Generation Linewidth of an Auto-Oscillator with a Nonlinear Frequency Shift: Spin-Torque Nano-Oscillator

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(Received 8 March 2007; published 10 January 2008)

It is shown that the generation linewidth of an auto-oscillator with a nonlinear frequency shift (i.e., an auto-oscillator in which frequency depends on the oscillation amplitude) is substantially larger than the linewidth of a conventional quasilinear auto-oscillator due to the renormalization of the phase noise caused by the nonlinearity of the oscillation frequency. The developed theory, when applied to a spin-torque auto-oscillator, gives a good description of experimentally measured angular and temperature dependences of the linewidth.

DOI: 10.1103/PhysRevLett.100.017207 PACS numbers: 85.75.−d, 05.10.Gg, 05.40.−a, 75.30.Ds

It is well known that the linewidth Γ_0 of a passive oscillating circuit is determined by the ratio of its dissipative element (e.g., resistance R) to its reactive element (e.g., inductance L): $\Gamma_0 = R/2L$. When the oscillating circuit is connected to an active element (transistor, vacuum tube, tunnel diode, etc.) and a source of a constant voltage (e.g., battery) the autogeneration of constant-amplitude oscillations at the resonance frequency of the oscillating circuit ($\omega = 1/\sqrt{LC}$, where C is the circuit capacitance) can take place [1,2]. The equilibrium amplitude of these auto-oscillations is determined by the dynamic balance between the positive nonlinear damping of the oscillating system and negative nonlinear damping introduced into the system by the active element [1,2].

It is also well established that the generation linewidth $\Delta\omega$ in a typical auto-oscillator is determined, for the most part, by thermal phase noise [see, e.g., Eq. (9.36) in [1]] and can be expressed in the following general form,

$$\Delta\omega = \Gamma_0 \frac{k_B T}{E(P)}, \qquad (1)$$

where k_B is the Boltzmann constant, T is the absolute temperature, $E(a) = \beta P = \beta|a|^2$ is the averaged energy of the auto-oscillation having the power $P = |a|^2$ and complex amplitude a, and β is the coefficient relating the averaged energy to the auto-oscillation power P. For example, in an auto-oscillator with a standard linear oscillating circuit, $\beta = C/2$, where C is the capacitance of the oscillating circuit and a is the amplitude of the voltage on this capacitance. Equation (1) is rather general and is equally applicable to any type of conventional auto-oscillator (transistor, vacuum tube, tunnel diode, laser, etc.) in which the oscillation frequency is not strongly dependent on the amplitude, i.e., in the limit $d\omega/dP \to 0$.

There exist, however, auto-oscillators for which the oscillation frequency exhibits a strong nonlinearity $N \equiv d\omega/dP$ that is too large to be neglected. In such systems, one expects that even small fluctuations in the amplitude (or power) at steady state can give important contributions to the phase noise. A pertinent example of present interest is the magnetic spin-torque nano-oscillator (STNO) [3–7], which consists of a nanosized metallic contact attached to a magnetic multilayer or a multilayered magnetic nanopillar. Direct electrical current passing through the multilayer can lead to a transfer of spin-angular momentum between magnetic layers in the stack [3,4], which in turn creates an effective negative damping for the magnetization of the thinner ("free") magnetic layer. This negative damping, analogous to the role played by an active element, can lead to self-sustained oscillations of magnetization in the free layer. The frequency of these auto-oscillations is determined by the applied magnetic field, static magnetization, etc., and is, in general, close to the ferromagnetic resonance frequency, while the oscillation amplitude is determined by the intrinsic nonlinearities of the system.

In contrast to traditional (e.g., transistor) auto-oscillators, the frequency of the STNO strongly depends on the power of the magnetization precession P: $\omega(P) = \omega_0 + NP$. The sign and magnitude of the nonlinear frequency shift coefficient N depend on the direction and magnitude of the bias magnetic field (see [6–9] for details) and can be varied over a range comparable to the oscillation frequency itself. Thus, the classical result (1) cannot describe quantitatively the generation linewidth in STNO, and a new theory that explicitly takes into account the nonlinear frequency shift of the auto-oscillator is necessary.

In this Letter, we develop a theory to describe the generation linewidth in an auto-oscillator with nonlinear frequency shift and show that this nonlinearity leads to a significant linewidth broadening. The theory is then applied to the STNO, and we demonstrate that the correct treatment of such nonlinearities is essential for even the *qualitative* description of the nonlinear auto-oscillator.

