Synchronization and Bistability of a Qubit Coupled to a Driven Dissipative Oscillator

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We study numerically the behavior of a qubit coupled to a quantum dissipative driven oscillator (resonator). Above a critical coupling strength the qubit rotations become synchronized with the oscillator phase. In the synchronized regime, at certain parameters, the qubit exhibits tunneling between two orientations with a macroscopic change of the number of photons in the resonator. The lifetimes in these metastable states can be enormously large. The synchronization leads to a drastic change of qubit radiation spectrum with the appearance of narrow lines corresponding to recently observed single artificial-atom lasing [O. Astafiev *et al.*, Nature (London) **449**, 588 (2007)].

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In physics there are not many simple quantum problems which are exactly solvable. Two of them are the monochromatically driven two-level atom (spin-half or qubit) [1] and quantum oscillator (unitary or dissipative) [2,3]. One atom weakly coupled to a field in a resonator is known as the Jaynes-Cummings model, which is also integrable [4–6]. At strong coupling the dynamics may become nontrivial with the emergence of classical [7,8] and quantum chaos [9], but in this case one should have many atoms which may absorb many photons. For one atom strongly coupled to a photonic field the dynamics is still rather simple due to a total energy balance [10].

This old problem recently regained a significant amount of interest due to the appearance of long living superconducting qubits [11] which can be strongly coupled to a microwave resonator [12–14]. There are also other possibilities of superconducting qubit coupling to a quantum oscillator [15,16]. The oscillator can be realized as a tank circuit tuned to the Rabi frequency [15] or as a current-biased dc SQUID [16] allowing efficient energy exchange with a qubit. Possibilities of qubit coupling to a cooled nanomechanical resonator are actively discussed [17,18] and coupling between micromechanical cantilever and atomic spin found impressive experimental implementations (see [19] and references therein). However, the most intriguing way seems to be the coupling with a microwave resonator where a lasing has been realized recently with 6–30 photons pumped into the resonator [14]. Contrary to recent interesting theoretical studies [19–21] where pumping is applied to a qubit, we concentrate here on the case where a monochromatic pumping is applied to a dissipative oscillator (resonator). Such an oscillator can also be viewed as a semiclassical detector which performs monitoring of a qubit. This continuous type of measurement is now actively discussed for superconducting qubits and other solid-state devices [22]. The continuous measurement of a superconducting qubit is realized in [15].

The Hamiltonian of our model reads

$$\hat{H} = \hbar\omega_0\hat{n} - \hbar\Omega\sigma_x/2 + g\hbar\omega_0(\hat{a} + \hat{a}^\dagger)\sigma_z + f\cos\omega t(\hat{a} + \hat{a}^\dagger), \tag{1}$$

where g is a dimensionless coupling constant, the driving force amplitude and frequency are $f = \hbar\lambda\sqrt{n_p}$ and ω, the oscillator frequency is ω_0, and $\hbar\Omega$ is the qubit energy spacing. We assume that the qubit lifetime is enormously long and that its dynamics is perturbed only by the coupling with the driven dissipative oscillator. The dissipation rate of the oscillator is λ and we assume the quality factor to be $Q = \omega_0/\lambda \sim 100$. The evolution of the whole system is described by the master equation for the density matrix $\hat{\rho}$ which has the standard form [3]

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \lambda\left(\hat{a}\,\hat{\rho}\,\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{\rho}\,\hat{a} - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a}\right). \tag{2}$$

The numerical simulations are done by direct solution of time evolution of $\hat{\rho}$ expanded in a finite basis of oscillator states n, by the state diffusion method [23], and by the method of quantum trajectories (QT) [24]. We ensured that these methods give the same results, but the majority of the data is obtained with QT, which we found to be more suitable for massive simulations. In addition, the QT have an advantage of providing a pictorial illustration of individual experimental runs. The numerical details are the same as in [25,26] and we use here up to $n = 70$ oscillator states, which gives good numerical convergence. Our results show that a coupling of two simple integrable models gives a nontrivial interesting behavior.

