# Multiple Components in Narrow Planetary Rings

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The phase-space volume of regions of regular or trapped motion, for bounded or scattering systems with 2 degrees of freedom, respectively, displays universal properties. In particular, drastic reductions in the volume (gaps) are observed at specific values of a control parameter. Using the stability resonances we show that they, and not the mean-motion resonances, account for the position of these gaps. For more degrees of freedom, exciting these resonances divides the regions of trapped motion. For planetary rings, we demonstrate that this mechanism yields rings with multiple components.

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The Cassini mission is providing unprecedented amounts of data on the structure of Saturn rings [1]. As usual, surprises and new questions emerge that call for theoretical understanding. Beyond the intrinsic interest on planetary rings [2], these are now the only examples of flat astrophysical systems for which such detailed data are available. This makes their analysis paradigmatic [2], which serves as an analog for other systems like galaxies and planet-forming disks. A first order understanding of the dynamics and processes in rings has been obtained, but important questions remain unanswered [2]. In particular, narrow planetary rings are eccentric and display multiple components (or strands), kinks, clumps, and arcs (or patches); the *F* ring of Saturn is a magnificent case [1,2]. Therefore, these narrow rings pose interesting questions in dynamical astronomy and nonlinear dynamics related to their stability, confinement, and structural properties.

Recently it became clear that the phase-space volume associated with regular orbits for bounded systems with 2 degrees of freedom (DOF) displays universal properties as a function of a control parameter [3,4]. For scattering systems such quantity is related to the trapped orbits. In particular, there are locations where a drastic reduction of the phase-space volume occupied by the regular or trapped orbits is observed, i.e., gaps. Here we show that these gaps are related to the occurrence of stability resonances (SR), and not to mean-motion resonances (MMR). The MMR correspond to locations where the period of the particle is a rational of the orbital period of a satellite or planet. Examples include Kirkwood gaps in the asteroid main belt [5,6] and the Cassini division in Saturn rings [7], where very few particles are observed. Other cases of MMR exist where particles accumulate locally, as the Jupiter Trojans or the Hilda group [8]. The SR are defined by a resonant condition in the (linear) stability exponents of a central stable periodic orbit. We show in this Letter that nonlinear and higher-dimensional effects related to the SR yield rings with two or more components, or strands, as those observed in the *F* ring [1,2]. We use the scattering approach to narrow planetary rings [9] and a disk rotating on a Kepler orbit as an example [10]. Our results can be used in other cases, ranging from particle accelerators [11] to galactic dynamics [12], where resonances and transport properties are important [13].

The scattering approach to narrow rings [9] considers the scattering dynamics of a (ring) particle in a planar system with some intrinsic rotation (external force). In the planetary case, this rotation comes from the orbital motion of one or several satellites. This rotation creates generically phase-space regions of *dynamically trapped motion*, in what otherwise is dominated by unbounded trajectories; "generic" implies that it holds for a wide class of rotating potentials [14], including gravitational interactions [9,15]. For a uniformly circular rotation, the system has 2 DOF and a constant of motion, the Jacobi integral. For these scattering systems the organizing periodic orbits generically appear in pairs through saddle-center bifurcations, one is stable and the other unstable. The manifolds of the unstable orbit intersect, isolating a region where trapped motion is of nonzero measure if stable orbits exist. By changing the Jacobi integral, the central *linearly* stable periodic orbit becomes unstable and a period doubling cascade sets in. This scenario eventually turns the horseshoe (invariant set) into a hyperbolic one, thus destroying any region of trapped motion [16]. Then, in the Jacobi constant space, the regions of trapped motion are bounded. Particles with initial conditions outside these regions escape rapidly along scattering trajectories; those inside remain dynamically trapped. Consider now an ensemble of initial conditions of independent particles distributed over the extended phase space, which contains entirely one region of trapped motion. Then, due to the intrinsic rotation, the pattern formed by projecting the trapped particles into the $X$-$Y$ plane at a given time forms a ring [14]. The ring is typically narrow, sharp-edged, and noncircular; this scenario is generic [14]. For more than two DOF, e.g., a nonuniform rotation on a Kepler elliptic orbit, the ring displays further structure [9].

