Testable Signatures of Quantum Nonlocality
in a Two-Dimensional Chiral *p*-Wave Superconductor

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A class of topological excitations—the odd-winding number vortices—in a spinless 2D chiral *p*-wave ($p_x + ip_y$) superconductor traps Majorana fermion states in the vortex cores. For a dilute gas of such vortices, the lowest energy fermionic eigenstates are intrinsically nonlocal. We predict two testable signatures of this unusual quantum nonlocality in quasiparticle tunneling experiments. We discuss why the associated teleportationlike phenomenon does not imply the violation of causality.

Introduction.—The intriguing physics of the Majorana fermion zero modes at the vortex cores of a spinless $p_x + ip_y$-wave superconductor or superfluid has recently attracted considerable attention [1–8]. This is because these zero modes endow the vortices with non-Abelian braiding statistics, and the braiding of such vortices [9] can, in principle, be exploited to build a fault-tolerant topological quantum computer [10]. Recently there has been considerable evidence [11,12] that the symmetry of the superconducting order parameter in strontium ruthenate (Sr_2RuO_4) [13,14] is spin-triplet $p_x + ip_y$. It has been proposed in Ref. [5] that when a thin film of this material is subjected to a magnetic field exceeding ~ 200 G, the most energetically favorable vortices enclose flux $hc/4e$ rather than $hc/2e$, and at the core of such a half-vortex there is a Majorana zero mode. For other proposals for the superconducting state in this material, see Refs. [15,16]. Some heavy fermion materials, such as UPt_3, are likely to be spin-triplet superconductors as well. Moreover, with the recent observations of *p*-wave Feshbach resonances in spin-polarized ^{40}K and ^{6}Li atoms in optical traps [17,18], these systems are promising candidates for realizing *p*-wave condensates as well [19,20]. Thus, vortices satisfying non-Abelian statistics are tantalizingly close to experimental reach. In the following, for brevity, we will use the term "superconductor" to indicate both superconductor and superfluid.

An inevitable consequence of the existence of Majorana zero modes is that the fermion quasiparticle excitations are inherently nonlocal. The zero energy bound state at the vortex core is described by the Majorana fermion operator $\gamma^\dagger = \gamma$ [1]. Other bound states are separated by an energy gap $\omega_0 \sim \frac{\Delta_0^2}{\epsilon_F} \ll \Delta_0$, where Δ_0 gives the gap in the bulk and ϵ_F is the Fermi energy. If the system has two Majorana zero modes, γ_1 and γ_2, one can define a composite operator $q^\dagger = (\gamma_1 + i\gamma_2)/\sqrt{2}$, and its Hermitian conjugate q, which satisfy Fermi anticommutation relations. If an elec-

tron is injected into the system with an energy $\ll \omega_0$, it can only go to the excited state described by q and $q^\dagger$. This intrinsic nonlocality of the quasiparticle wave function was first discussed in Refs. [21,22] for a 1D quantum wire embedded in a *p*-wave superconductor. In this Letter, we use it to make interesting predictions for quasiparticle tunneling experiments in a 2D $p_x + ip_y$ superconductor.

To distinguish the nonlocality discussed above from that in a quantum double well problem, let $|l\rangle$ and $|r\rangle$ represent the two states localized in the left and the right well. In the presence of quantum tunneling, $-t(c_l^\dagger c_r + \text{H.c.})$, the ground and the excited states are given by the symmetric and the antisymmetric states, $|s\rangle = \frac{1}{\sqrt{2}}(c_l^\dagger + c_r^\dagger)|0\rangle$ and $|a\rangle = \frac{1}{\sqrt{2}}(c_l^\dagger - c_r^\dagger)|0\rangle$, respectively. Thus, it might appear that in this case also the quasiparticle operators are delocalized between the two wells even when they are very far separated. However, if we inject an electron into the left well, it will enter into the linear combination of $|s\rangle$ and $|a\rangle$, the state localized in the left well [21,22]. Only after a time of the order of $\hbar$ divided by the tunnel splitting does the electron acquire appreciable probability of appearing at the right well. Further, because of the exponentially small
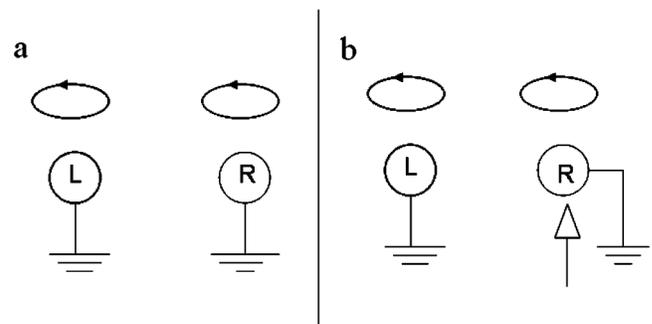

