Scattering Theory Derivation of a 3D Acoustic Cloaking Shell

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Through acoustic scattering theory we derive the mass density and bulk modulus of a spherical shell that can eliminate scattering from an arbitrary object in the interior of the shell—in other words, a 3D acoustic cloaking shell. Calculations confirm that the pressure and velocity fields are smoothly bent and excluded from the central region as for previously reported electromagnetic cloaking shells. The shell requires an anisotropic mass density with principal axes in the spherical coordinate directions and a radially dependent bulk modulus. The existence of this 3D cloaking shell indicates that such reflectionless solutions may also exist for other wave systems that are not isomorphic with electromagnetics.

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Pendry *et al.* [1] have shown that arbitrary coordinate transformations of Maxwell's equations can be interpreted in terms of an electromagnetic material in the original coordinates with transformed permittivity and permeability values. Consequently, the bending and stretching of electromagnetic fields specified by coordinate transformations can be implemented with electromagnetic materials, enabling unexpected and interesting solutions such as electromagnetic cloaking [1,2].

The degree to which this cloaking concept can be extended to other classes of waves is not known in general. Miller [3] described an active approach for general wave cloaking based on sensing and nonlocal retransmission of signals on the surface of an object. Leonhardt [4] has described how a 2D object can be cloaked in the short wavelength limit in a way that also applies to different classes of waves. Milton *et al.* [5] showed that the coordinate transform approach cannot be extended in general to elastic media. Cummer and Schurig [6] showed, however, an exact analogy exists between 2D electromagnetics and acoustics for anisotropic materials and therefore that a 2D acoustic cloaking shell exists. This isomorphism does not extend to three dimensions, however, which means that a 3D acoustic cloaking shell, if it exists, is different from its electromagnetic counterpart.

Chen *et al.* [7] performed a spherical harmonic scattering analysis of the 3D spherical cloaking shell described in [1] and confirmed that this shell renders any object in its interior free of scattering in all directions. Here we use scattering analysis as the starting point to derive a set of acoustic material parameters for a shell that renders a 3D object in the interior of the shell free of acoustic scattering.

For an inviscid fluid with zero shear modulus, the small perturbation dynamics are described by conservation of momentum and the stress-strain relation. Because [6] showed that mass density anisotropy is required for 2D acoustic cloaking, we assume this anisotropy from the outset. With a $\exp(-i\omega t)$ time dependence, these equations of motion are

$$\nabla p = i\omega \bar{\bar{\rho}}(\bar{r})\rho_0 \mathbf{v}, \tag{1}$$

$$i\omega p = \lambda(\bar{r})\lambda_0 \nabla \cdot \mathbf{v}, \tag{2}$$

where p is scalar pressure, $\mathbf{v}$ is vector fluid velocity, λ is the inhomogeneous fluid bulk modulus relative to λ_0, and $\bar{\bar{\rho}}$ is an inhomogeneous generalized fluid mass density tensor relative to ρ_0. These equations are the fluid version of the more general elastodynamic equations considered in [5]. Although anisotropic mass density is not a property commonly encountered in natural materials, it naturally arises in the analysis of elastodynamics of strongly inhomogeneous composite materials [8]. It is important to note that the anisotropic mass density in (1) is an effective dynamic mass density that is not necessarily tied to the physical mass density of any of the individual elements of a composite material [9,10]. Milton *et al.* [5] describe a simple conceptual model based on mass-loaded springs embedded in a host matrix that is described by an anisotropic mass density. An applied force near the internal resonant frequency of such an inclusion will produce large motions thereby giving the medium a small effective mass density. If the internal springs vary with direction, the effective mass density will also vary with direction and thus be anisotropic. A composite material containing anisotropic mechanically resonant inclusions in a fluid host matrix will obey the scalar stress-strain relation because of the fluid matrix yet will have an anisotropic dynamic response, and therefore an anisotropic dynamic mass density, when forces are applied.