The general equation describing the time evolution of the oscillation amplitude a in a nonlinear auto-oscillator in the presence of noise can be written in the form

$$\frac{\partial a}{\partial t} + i\omega(P)a + \Gamma_+(P)a - \Gamma_-(P)a = f_n(t), \qquad (2)$$

where $\omega(P)$ is the nonlinearly shifted frequency of the excited oscillation mode, $P = |a|^2$, $\Gamma_+(P)$ is the natural positive damping of the oscillator, $\Gamma_-(P)$ is the effective negative damping introduced by an active element, and $f_n(t)$ is a random white Gaussian process that describes the influence of the thermal noise. The correlation function of this random noise can be written as $\langle f_n(t) f_n^*(t')\rangle = 2\Gamma_+ P_n \delta(t - t')$, where $P_n = k_B T/\beta$ is the oscillator power at thermal equilibrium.

The stationary solution of Eq. (2) in the absence of noise $[f_n(t) = 0]$ can be easily obtained in the form

$$a(t) = \sqrt{P_0} e^{-i\omega(P_0)t + i\phi}, \qquad (3)$$

where the equilibrium oscillation power P_0 is determined by the condition $\Gamma_+(P_0) = \Gamma_-(P_0)$ and ϕ is a constant oscillation phase.

Sufficiently far above the auto-oscillation threshold (i.e., for $P_0 \gg P_n$) the solution of Eq. (2) with the noise term included will be similar to the noise-free solution Eq. (3) in the sense that the oscillation amplitude will be close to the mean value of $\sqrt{P_0}$, i.e., $|a(t)| = \sqrt{P_0} + \delta A(t)$, $|\delta A(t)|^2 \ll P_0$, and the phase ϕ will be a slow function of time. Substituting the expression

$$a(t) = [\sqrt{P_0} + \delta A(t)] e^{-i\omega(P_0)t + i\phi(t)} \qquad (4)$$

for $a(t)$ in Eq. (2), and retaining only the terms of the first order in δA, we find equations for fluctuations of the amplitude

$$\frac{\partial \delta A}{\partial t} + 2\Gamma_{\rm eff} P_0 \delta A = \mathrm{Re}(\tilde{f}_n(t) e^{-i\phi}) \qquad (5a)$$

and phase

$$\frac{\partial \phi}{\partial t} + 2N\sqrt{P_0}\delta A = \frac{1}{\sqrt{P_0}} \mathrm{Im}(\tilde{f}_n(t) e^{-i\phi}). \qquad (5b)$$

Here $\Gamma_{\rm eff}$ and N are the effective nonlinear damping and nonlinear frequency shift, respectively:

$$\Gamma_{\rm eff} = \frac{d\Gamma_+(P)}{dP} - \frac{d\Gamma_-(P)}{dP}, \qquad N = \frac{d\omega(P)}{dP}, \qquad (6)$$

where the derivatives are taken at $P = P_0$. In Eqs. (5a) and (5b) $\tilde{f}_n(t) = f_n(t) e^{i\omega(P_0)t}$. Note that the statistical properties of $f_n(t)$ and $\tilde{f}_n(t)$ are identical. Therefore, the tilde will be omitted in the following text for simplicity.

There is a significant qualitative difference between the behavior of the amplitude and the phase. Since the oscillation amplitude at steady state remains practically constant, $|a| \approx \sqrt{P_0}$, the correlation function for the amplitude fluctuations $K_A(\tau) \equiv \langle |a(t)||a(t+\tau)|\rangle$ remains finite even if $\tau \to \infty$, i.e., $K_A \to P_0$. Therefore, for large τ the behavior of the full correlation function $K(\tau) \equiv \langle a(t) a^*(t+\tau)\rangle$ will be determined solely by the phase fluctuations,

$$K(\tau) \approx P_0 \langle e^{i[\phi(t) - \phi(t+\tau)]}\rangle e^{i\omega(P_0)\tau}. \qquad (7)$$

For the frequency linewidth of the auto-oscillation, we are interested only in the fluctuations taking place inside a narrow frequency region $\Delta\omega \ll \Gamma_{\rm eff} P_0$, in which $|\partial \delta A/\partial t| \sim \Delta\omega |\delta A| \ll 2\Gamma_{\rm eff} P_0 |\delta A|$. As such, the first (derivative) term in the left-hand side of Eq. (5a) can be neglected compared to the second term, and an explicit expression for $\delta A(t)$ can be obtained,

$$\delta A = \frac{1}{2\Gamma_{\rm eff} P_0} \mathrm{Re}(f_n e^{-i\phi}). \qquad (8)$$