FIG. 1.   (a) Two quantum dots in tunneling contact with two vortices. (b) A quantum dot and a STM tip in tunneling contact with two vortices.

tunnel splitting, by adding a small perturbation, e.g., $\epsilon_L c_L^\dagger c_L$, to the Hamiltonian, the eigenwave function will become localized near the left or the right well. Motivated by these considerations we consider the following thought experiments to test the explicit nonlocality of a quasiparticle excitation in the Majorana fermion system.

Teleportation.—We consider the setup of Fig. 1(a) where two voltage-biased quantum dots are in tunneling contact with two non-Abelian vortices of a $p_x + ip_y$ superconductor. The effective Hamiltonian describing the system at energy $\ll \omega_0$ is given by

$$H = \sum_{\alpha=L,R} \epsilon_\alpha \psi_\alpha^\dagger \psi_\alpha + iJ\gamma_L\gamma_R - \sqrt{2}t_L\gamma_L(\psi_L^\dagger - \psi_L)$$
$$- \sqrt{2}t_R\gamma_R(\psi_R^\dagger - \psi_R). \tag{1}$$

Here, γ_L and γ_R are the two Majorana fermion operators and ψ_L and ψ_R are the electron annihilation operators for the lowest energy state of the quantum dots. The parameter J describes the hybridization between the Majorana fermion modes. It is exponentially small if the distance between the vortices is large.

After introducing fermion operators $q^\dagger = (\gamma_R + i\gamma_L)/\sqrt{2}$, $q = (\gamma_R - i\gamma_L)/\sqrt{2}$, and performing the local gauge transformation $\psi_L \to i\psi_L$, Eq. (1) becomes [23]

$$H = \sum_{\alpha=L,R} \epsilon_\alpha \psi_\alpha^\dagger \psi_\alpha + Jq^\dagger q + \sum_{\alpha=L,R} t_\alpha(q^\dagger\psi_\alpha + \text{H.c.})$$
$$- t_L(\psi_L^\dagger q^\dagger + \text{H.c.}) + t_R(\psi_R^\dagger q^\dagger + \text{H.c.}). \tag{2}$$

In this form, it is clear that the tunneling terms only conserve charge modulo $2e$. (The extra or deficit charge is taken away or provided by the superconducting condensate.) The lowest energy Hilbert space upon which Eq. (2) acts is eight dimensional. Its basis states are labeled by $|n_L, n_q, n_R\rangle$, where $n_{L,q,R} = 0, 1$ denote the number of particles in the single particle states associated with the ψ_L, q, and ψ_R operators, respectively. Because of charge conservation modulo $2e$, this eight dimensional Hilbert space decouples into two four-dimensional ones spanned by $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle, |1,1,1\rangle\}$ and $\{|0,1,1\rangle, |1,0,1\rangle, |1,1,0\rangle, |0,0,0\rangle\}$, respectively. In the demonstration of the teleportation phenomenon, we shall consider one extra electron at the left dot at time zero and see how it is transported to the right dot at a later time. For that purpose, we only need to consider the subspace where the number of electrons is odd. In this subspace the Hamiltonian is a 4×4 matrix given by

$$H = \begin{pmatrix} \epsilon_L & t_L & 0 & -t_R \\ t_L & J & t_R & 0 \\ 0 & t_R & \epsilon_R & -t_L \\ -t_R & 0 & -t_L & J + \epsilon_L + \epsilon_R \end{pmatrix}. \tag{3}$$

To simplify the calculations, in the following we shall consider the situation where $J \to 0^+$, $t_L = t_R = t$, $\epsilon_R = 0$, and study the eigenstates of Eq. (1) as a function of ϵ_L. Physically, we expect that when ϵ_L is large and negative,

the extra electron initially at the left dot will stay there. On the contrary, by tuning ϵ_L to zero, the extra electron can delocalize by tunneling to the right dot via the conduit provided by the extended q state. Notice that this is very different from ordinary quantum tunneling of the electron which is nonzero only when J is appreciable.