We consider these fields in a domain shown in Fig. 1, in which a uniform fluid of isotropic density ρ_0 and bulk modulus λ_0 is present for $r > b$ and $r < a$, while for $a < r < b$ the material is an inhomogeneous and anisotropic shell to be specified. A uniform plane wave is incident on

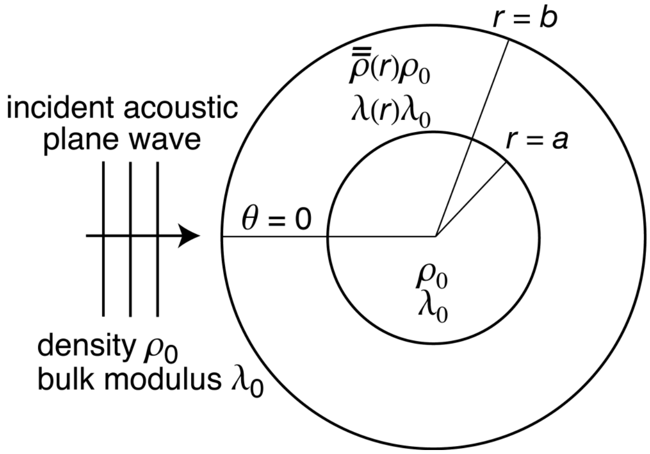

FIG. 1. The problem domain, in which a compressional uniform plane wave propagating along the $\theta = 0$ direction is incident on a material shell with inner radius a and outer radius b.

the shell from the $\theta = 0$ direction, without loss of generality.

Symmetry dictates that if a cloaking shell exists, its scattering parameters must be independent of the incident wave direction. Therefore, its material properties must depend only on radius, and the principal axes of the mass density tensor must be aligned with spherical coordinate directions. The material properties to be determined are $\rho_r(r)$, $\rho_\theta(r)$, $\rho_\phi(r)$, and $\lambda(r)$. Moreover, symmetry implies that $\rho_\theta(r) = \rho_\phi(r)$. These parameters are all assumed to be normalized with respect to the background density ρ_0 and bulk modulus λ_0.

It is straightforward to show that p satisfies the equation

$$\nabla \cdot (\bar{\bar{\rho}}^{-1}\nabla p) + \frac{\omega^2}{\lambda} p = 0. \tag{3}$$

We first focus on the solution to (3) inside the anisotropic shell. In this region and in spherical coordinates, this equation becomes

$$\frac{\omega^2}{v_{p0}^2} r^2 p + \lambda \frac{\partial}{\partial r}\left(\frac{r^2}{\rho_r}\frac{\partial p}{\partial r}\right) + \frac{\lambda}{\rho_\phi \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial p}{\partial \theta}\right) + \frac{\lambda}{\rho_\phi \sin^2\theta}\frac{\partial^2 p}{\partial \phi^2} = 0, \tag{4}$$

where we have used the compressional wave velocity in the background medium $v_{p0} = \sqrt{\rho_0/\lambda_0}$ and $\rho_\theta(r) = \rho_\phi(r)$. The azimuthal mass density elements have been moved outside their derivatives because of their purely radial dependence.

Letting $p(r, \theta, \phi) = f(r)g(\theta)h(\phi)$ through separation of variables and after multiplying all terms by $\lambda^{-1}\rho_\phi \sin\theta$, the ordinary differential equation for $f(r)$ becomes

$$\rho_\phi \frac{\partial}{\partial r}\left(\frac{r^2}{\rho_r}\frac{\partial f}{\partial r}\right) + \left[k_0^2 \frac{\rho_\phi}{\lambda} r^2 - n(n+1)\right] f = 0, \tag{5}$$

where we have used the wave number in the background medium $k_0^2 = \omega^2/v_{p0}^2$. The resulting equations for $g(\theta)$ and $h(\phi)$ are in standard form (e.g., [11]) and their solutions are the associated Legendre functions $g(\theta) = K_0 P_n^m(\cos\theta)$ (the other Legendre function has been excluded because of the domain) and the azimuthal harmonics $h(\phi) = K_1 \cos m\phi + K_2 \sin m\phi$. However, (5) is not in general the spherical Bessel equation because of the r dependence of the acoustic properties of the shell.