A typical example of QT is shown in Fig. 1. It shows two main properties of the evolution: the oscillator spends a very long time at some average level $\langle n\rangle = n_-$ and then jumps to another significantly different value n_+. At the same time the polarization vector of qubit $\vec{\xi}$ defined as $\vec{\xi} = \mathrm{Tr}(\hat{\rho}\,\vec{\sigma})$ also changes its orientation direction with a clear

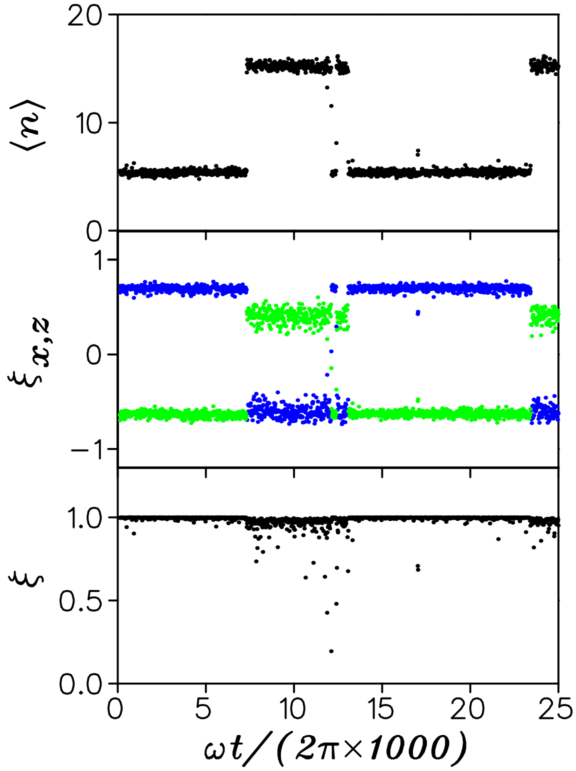

FIG. 1 (color online). Bistability of a qubit coupled to a driven oscillator with jumps between two metastable states. The top panel shows the average oscillator level number $\langle n \rangle$ as a function of time t at stroboscopic integer values $\omega t/2\pi$; the middle panel shows the qubit polarization vector components ξ_x [black (blue)] and ξ_z [gray (green)] at the same moments of time; the bottom panel shows the degree of qubit polarization ξ. Here the system parameters are $\lambda/\omega_0 = 0.02$, $\omega/\omega_0 = 1.01$, $\Omega/\omega_0 = 1.2$, $f = \hbar\lambda\sqrt{n_p}$, $n_p = 20$, and $g = 0.04$.

change of sign of ξ_x from $\xi_x > 0$ to $\xi_x < 0$. The time averaged values of $\xi_{y,z}$ are zero, but when they are taken at stroboscopic integer moments $\omega t/2\pi$ they also show transitions between two metastable states. The transition time is approximately $t_{\mathrm{tr}} \sim 1/\lambda$ being rather small compared to the lifetime in a metastable state. Inside such a state the degree of qubit polarization $\xi = |\vec{\xi}|$ is very close to unity showing that the qubit remains mainly in a pure state. The drops of ξ appear only during transitions between metastable states. Special checks show that an inversion of ξ_x by an additional pulse (e.g., from $\xi_x > 0$ to $\xi_x < 0$) produces a transition of oscillator to a corresponding state (from n_- to n_+) after time $t_{\mathrm{tr}} \sim 1/\lambda$. Thus we have here an interesting situation when a quantum flip of qubit produces a macroscopic change of a state of detector (oscillator) which is continuously coupled to a qubit (we checked that even larger variation $n_\pm \sim n_p$ is possible by taking $n_p = 40$). Inside a metastable state the coupling induces a *synchronization* of the qubit rotation phase with the oscillator phase which in its turn is fixed by the phase of the driving field. The synchronization is a universal phenomenon for classical dissipative systems [27]. It also exists for dissipative quantum systems at small effective values of $\hbar$ [26]. However, here we have a new unusual case of qubit synchronization when a semiclassical system produces synchronization of a pure quantum two-level system. Qualitatively, due to qubit coupling the oscillator becomes effectively nonlinear [6] and synchronized by dissipation with the driving phase, in its turn a Hamiltonian coupling of the oscillator with the qubit induces the qubit synchronization with oscillator phase [28].