A simple calculation shows that if the initial state is $|1, 0, 0\rangle$, the probability of observing an electron at the left and the right dots at a later time T is given by

$$P_L(T) = |\langle 1,0,0|e^{-iHT}|1,0,0\rangle|^2 + |\langle 1,1,1|e^{-iHT}|1,0,0\rangle|^2$$
$$= 1 - \frac{2t^2}{4t^2 + \epsilon_L^2}\left[1 - \cos(T\sqrt{4t^2 + \epsilon_L^2})\right],$$
$$P_R(T) = |\langle 0,0,1|e^{-iHT}|1,0,0\rangle|^2 + |\langle 1,1,1|e^{-iHT}|1,0,0\rangle|^2$$
$$= \sin^2(Tt). \tag{4}$$

Equation (4) predicts that $P_L \to 1$ for large negative ϵ_L. Therefore, as expected, the electron initially on the left dot will remain there. What is more important is the fact that the probability of finding an electron on the right quantum dot is independent of ϵ_L.

If one separates $P_R(T)$ into the sum of the following probabilities,

$$P_{R1}(T) = |\langle 0,0,1|e^{-iHT}|1,0,0\rangle|^2,$$
$$P_{R2}(T) = |\langle 1,1,1|e^{-iHT}|1,0,0\rangle|^2, \tag{5}$$

one would obtain

$$P_{R1}(T) = \frac{4t^2}{4t^2 + \epsilon_L^2}\sin^2(Tt)\sin^2\left(\frac{T\sqrt{4t^2 + \epsilon_L^2}}{2}\right), \tag{6}$$
$$P_{R2}(T) = \sin^2(Tt) - P_{R1}(T).$$

The first term, $P_{R1}(T)$, is the probability of transporting the electron from the left to the right dot without disturbing the superconductor. For $\epsilon_L = 0$ it is equal to $P_{R1}(T) = \sin^4(Tt)$; hence, it equals unity when $T = \pi/2t$. The fact that the above time is finite even in the limit of $J \to 0^+$ suggests that it is independent of the (large) separation between the vortices, thus justifying the word "teleportation."

Note that the total probability of observing an electron at the right dot, Eq. (4), is independent of what is being done at the left dot. Since there is no way to distinguish between the processes responsible for P_{R1} and P_{R2} by a measurement only on the right dot, there is no way to know if the electron has been teleported from the left dot or has arisen out of the condensate. As a result, no information can be sent from the left to the right dot; hence, causality is maintained. To prove the existence of teleportation, it is necessary to differentiate the processes responsible for P_{R1} and P_{R2}. For example, one could perform a coincident measurement of both the occupation numbers of the left and the right dots at time t. This measurement will give zero for the process responsible for P_{R1} and nonzero for

that responsible for P_{R2}. However, since a "classical" exchange of information (the result of the coincident measurement) is necessary, there is no superluminal transfer of information in the observation of the teleportation effect. This experiment is possible in principle, but will be difficult in practice. In the following, we describe a much easier STM experiment which reveals the nonlocal nature of the q-quasiparticle state discussed earlier.

Action over distance.—Now let us consider the setup in Fig. 1(b) where a voltage-biased quantum dot is in tunneling contact with the left vortex, while an STM tip probes the differential conductance dI/dV of the right dot. The purpose is to study the effect of biasing the left dot on the tunneling curve at the right dot. Again one might expect a nonzero effect due to the extended q state.