Chen *et al.* [7] showed that an important element in making the 3D electromagnetic cloaking shell scatter-free is a radial shift from r to $(r - a)$ in the equation for $f(r)$ (which is the Riccati-Bessel equation in the electromagnetic case) that is produced by the radially dependent parameters of the medium. Recognizing this as one potential path for realizing a scatter-free shell, we now find the conditions on ρ_r, ρ_ϕ, and λ that transform (5) into the standard spherical Bessel equation in $(r - a)$. These conditions are

$$\frac{r^2}{\rho_r} = \frac{(r-a)^2}{k_1}, \tag{6}$$

$$\rho_\phi = k_1, \tag{7}$$

$$\frac{\rho_\phi}{\lambda} k_0^2 r^2 = k_{\rm sh}^2 (r-a)^2, \tag{8}$$

where k_1 and $k_{\rm sh}$ are constants to be determined later. Under these conditions (5) becomes

$$\frac{\partial}{\partial r}\left((r-a)^2 \frac{\partial f}{\partial r}\right) + [k_{\rm sh}^2 (r-a)^2 - n(n+1)] f = 0, \tag{9}$$

which has the solution $f(r) = b_n[k_{\rm sh}(r-a)]$, where $b_n(x)$ is a spherical Bessel or Hankel function of order n.

The total pressure field in all regions can now be expressed. For $r > b$, the spherical expansion of the incident compressional plane wave gives

$$p^{\rm inc} = \sum_{n=0}^{\infty} K_n j_n(k_0 r) P_n(\cos\theta), \tag{10}$$

where $K_n = i^n(2n+1)$ and $P_n(\cos\theta)$ is the nth degree Legendre polynomial. The scattered field is subject to the radiation condition and, because of the azimuthal invariance of the source and geometry, can be written as

$$p^{\rm scat} = \sum_{n=0}^{\infty} A_n h_n^{(1)}(k_0 r) P_n(\cos\theta), \tag{11}$$

with $h_n^{(1)}(k_0 r)$ the spherical Hankel function of the first kind, and A_n are constants to be determined by the boundary conditions. For $a < r < b$, the azimuthal invariance means that the pressure is given by

$$p^{\rm sh} = \sum_{n=0}^{\infty} B_n j_n[k_{\rm sh}(r-a)]P_n(\cos\theta), \tag{12}$$

where the j_n spherical Bessel function is used to ensure the fields remain finite at $r = a$. This assumption will have to be slightly modified below to handle subtleties associated with the $n = 0$ harmonic. In the interior of the shell where $r < a$, the pressure field is given by

$$p^{\rm int} = \sum_{n=0}^{\infty} C_n j_n(k_0 r)P_n(\cos\theta). \tag{13}$$

The radial velocity (normal to all interfaces) is continuous and therefore is needed to complete the problem. From (1) we have

$$v_r = -\frac{1}{i\omega\rho_r\rho_0}\frac{\partial p}{\partial r}, \tag{14}$$

and therefore in the three regions the expressions for the radial velocity are

$$v_r^{\rm inc} = \frac{-k_0}{i\omega\rho_0}\sum_{n=0}^{\infty} K_n j_n'(k_0 r)P_n(\cos\theta), \tag{15}$$

$$v_r^{\rm scat} = \frac{-k_0}{i\omega\rho_0}\sum_{n=0}^{\infty} A_n h_n^{(1)\prime}(k_0 r)P_n(\cos\theta), \tag{16}$$

$$v_r^{\rm sh} = \frac{-k_{\rm sh}}{i\omega\rho_0\rho_r}\sum_{n=0}^{\infty} B_n j_n'[k_{\rm sh}(r-a)]P_n(\cos\theta), \tag{17}$$

