenberg-picture manifestation of the fact that $\mathcal{E}$ maps states on $\overline{\mathcal{P}}$ to $\mathcal{P}$.

Finding the noiseless IPS.—By construction, $\mathcal{E}$'s noiseless IPS contains all of $\mathcal{E}$'s NS. To find this IPS:

(1) Write $\mathcal{E}$ as a $d^2 \times d^2$ matrix.

(2) Diagonalize it, and extract the $\lambda = 1$ right and left eigenspaces (Σ and $\mathcal{B}$, respectively).

(3) Compute $\mathcal{P}$, the joint support of all $\rho \in \Sigma$, and project $\mathcal{B}$ onto $\mathcal{P}$ to obtain a basis for $\mathcal{A}$.

(4) Find the shape of $\mathcal{A}$, using (for example) tools in [30].

Our algorithm runs in time $O(d^6)$ [matrix diagonalization is $O((d^2)^3)$] and uses standard numerical tools. As such, it is more efficient than algorithms (e.g., [16,17]) that require exhaustive search over states or subspaces in $\mathcal{H}$—for these sets grow exponentially in volume with d.

The above algorithm may be easily generalized to unitarily noiseless IPS, provided that we shift our focus from $\mathcal{E}$'s fixed points to its rotating points, defined as follows: The rotating points of $\mathcal{E}$ comprise the span of its unit-modulus eigenoperators. We then have

Theorem 4 *Every maximal unitarily noiseless code for* $\mathcal{E}$ *is isometric to the [positive trace-1 states in the] rotating points of* $\mathcal{E}$.

The key observation for the proof (deferred to [23]) is that there exist high powers of $\mathcal{E}$ that project onto its rotating points. Thus, $\mathcal{E}$ has a unique unitarily noiseless IPS, which can be found using the algorithm above provided that "the $\lambda = 1$ eigenspace" is replaced with "the span of all the unit-modulus ($\lambda = e^{i\phi}$) eigenoperators."

Discussion: Our IPS framework can be used in multiple ways. An experimentalist who has characterized a system using quantum process tomography can apply our algorithm to find noiseless and unitarily NS, then use the IPS shape as a concise language to report the results. On a theoretical front, we have classified all maximal preserved codes. This rules out certain kinds of information, as unphysical; e.g., no process acting on a single qubit can perfectly preserve only $\mathbb{1}$, σ_x, and σ_y (a "rebit").

Physically, the IPS shape distills the invariant properties of a process (what kind of information is preserved), discarding the details (which states are preserved) that are needed to design quantum hardware, but not to understand what it can do. It is closely related to $\mathcal{E}$'s eigenvalues, but is both more concise and more informative [31]. One might hope to generalize our algorithm to find *all* correctable codes, not just noiseless ones. However, a constructive algorithm seems difficult, and finding the best codes for even a classical process is NP-hard. Thus, while we now know what every code must look like, finding one may be intractable.

We thank H. Barnum, P. Hayden, M.-B. Ruskai, R. Spekkens, J. Yard, P. Zanardi, and W. Zurek for helpful discussions. This work was supported in part by the Gordon and Betty Moore Foundation, by the NSF under Grants No. PHY-0456720 and No. PHY-0555417, and by the NSERC.

*Current address: Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2L 2Y5.

[1] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[3] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[4] E. Knill *et al.*, Phys. Rev. Lett. **84**, 2525 (2000); P. Zanardi, Phys. Rev. A **63**, 012301 (2000); J. Kempe *et al.*, *ibid.* **63**, 042307 (2001).
[5] G. M. Palma *et al.*, Proc. R. Soc. A **452**, 567 (1996); P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997); L.-M. Duan and G.-C. Guo, *ibid.* **79**, 1953 (1997); D. A. Lidar *et al.*, *ibid.* **81**, 2594 (1998).
[6] P. W. Shor, Phys. Rev. A **52**, R2493 (1995); A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996); C. H. Bennett *et al.*, Phys. Rev. A **54**, 3824 (1996); E. Knill and R. Laflamme, *ibid.* **55**, 900 (1997).
[7] D. Kribs *et al.*, Phys. Rev. Lett. **94**, 180501 (2005).
[8] P. G. Kwiat *et al.*, Science **290**, 498 (2000); D. Kielpinski *et al.*, *ibid.* **291**, 1013 (2001); E. M. Fortunato *et al.*, New J. Phys. **4**, 5 (2002); M. Carravetta *et al.*, Phys. Rev. Lett. **92**, 153003 (2004).
[9] L. Viola *et al.*, Science **293**, 2059 (2001).
[10] D. G. Cory *et al.*, Phys. Rev. Lett. **81**, 2152 (1998); E. Knill *et al.*, *ibid.* **86**, 5811 (2001); N. Boulant *et al.*, *ibid.* **94**, 130501 (2005); J. Chiaverini *et al.*, Nature (London) **432**, 602 (2004).
[11] An isometry is a 1:1 distance-preserving map; the relevant distance here is the 1-norm, $\|A\|_1 = \mathrm{tr}|A|$.
[12] A. Arias *et al.*, J. Math. Phys. (N.Y.) **43**, 5872 (2002).
[13] D. W. Kribs, Proc. Edinb. Math. Soc. **46**, 421 (2003).
[14] A. Frigerio, Lett. Math. Phys. **2**, 79 (1977); Commun. Math. Phys. **63**, 269 (1978).
[15] C. Beny *et al.*, Phys. Rev. Lett. **98**, 100502 (2007).
[16] W. H. Zurek, Prog. Theor. Phys. **89**, 281 (1993).
[17] M.-D. Choi and D. W. Kribs, Phys. Rev. Lett. **96**, 050501 (2006).
[18] E. Knill, Phys. Rev. A **74**, 042301 (2006).
[19] D. W. Kribs and R. W. Spekkens, Phys. Rev. A **74**, 042329 (2006).
[20] D. Bacon *et al.*, Phys. Rev. A **60**, 1944 (1999).
[21] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, London, 1976).
[22] Our "unitarily noiseless" differs from that in D. W. Kribs *et al.*, Quantum Inf. Comput. **6**, 382 (2006) (better described as "unitarily correctable," as in [19]).
[23] R. Blume-Kohout *et al.* (to be published).
[24] H. Barnum and E. Knill, J. Math. Phys. (N.Y.) **43**, 2097 (2002).
[25] G. Lindblad, Lett. Math. Phys. **47**, 189 (1999).
[26] K. Davidson, C*-*Algebras by Example, Fields Institute Monographs* (American Mathematical Society, Providence, RI, 1996).
[27] G. Kuperberg, IEEE Trans. Inf. Theory **49**, 1465 (2003).
[28] "Initialization-free" DFS/NS, introduced in A. Shabani and D. A. Lidar, Phys. Rev. A **72**, 042303 (2005), can also be described by fixed points; see [23] for discussion.
[29] A. Granas and J. Dugundji, *Fixed Point Theory* (Springer, New York, 2003).
[30] J. Holbrook *et al.*, Quant. Info. Proc. **2**, 381 (2004).
[31] For example, processes that preserve a 1-qubit DFS and a 4-state pointer basis have identical eigenvalues ($\{1, 1, 1, 1, x_1, x_2, \ldots\}$, $|x_j| < 1$) but different IPS shape (respectively, $\{2\}$ and $\{1, 1, 1, 1\}$).