Substituting this expression for $\delta A(t)$ in Eq. (5b) leads to a closed-form equation for the phase fluctuations $\phi(t)$ in the system,

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{1}{\sqrt{P_0}} \left[-\frac{N}{\Gamma_{\rm eff}} \mathrm{Re}(f_n e^{-i\phi}) + \mathrm{Im}(f_n e^{-i\phi}) \right], \\ &= \frac{1}{\sqrt{P_0}} \sqrt{1 + \left(\frac{N}{\Gamma_{\rm eff}}\right)^2} \mathrm{Im}(f_n e^{-i\alpha - i\phi}), \end{aligned} \qquad (9)$$

where $\alpha = \arctan(N/\Gamma_{\rm eff})$.

Equation (9) is formally identical to the equation for phase fluctuations in a system without a nonlinear frequency shift [see, e.g., second Eq. (9.8) in [1]], but with the increased noise level

$$f_n(t) \to f_n'(t) = \sqrt{1 + \left(\frac{N}{\Gamma_{\rm eff}}\right)^2} e^{-i\alpha} f_n(t). \qquad (10)$$

Application of the general methodology to compute auto-oscillator linewidths (see, e.g., Chap. 9 in [1] or [10]) to Eq. (9) leads to the following expression for the Lorentzian linewidth of the auto-oscillator with a nonlinear frequency shift N,

$$\Delta\omega = \Gamma_0 \left(\frac{k_B T}{E_0}\right) \left[1 + \left(\frac{N}{\Gamma_{\rm eff}}\right)^2\right], \qquad (11)$$

where $\Gamma_0 = \Gamma_+(P_0)$, $E_0 = \langle E(a)\rangle = \beta P_0$ is the average oscillator energy, and we have rewritten the ratio P_n/P_0 as $k_B T/E_0$. The comparison of the classical result (1) with the generalized Eq. (11) shows clearly that the nonlinear frequency shift in the auto-oscillator leads to a significant linewidth broadening that is caused by effective renormalization of the phase noise (10) due to the frequency nonlinearity N.

The result in (11) is the principal result of this Letter and illustrates the fact that three key parameters determine the linewidth of an auto-oscillator with a nonlinear frequency shift. First, the relaxation rate of the oscillator Γ_0 determines the overall scale of the possible linewidth variations. Second, the generation linewidth is proportional to the ratio of the noise energy (which increases with temperature) to the average energy of the auto-oscillation. Third, the ratio of the nonlinear frequency shift coefficient N to the effective nonlinear damping Γ_{eff} gives a measure of the phase-noise renormalization due to amplitude fluctuations.

It should be noted, that the importance of the nonlinear frequency shift for the STNO generation linewidth was explicitly pointed out in the pioneering paper [11], where experimental measurements and numerical calculations of the linewidth in a temperature interval were performed. However, the empirical linewidth expression [see Eq. (2) in [11]] and the numerical calculations performed in [11] give the value of the linewidth that is about 1 order of magnitude larger than the experimentally measured one and a $T^{1/2}$ linewidth dependence on the temperature. We believe that both these results are caused by the approximation of a long correlation time ($\Delta\omega \gg \Gamma_{\mathrm{eff}} P_0$) adopted in [11] that is not valid for typical STNO parameters.

Another attempt to calculate the STNO linewidth was undertaken by one of the authors in [10], but the nonlinear frequency shift was neglected. The calculation resulted in an expression for the linewidth [see Eq. (28) in [10]] that can be cast in the classical form (1), where the constant β is given by $\beta = (M_0/\gamma)\omega_0 V_{\mathrm{eff}}$, where γ is the gyromagnetic ratio, ω_0 is the oscillation frequency, and V_{eff} is the effective volume of the magnetic material of the free layer involved in the auto-oscillation [see Eq. (4) in [12]]. When compared to experiments, however, the result [10] underestimates the generation linewidth by 20–40 times.