$$v_r^{\rm int} = \frac{-k_0}{i\omega\rho_0}\sum_{n=0}^{\infty} C_n j_n'(k_0 r)P_n(\cos\theta), \tag{18}$$

where the prime denotes differentiation with respect to the entire argument of the Bessel functions. After exploiting the orthogonality of $P_n(\cos\theta)$, continuity of p and v_r at $r = b$ means that the expressions for A_n and B_n that must be satisfied by a solution are

$$K_n j_n(k_0 b) + A_n h_n^{(1)}(k_0 b) = B_n j_n[k_{\rm sh}(b-a)], \tag{19}$$

$$\frac{k_0}{i\omega\rho_0}[K_n j_n'(k_0 b) + A_n h_n^{(1)\prime}(k_0 b)] = \frac{k_{\rm sh}}{i\omega\rho_0\rho_r(b)} \times B_n j_n'[k_{\rm sh}(b-a)]. \tag{20}$$

Solving these simultaneously for the scattered field coefficients A_n yields

$$\frac{A_n}{K_n} = \frac{k_0 j_n'(k_0 b) j_n[k_{\rm sh}(b-a)] - \frac{k_{\rm sh}}{\rho_r(b)} j_n(k_0 b) j_n'[k_{\rm sh}(b-a)]}{-k_0 h_n^{(1)\prime}(k_0 b) j_n[k_{\rm sh}(b-a)] + \frac{k_{\rm sh}}{\rho_r(b)} h_n^{(1)}(k_0 b) j_n'[k_{\rm sh}(b-a)]}. \tag{21}$$

The A_n can be made identically zero if two conditions are met, namely,

$$k_0 b = k_{\rm sh}(b-a), \tag{22}$$

$$k_0\rho_r(b) = k_{\rm sh}. \tag{23}$$

Physically, (22) says that the number of wavelengths in the shell over a distance $b - a$ must be the number of wavelengths in the background medium over a distance b. This wavelength compression is exactly what is produced by coordinate transformation cloaking [1]. Equation (23) simply says that the wave impedances must be appropriately matched at the outer edge of the shell to eliminate reflections. When combined with (6), these expressions determine the two unknown constants in the specification of the acoustic cloaking shell, namely, $k_1 = b^{-1}(b-a)$ and $k_{\rm sh} = b(b-a)^{-1}k_0$. A complete material shell specification that renders the interior acoustically invisible is thus

$$\rho_\phi = \rho_\theta = \frac{b-a}{b}, \tag{24}$$

$$\rho_r = \frac{b-a}{b}\frac{r^2}{(r-a)^2}, \tag{25}$$

$$\lambda = \frac{(b-a)^3}{b^3}\frac{r^2}{(r-a)^2}. \tag{26}$$

We must now ensure that this solution yields meaningful fields in the shell and interior regions. From the $r = b$ boundary conditions we have $B_n = K_n$, and thus continuity of p and v_r at the $r = a$ interface give, with $r \to a$ and using (24),

$$K_n j_n[k_{\rm sh}(r-a)] = C_n j_n(k_0 r), \tag{27}$$

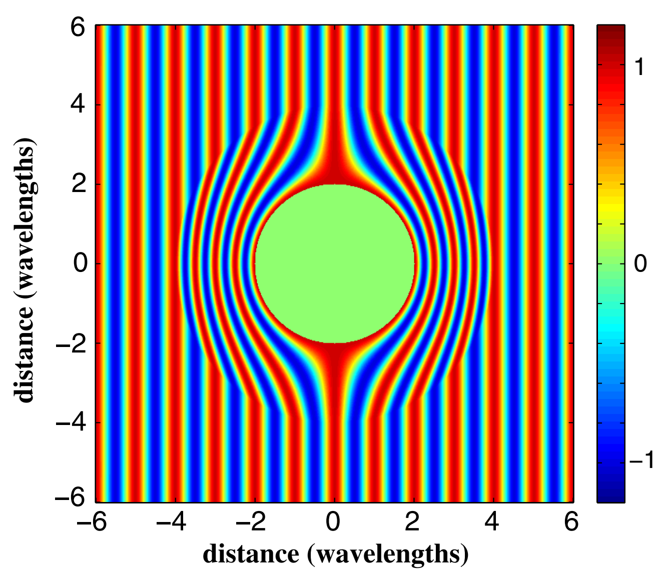

FIG. 2 (color online). The real part of the pressure field in the $r - \theta$ plane of the problem domain computed from the series solution. The plane wave is incident from the left.