The simplest example to illustrate the occurrence of rings is a planar scattering billiard: a disk on a Kepler orbit [10]. This system consists of a point particle moving freely unless it collides with an impenetrable disk of radius $d$, which orbits around the origin on a Kepler orbit of semi-major axis $R$ ($d < R$). Collisions with the disk are treated as usual [10]; no collisions lead to escape. For the disk on a circular orbit, the simple periodic orbits are the radial collision orbits. They and their linear stability are given by [9]

$$J = 2\omega_d^2(R - d)^2(1 + \Delta\phi\tan\theta)\cos^2\theta(\Delta\phi)^{-2}, \quad (1)$$

$$\mathrm{Tr}\, D\mathcal{P}_J = 2 + [(\Delta\phi)^2(1 - \tan^2\theta) - 4(1 + \Delta\phi\tan\theta)]R/d. \quad (2)$$

Equation (1) defines the value of the Jacobi integral $J$ for the radial collision orbits in terms of $\theta$, the outgoing angle of the particle's velocity after a collision with the disk. Here, $\Delta\phi = (2n - 1)\pi + 2\theta$ is the angular displacement of the disk between consecutive radial collisions, and $n = 0, 1, 2, \ldots$ is the number of full turns completed by the disk before the next collision. The period of the disk's orbit is $T_d = 2\pi$ ($\omega_d = 1$). Equation (2) provides the trace of the linearized matrix $D\mathcal{P}_J$ around the radial collision orbits. Being a two DOF system, periodic orbits are *linearly* stable iff $|\mathrm{Tr} D\mathcal{P}_J| \leq 2$.

To understand the phase-space properties that define the dynamics, in particular, for many DOF, we consider a relative measure of the phase-space volume occupied by the regions of trapped motion [3,4]. This quantity is a function of some control parameter and tunes the horseshoe development [16]. For the disk on a circular orbit a good choice is $J$. However, this quantity is not conserved for nonzero eccentricity $\varepsilon$, thus being useless for more than two DOF. A convenient quantity is the average time between consecutive collisions with the disk, $\langle\Delta t\rangle$. The average is defined, for a given initial condition (ring particle), over the successive consecutive collision times; in addition, we consider an ensemble of them. In our numerical calculations we considered an orbit to be trapped if it displays more than 20 000 collisions with the disk; the next 200 000 reflections were used to improve the statistics.

In Fig. 1(a) we show the histogram of $\langle\Delta t\rangle$ for the $n = 0$ trapped region; Fig. 1(b) shows a detail of the corresponding ring [9]. We note that Fig. 1(a) displays some structure, namely, some rather well localized positions where there is essentially no trapped motion in phase space or, at least, a drastic reduction of its size. This structure is similar to those in Ref. [3] and is universal [3,4]. Then, universality extends to scattering systems in terms of the phase-space volume of the regions of trapped motion. We note that the structure of Fig. 1(a) also resembles the structure of the population histogram of asteroids in terms of their semi-major axis [6], which uncovers the Kirkwood gaps and the role of MMR.

Guided by the similarity with the asteroids, we ask whether the gaps in Fig. 1(a) are related to MMR. The question can be answered analytically for the radial collision periodic orbits that are the organizers of the dynamics. Their collision time is $t_{\mathrm{col}} = \Delta\phi/\omega_d$; hence, the MMR