Now, it will be interesting to apply our new general result (11), where the frequency nonlinearity has been taken into account, to calculate the linewidth of a STNO. It has been shown previously [9,12,13] that the nonlinear oscillator equation (2) for the case of STNO can be derived from the Landau-Lifshitz-Gilbert equation with the Slonczewski term [3] describing the spin transfer torque. For the case of STNO the dimensionless complex amplitude a can be defined as $|a|^2 = (M_0 - M_z)/2M_0$, where M_0 is the length of the magnetization vector in the free magnetic layer, and M_z is the projection of this vector on the equilibrium magnetization direction $\mathbf{z}$ (see [9] for details), the negative damping caused by spin torque is given by $\Gamma_-(P) = \sigma I(1 - P)$, where I is the bias current and σ is the spin-polarization efficiency defined in Eq. (2) of [12], and the positive damping equals to $\Gamma_+(P) = \Gamma(1 + QP)$, where Γ characterizes the oscillator equilibrium linewidth in the passive regime [see Eq. (31) in [9]] and $Q > 0$ is a phenomenological coefficient characterizing the nonlinearity of the positive damping (see [14] for details).

The dependences of the STNO generation linewidth on the angle θ_e, that the external bias magnetic field H_e makes with the plane of the STNO free layer, calculated using Eq. (11) for $Q = 3$ and typical parameters of STNO [6] are shown in Fig. 1. An important result that follows from Eq. (11) and Fig. 1 is the prediction of a linewidth minimum that follows from a change in sign in the frequency shift [e.g., from "red" ($N < 0$) to "blue" ($N > 0$)] as the magnetization is tilted out of the film plane. Across this transition the nonlinear frequency shift coefficient N passes through zero (see, e.g., Fig. 8 in [9]) at which one recovers the smallest value of the generation linewidth.

In Fig. 2 we directly compare the generation linewidth calculated using Eq. (11) with the results of experimental measurements of the temperature dependence of the STNO linewidth $\Delta\omega(T)$ performed on the nanopillar devices no. 1 [Fig. 2(a)] and no. 2 [Fig. 2(b)] in Ref. [11] (see Fig. 2 in [11]), and with the angular dependence of the STNO linewidth $\Delta\omega(\theta_e)$ [Fig. 2(c)] experimentally measured on the nanocontact device in [15] (see Fig. 6 in [15]). Geometrical parameters of the nanopillar device [see Figs. 2(a) and 2(b)] were taken from Ref. [11] and it was assumed that the excited magnetization oscillation is pinned at the pillar lateral boundaries (see [16] for details); the Gilbert damping parameter $\alpha_G = 0.01$, the nonlinearity parameter of positive damping $Q = 3$, and the polarization efficiency $\varepsilon = 0.4$ [see Eq. (2) of [12]] were assumed to be the same for both devices. Similarly, all

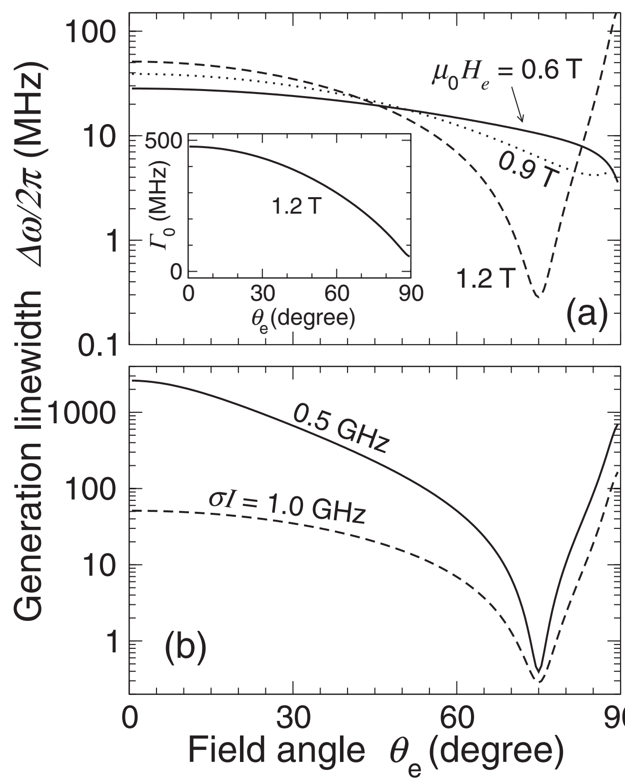

FIG. 1. Generation linewidth as a function of applied field angle θ_e for (a) three applied fields at constant $\sigma I = 1$ GHz and (b) two bias currents at constant $\mu_0 H_e = 1.2$ T. Inset of (a): Equilibrium linewidth Γ as a function of θ_e for $\mu_0 H_e = 1.2$ T.