$$\frac{b^2(r-a)^2}{(b-a)^2 r^2} K_n j_n'[k_{\mathrm{sh}}(r-a)] = C_n j_n'(k_0 r). \tag{28}$$

Because of the $(r-a)^2$ term from ρ_r, the left hand side of (28) is zero for all n, giving $C_n = 0$ and thus zero internal fields from the continuity of v_r. Physically, this is because the radial mass density tends to infinity at the inner edge of the shell which reduces all radial particle motion to zero. Equation (26) is consistent with this solution except for the $n = 0$ term for which $j_0(0)$ does not vanish. However, it can be shown that C_0 does vanish in the limit of the ideal acoustic cloaking parameters by following an argument employed by Ruan *et al.* [12] in analyzing the 2D electromagnetic cloak. We only need to modify the expressions for the $n = 0$ fields (remember that scattering vanishes for all higher order harmonics without any tricks) in the cloaking shell by including the Hankel function solution that was removed on account of its singularity at the origin. By including this term, it is straightforward to show that C_0 tends to zero as the shell approaches the ideal cloak.

Figure 2 shows the real part of the pressure field for the incident plane wave in the entire domain of Fig. 1 computed from the series solution. The real part is plotted so that the individual phase fronts are visible. As for the 2D [13] and 3D [7] electromagnetic solutions, the cloaking shell smoothly bends the phase fronts around the interior with no scattering in any direction, including the forward direction.

Thus the material specifications in (24)–(26) are analytically an ideal 3D acoustic cloak that excludes all incident fields from the interior of the shell and does not scatter in any direction. We note that, because the same limiting argument employed in [12] is needed here, the $n = 0$ scattered fields may decrease slowly as the ideal cloak parameters are approached. It is also interesting to note that the ideal 3D acoustic cloaking parameters (24)–(26) are similar in structure to the 2D electromagnetic and 2D acoustic [6] parameters in that they contain singularities at the interior edge of the cloak. This is in contrast to the 3D electromagnetic cloak [1], which does not contain singularities and for which scattering analysis does not require limiting arguments to show that the scattering is identically zero [7]. This is because the 2D electromagnetic, 2D acoustic, and 3D acoustic cases directly involve solutions of the Helmholtz equation and therefore Bessel functions and spherical Bessel functions that do not all go to zero at the origin. The 3D electromagnetic case involves Riccati-Bessel functions that do all go to zero at the origin.

Physical realization of the mass anisotropy needed for such an acoustic cloak will require some type of engineered material. We note that significant progress has been made in theoretical and experimental acoustic metamaterials [14,15]. Specifically, Milton *et al.* [5] describe conceptually how anisotropic effective mass can be achieved with spring loaded masses, and Torrent and Sanchez-Dehesa [10] have shown how effective density and bulk modulus can be controlled in an acoustic metamaterial by embedding solid inclusions in a fluid matrix. There is thus some hope that such an acoustic cloak could be physically realized. Because the strong limits on wave velocity and thus causal dispersion relations for electromagnetic materials do not apply to acoustic materials, certain properties, such as a wide bandwidth, may be easier to realize in practice for an acoustic cloak. The existence of this 3D cloaking shell indicates that related reflectionless solutions may also exist for other wave systems that are not isomorphic with electromagnetics.

Note added.—Chen and Chan [16] recently reported the same material specifications for a 3D spherical acoustic cloaking shell through a different approach that relies on the isomorphism between the acoustic equations and the electric conductivity equation.

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