The phenomenon of qubit synchronization is illustrated more clearly in Fig. 2. The top panels taken at integer values $\omega t/2\pi$ show the existence of two fixed points in the phase space of the oscillator (left) and the qubit (right) coupled by quantum tunneling (the angles are determined as $\xi_x = \xi\cos\theta$, $\xi_y = \xi\sin\theta\sin\phi$, $\xi_z = \xi\sin\theta\cos\phi$). A

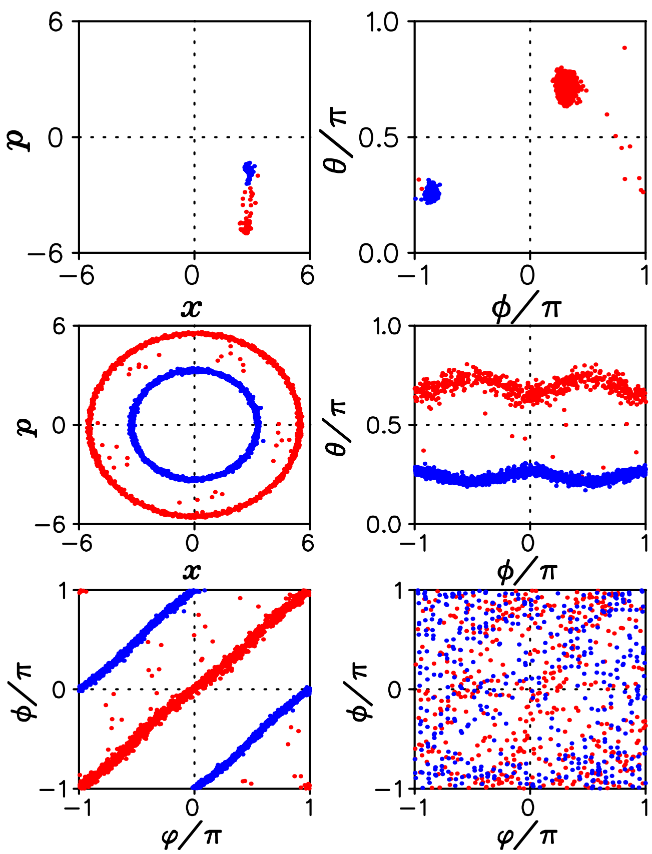

FIG. 2 (color online). Top panels: The Poincaré section taken at integer values of $\omega t/2\pi$ for an oscillator with $x = \langle(\hat{a} + \hat{a}^\dagger)/\sqrt{2}\rangle$, $p = \langle(\hat{a} - \hat{a}^\dagger)/\sqrt{2}i\rangle$ (left) and for qubit polarization with polarization angles (θ, ϕ) defined in text (right). Middle panels: Same quantities shown at irrational moments of $\omega t/2\pi$. Bottom panels: Qubit phase ϕ versus oscillator phase φ ($p/x = -\tan\varphi$) at time moments as in the middle panels for $g = 0.04$ (left) and $g = 0.004$ (right). Other parameters and the time interval are as in Fig. 1. The color of points is black (blue) for $\xi_x > 0$ and gray (red) for $\xi_x < 0$.