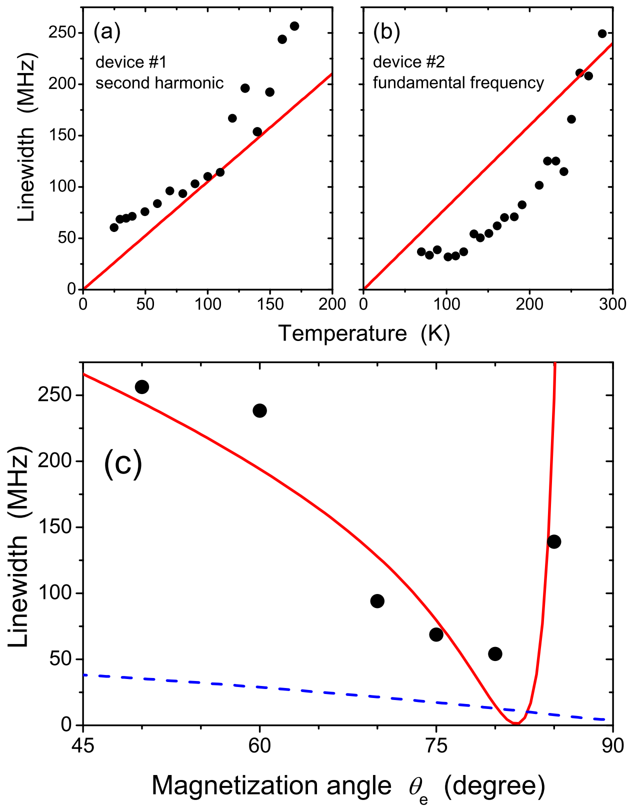

FIG. 2 (color online). Generation linewidth of a spin-torque auto-oscillator calculated from Eq. (11) (solid line) in comparison with the temperature dependence of linewidth in a nanopillar device no. 1 (a) and device no. 2 (b) measured in [11] (black dots) at $\theta_e = 0$, and (c) the angular dependence of the nanocontact STNO linewidth measured at a room temperature, taken from Fig. 6(a) in Ref. [15] (black dots). (a) The linewidth measured at the *second* harmonic of the signal $\Delta\omega_2 = 4\Delta\omega$. The dashed line in (c) represents the multiplied by ten classical result for the oscillator linewidth calculated from Eq. (1).

the parameters of the nanocontact device Fig. 2(c) were taken from [15]; current $I = 9$ mA and magnetic field $\mu_0 H_e = 0.9$ T correspond to the center of the experimentally studied region, and the nonlinearity parameter of positive damping was again chosen to be equal to $Q = 3$.

As it is clear from Figs. 2(a) and 2(b), our simple analytical expression (11) gives a reasonably good estimate of the observed linewidths at different temperatures for *both* nanopillar devices with *the same* parameters. Assuming that the parameters of the two devices are slightly different (which is possible due to different nanopatterning and different thicknesses of the "free" magnetic layer), one can obtain much better quantitative agreement with the experiment [11].

It is also clear from Fig. 2(c) that the linewidth dependence on the bias field orientation calculated using our result Eq. (11) is in good quantitative agreement with the experimental results from Ref. [15]. In contrast to the classical result (1), which predicts much narrower lines and a monotonic decrease in the linewidth as a function of θ_e, the renormalized phase-noise result (11) gives a reasonable qualitative and quantitative description of the experimentally observed behavior, in particular, the linewidth minimum around $\theta_e \approx 80°$.

In summary, we have developed a theory of the generation linewidth of an auto-oscillator with a nonlinear frequency shift which generalizes the classical result (1). The additional nonlinearity in the oscillator frequency leads to a renormalization of the phase noise far above threshold. Applied to the particular case of a spin-torque nano-oscillator, the theory accounts for a number of characteristic, but previously unexplained, features observed in experiment: (i) general linewidth narrowing with increases in the bias current and oscillation amplitude (see Fig. 4 in Ref. [15]), (ii) presence of a linewidth minimum as a function of the external magnetic field orientation (see Fig. 6 in Ref. [15]), and (iii) linear dependence of the linewidth on temperature.

This work was in part supported by MURI Grant No. W911NF-04-1-0247 from the U.S. Department of Defense, by Contract No. W56HZV-07-P-L612 from the U.S. Army TARDEC, RDECOM, by Grant No. ECCS-0653901 from the U.S. National Science Foundation, and by the Oakland University Foundation. J. K. acknowledges support from the European Communities program IST under Contract No. IST-016939 TUNAMOS.

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