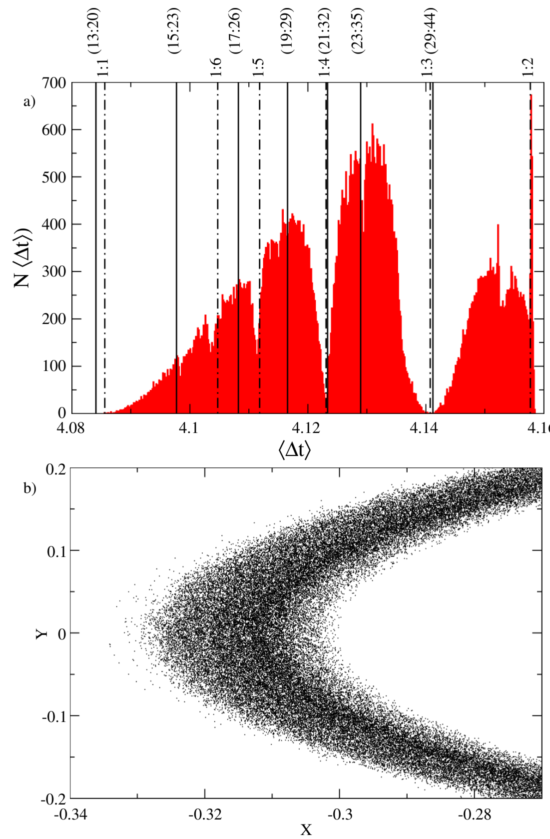


FIG. 1 (color online). (a) Histogram of $\langle\Delta t\rangle$ showing the relative phase-space volume of the region of trapped motion for the scattering billiard on a circular orbit. The continuous lines represent the position of the lowest MMR, indicated in parentheses, that occur in the interval of $\langle\Delta t\rangle$; dash-dotted lines correspond to the lower-order SR. (b) Detail of the ring corresponding to this region of trapped motion.

condition is $t_{\mathrm{col}}/T_d = p/q$, with $p$ and $q$ incommensurate integers. In Fig. 1(a), we have indicated the location of the *lowest* mean-motion resonances in the $\langle\Delta t\rangle$ interval of interest; the actual resonances are indicated in parentheses. We have also included the 29:44 resonance because of its proximity to the position of the main gap. While these resonances are definitely not low-order resonances, we note that some may be very close to the gaps, while others are not. Therefore, low-order mean-motion resonances are not related to the reduction of the phase-space volume (gaps) of the regions of trapped motion.

To understand the gaps of Fig. 1(a), we consider the phase-space structure defining a surface of section at the collisions with the disk in a rotating frame (see [9]). Thus, the average return time to the surface of the section is precisely $\langle\Delta t\rangle$. Figure 2 displays the surfaces of the section for three values of the Jacobi integral around the location of the main gap in Fig. 1(a). Satellite period-three islands squeeze the central stable fixed point, reducing efficiently the region of trapped motion. At some value of $J$ there is a bifurcation with the central point being unstable. After this value, the whole structure reappears with some symmetric inversion. This phenomenon is intrinsically nonlinear and is universal [17]. Note that the existence of the period-three islands in the surface of section is not equivalent to a 1:3 MMR. The period-three fixed points involve different collision times. Adding these individual times matches a multiple of the collision time of the central fixed point, which in general is not a rational of $T_d$.

The gaps in Fig. 1(a) are thus related to a reduction of the phase-space volume occupied by trapped orbits due to satellite islands (see also [18]). In [17] it is shown that the passage through such 1:3 resonances (Fig. 2) leads universally to instabilities, while resonances of order higher than 4 do not induce instability; for the resonance of order 4 the behavior depends upon which contribution (resonant or nonresonant) dominates the normal form [17,18]. These results follow from the structure of the resonant normal form, i.e., the relevant (nonlinear) contributions to the stability analysis of a central linearly stable periodic orbit. We thus consider whether these resonances are related to the structure of Fig. 1(a). We write the eigenvalues of the linearized map $D\mathcal{P}_J$ around the stable radial collision orbit as $\lambda_\pm = e^{\pm i\alpha}$. We are interested in values of $\alpha$ that are rational multiples of $2\pi$, i.e., $\alpha/(2\pi) = p/q$ with $p$ and $q$ incommensurate integers. The eigenvalues $\lambda_\pm$ are related to each other by complex conjugation; by consequence, the $p$:$q$ resonance is related to the $q - p$:$q$ resonance. Since these resonances involve the stability exponents, we refer to them as stability resonances.