In computing the STM spectral function, we use the formulas

$$A(\omega) = \sum_\alpha |\langle \alpha_{-\eta} | \psi_R^\dagger | 0_\eta \rangle|^2 \delta(\omega - E_\alpha^{-\eta} + E_0^\eta) \quad (7)$$

for $\omega > 0$ and

$$A(\omega) = \sum_\alpha |\langle \alpha_{-\eta} | \psi_R | 0_\eta \rangle|^2 \delta(\omega - E_\alpha^{-\eta} + E_0^\eta) \quad (8)$$

for $\omega < 0$. Here $\eta = \pm 1$ labels the even and odd particle number subspaces, respectively, and $|\alpha_\eta\rangle$, E_α^η are the eigenstates and the eigenenergies of Eq. (2) ($\alpha = 0$ indicates the ground state). In the explicit calculation, we diagonalize the following 8×8 Hamiltonian matrix,

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad (9)$$

where H_1 is the 4×4 matrix given in Eq. (3) and H_2 is given by

$$H_2 = \begin{pmatrix} \epsilon_R + J & t_L & 0 & -t_R \\ t_L & \epsilon_L + \epsilon_R & t_R & 0 \\ 0 & t_R & \epsilon_L + J & -t_L \\ -t_R & 0 & -t_L & 0 \end{pmatrix}. \quad (10)$$

In this eight-dimensional space $\psi_R^\dagger$ is represented by

$$\psi_R^\dagger = \begin{pmatrix} 0 & C_{oe} \\ C_{eo} & 0 \end{pmatrix}, \quad (11)$$

where

$$C_{oe} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_{eo} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$
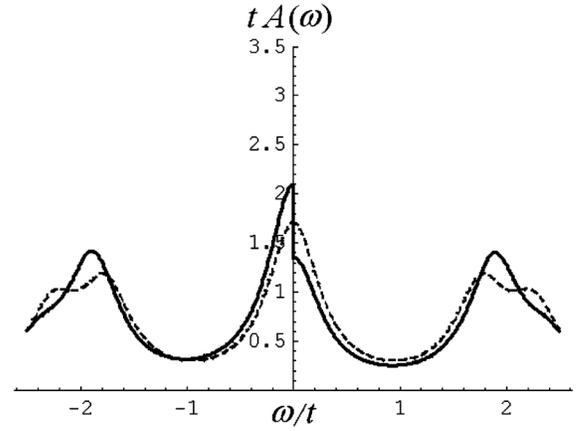

FIG. 2.   The tunneling curves for $J = 0.5t$. Dashed curve, $\epsilon_L = 0$; solid curve, $\epsilon_L = t$.

For $J = 0$, straightforward computation gives

$$A(\omega) = \frac{1}{2\pi} \frac{\Gamma[2t^2 + \Gamma^2 + |\omega|(|\omega| - 2t)]}{(\omega^2 + \Gamma^2)[(|\omega| - 2t)^2 + \Gamma^2]}, \quad (13)$$

where Γ is a Lorentzian broadening of the energy levels. The fact that the result is independent of ϵ_L is expected because, for zero J, there is no communication between the two Majorana fermion states associated with γ_L and γ_R. As a result, Eq. (1) decouples into two independent quantum dots in tunneling contact with two independent vortices. Because of this, the tunneling spectrum of the right dot is independent of the left dot.

Next let us consider the case of nonzero J. In this case, the expression for $A(\omega)$ is lengthy and we shall just present the numerical results for $J = 0.5t$. In Fig. 2 the dashed curve is the tunneling spectrum for $\epsilon_L = 0$ while the solid curve is that for $\epsilon_L = t$. In constructing this figure we used $\Gamma = 0.15t$. It is clear that the tunneling spectrum at the right dot can be qualitatively altered by tuning ϵ_L only on the left dot. The most surprising result is that, when $\epsilon_L \neq 0$, there is a discontinuity in $A(\omega)$ at $\omega = 0$. The presence of zero energy spectral weight is due to the relation

$$E_0^{-\eta} = E_0^\eta. \quad (14)$$

The discontinuity arises from the fact that

$$|\langle 0_{-\eta} | \psi_R^\dagger | 0_\eta \rangle|^2 \neq |\langle 0_{-\eta} | \psi_R | 0_\eta \rangle|^2. \quad (15)$$

Equation (14) is due to the fact that, for $\epsilon_R = 0$, H_1 and H_2 have identical eigenvalues. As for Eq. (15), it can be shown analytically that

$$|\langle 0_{-\eta} | \psi_R^\dagger | 0_\eta \rangle|^2 - |\langle 0_{-\eta} | \psi_R | 0_\eta \rangle|^2 = |\langle 0_\eta | \psi_R | 0_{-\eta} \rangle|^2 - |\langle 0_{-\eta} | \psi_R | 0_\eta \rangle|^2 = \frac{J \epsilon_L}{\sqrt{16t^4 + \epsilon_L^2 (4t^2 + J^2)}}, \quad (16)$$

which yields zero (no discontinuity) if either J or ϵ_L is zero.