From the definition of the purely imaginary eigenvalues $\lambda_\pm$, $|\mathrm{Tr} D\mathcal{P}_J| \leq 2$, we have $\cos\alpha = \mathrm{Tr} D\mathcal{P}_J/2$. This can be written in terms of the collision time $t_{\mathrm{col}} = \Delta\phi/\omega_d$ using Eq. (2). In Fig. 1(a), we have indicated the lower-order SR as dash-dotted lines. The correspondence with the gaps is astonishing. Therefore, we attribute the gaps in Fig. 1(a) to crossing a SR.

The SR, and, in particular, the 1:3, have interesting consequences in the context of rings when we consider a small nonvanishing eccentricity of the disk's orbit. Note that, for nonzero $\varepsilon$, the system is explicitly time dependent, has two-and-half DOF, and cannot be handled with the usual techniques. In Fig. 3(a) we illustrate the phase-space volume of the trapped region for $\varepsilon = 0.0001$, indicating the location of some $\varepsilon = 0$ resonances as a guide, and in Fig. 3(b) we show a detail of the resulting ring. In comparison to Fig. 1(a), the gaps at the 1:3 and 1:6 SR are wider, but the overall structure remains. The histogram is divided into three distinct regions separated by those SR; $\varepsilon$ tunes the width of the gaps. We construct the ring using different colors for the initial conditions in the different $\langle \Delta t \rangle$ regions. The ring displays two well-separated components. Each one is related to a different region in phase space; this is a consequence of the large gap opened by the 1:3 SR. These components are entangled and form a braided ring [9].

We note in Fig. 3(b) that there are only two independent ring components instead of three, as expected from the three disjoint intervals in $\langle \Delta t \rangle$ defined by the 1:3 and 1:6 resonance. This follows from the fact that the 1:6 resonance does not separate enough the phase-space regions around it, so the projections into the $X$-$Y$ plane of these two intervals overlap and do not manifest two components. Yet, as observed in Fig. 3(b), the corresponding ring component displays a clear separation of the ring particles, except for a thin common strip, depending on which side of the 1:6 resonance they come from. For larger values of $\varepsilon$ we have observed rings with three strands [9]. Therefore, multiple ring components are obtained by exciting SR through nonzero eccentricity. Note that these properties follow from the higher dimensionality of phase space and nonlinear effects.

The fact that the overall structure of Fig. 3(a) is similar to the two DOF case and to the population histogram of asteroids may be an indication of universality for higher dimensions. Yet, our results are inconclusive for this issue: Fig. 3(a) represents the first results of the phase-space volume occupied by trapped orbits for a system of more than two DOF.

To summarize, we have studied the phase-space volume occupied by trapped orbits using a scattering billiard, a disk rotating on a Kepler orbit, and succeeded in relating some of its structure to the occurrence of SR. For two DOF the universal structure of the phase-space volume occupied by trapped orbits is extended to scattering systems [3,4].