A discontinuity in the tunneling spectrum is very unusual. The presence of a nonzero tunneling spectral weight at $\omega = 0$ requires the absence of an energy gap. This occurs, for example, in metals and Anderson insulators. The discontinuity in

the spectral weight at $\omega = 0$ then requires $|\langle 0_{N-1}|\psi_R|0_N\rangle| \neq |\langle 0_N|\psi_R|0_{N+1}\rangle|$, where $|0_M\rangle$ is the M-particle ground state. This would not occur in a metal or an Anderson insulator.

Experiments in atomic superfluid.—The schemes proposed above for testing the nonlocal properties of the Majorana zero modes can be implemented in a $(p_x + ip_y)$-wave atomic superfluid through a recently proposed laser probing scheme [24,25]. This scheme is similar to the tunneling microscopy of a superconductor in that it relies on induced tunneling between a superfluid and a normal phase. Here, we assume that the atoms in the superfluid are prepared in a hyperfine state $|\downarrow\rangle$, while the atoms in the other hyperfine state $|\uparrow\rangle$ do not participate in the p-wave pairing, and therefore constitute the normal phase. The interference between the superfluid and normal phases can be realized through a two-photon Raman process that couples the two states $|\downarrow\rangle$ and $|\uparrow\rangle$ of atoms with two local laser fields. An optical dipole trap with spin-dependent potential depth can be used to hold the normal atoms near the vortex cores. The bias voltage ε_L can be adjusted by varying the intensity of the optical dipole trap, which changes the energy of the normal atoms with respect to atoms in the superfluid. The number of atoms in the normal phase can be measured through fluorescence signals of resonance lasers. The tunneling strengths t_L and t_R can be adjusted by varying the intensities of the Raman laser fields, which changes the coupling strength between the two hyperfine states. In the experiment, to detect the action over distance, the spectral function $A(\omega)$ can be measured by varying the detuning of the laser fields to the atomic transition $\delta = \omega_L - \omega_A$ [24,25], where ω_L is the frequency difference between the two laser fields for the two-photon Raman process, and ω_A is the frequency splitting between the two hyperfine levels $|\downarrow\rangle$ and $|\uparrow\rangle$. In this way, the proposed experiments to test the quantum teleportation and the action over distance can be performed on an atomic superfluid as well. The mathematical illustrations, including the form of the Hamiltonian and the measurable quantities, remain the same as in an electronic superconductor.

Experimental feasibility.—Here we comment on the cutoff energy scales below which the interesting effects discussed above would be observable. The tunneling amplitudes t_L and t_R (the bias voltages) need to be much smaller than ω_0. For Sr_2RuO_4 (assuming it is a spin-triplet p-wave superconductor) using $T_c \sim 1.5$ K and the effective Fermi temperature $T_F \sim 50$ K [13], this requires $t_R, t_L \ll \omega_0 \sim 50$ mK. In the case of Feshbach superfluids, where $\Delta_0 \sim \epsilon_F$ in the unitary regime [26], $\omega_0 \sim \epsilon_F$ and the effects due to the Majorana fermions should be observable below the scale of the Fermi temperature itself. It is also important that the experiment be done on a time scale short compared to the decoherence time. The relevant processes are (1) thermal excitation of a fermion zero mode to a higher-energy state and (2) quantum tunneling of the zero mode from the vortex probed in our experiment to some other vortex induced by disorder. Process (1) will occur

with probability $\sim e^{-\omega_0/T}$. For $T \ll 50$ mK, this will be small in Sr_2RuO_4 (note that disorder may also suppress the excitation gap ω_0 locally, which would require the temperature to be reduced further); in a cold atom system, we merely need $T \ll \epsilon_F$. The second process can be avoided by performing the experiment with vortices which are well isolated from any other vortices (which can be detected by various imaging techniques).

To conclude, the observation of our predicted nonlocal quantum entanglement behavior would be direct evidence supporting the existence of non-Abelian topological anyons in 2D chiral p-wave superfluids and superconductors.

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