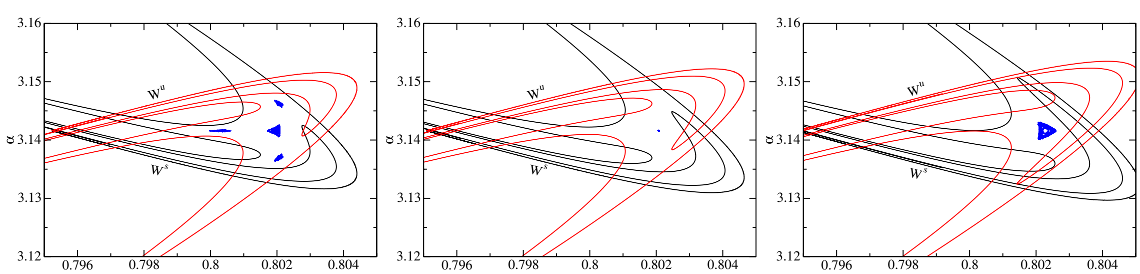


FIG. 2 (color online). Surfaces of section for the scattering billiard on a circular orbit for decreasing values of $J$. The sequence illustrates the reduction of the phase-space volume of the trapped region due to the 1:3 SR corresponding in Fig. 1(a), from left to right, to $\langle \Delta t \rangle \approx 4.1382$, $\langle \Delta t \rangle \approx 4.1409$, and $\langle \Delta t \rangle \approx 4.1445$. The black/red curves are the stable/unstable manifolds of the unstable periodic orbit. The blue regions correspond to regions that represent more than 90% of the phase-space volume occupied by the trapped orbits.

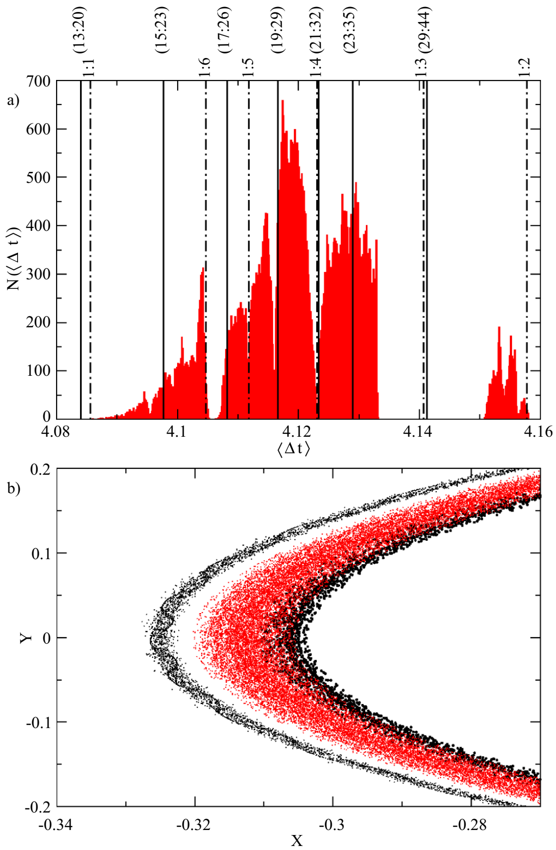


FIG. 3 (color online). Same as Fig. 1 for the billiard on an elliptic Kepler orbit with eccentricity $\varepsilon = 0.0001$. The resonances indicated correspond to $\varepsilon = 0$. In comparison to the circular case, the gap of the 1:3 SR is wider. This causes a true opening in the corresponding ring, thus forming a two-component ring. The black inner component of the ring corresponds to the region on the left of the 1:6 SR; the gray (red) is associated with the region between the 1:6 and 1:3 resonances. Note that the 1:6 resonance gap is not wide enough to have a projection with another (distinct) strand.

For more DOF the qualitative similarity of Fig. 3(a) and Fig. 1(a) is the first indication that universality may hold also for higher dimensions; yet, this issue remains open. Stability resonances are defined by a resonant condition on the stability exponents of the linearized dynamics around a central stable periodic orbit. These resonances, and not the mean-motion resonances, manifest locally as a reduction of the phase-space volume of the trapped trajectories due to nonlinear effects. While SR have a local manifestation, they have global consequences: Nonlinear and higher-dimensional effects (nonvanishing eccentricity of the disk's motion) lead to an effective separation of the trapping region in phase space [Fig. 3(a)]. In the context of planetary rings, if this separation is large enough, this yields a ring with two or more components or strands, which may entangle and form a braided ring. This provides a simple explanation of recent observations of planetary rings with multiple components [2]. These results should be interesting beyond the context of planetary rings, in systems where resonances and the phase-space structure are significant; examples include particle accelerators [11] and galactic dynamics [12].

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