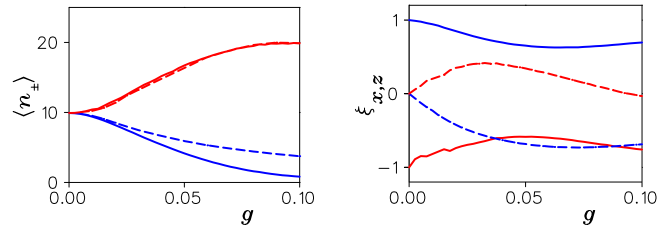

FIG. 3 (color online). Right: Dependence of average qubit polarization components ξ_x and ξ_z (full curves and dashed curves) on g, averaging is done over stroboscopic times (see Fig. 1) in the interval $100 \leq \omega t/2\pi \leq 2 \times 10^4$; color is fixed as in Fig. 2. Left: Dependence of average level of oscillator in two metastable states on coupling g, color is fixed by ξ_x sign on the right panel [gray (red) for large n_+ and black (blue) for small n_-]; average is done over the quantum state and stroboscopic times as in the right panel; dashed curves show theory dependence (see text). Two QT are used with initial value $\xi_x = \pm 1$. All parameters are as in Fig. 1 except g.

certain scattering of points in a spot of finite size should be attributed to quantum fluctuations. But the fact that on enormously long time (Fig. 1) the spot size remains finite clearly implies that the oscillator phase φ is locked with the driving phase ωt inducing the qubit synchronization with φ and ωt. The plot at t values incommensurate with $2\pi/\omega$ (middle panels) shows that in time the oscillator performs circle rotations in the (p, x) plane with frequency ω while qubit polarization rotates around the x axis with the same frequency. Quantum tunneling gives transitions between two metastable states. The synchronization of the qubit phase ϕ with the oscillator phase φ is clearly seen in the bottom left panel where points form two lines corresponding to two metastable states. This synchronization disappears below a certain critical coupling g_c where the points become scattered over the whole plane (bottom right panel). It is clear that quantum fluctuations destroy synchronization for $g < g_c$. Our data give $g_c \simeq 0.008$ for parameters of Fig. 1.

The variation of bistable states with coupling g is shown in Fig. 3. The difference between n_+ and n_- grows with g. It is striking that n_- may become close to zero. The direction of qubit polarization also changes in a smooth but nontrivial way. It is also important to note that according to our data the dispersion of the oscillator wave function in metastable states is compatible with the dispersion of a coherent state with $n_\pm$. This corresponds to a wave packet collapse induced by dissipation (see [25,26] and references therein). The dependence of $n_\pm$ on ω is shown in Fig. 4. A symmetric double peak structure is evident: for $\omega > \omega_0$ the metastable state with $\xi_x < 0$ has maximal n value, while for $\omega < \omega_0$ the state with maximal n has $\xi_x > 0$ (note color interchange). The peak width is approximately equal to the dissipation rate λ. With the increase of g their form becomes asymmetric indicating the importance of nonlinear effects. The splitting of peaks grows

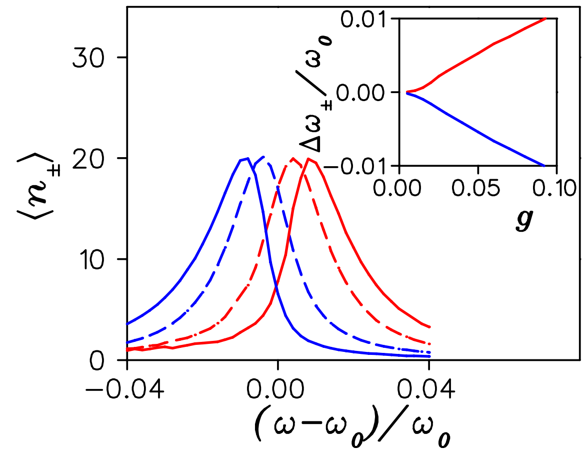

FIG. 4 (color online). Dependence of average level $n_\pm$ of oscillator in two metastable states on the driving frequency ω (average and color choice are the same as in the right panel of Fig. 3); coupling is $g = 0.04$ and $g = 0.08$ (dashed curves and full curves). Inset shows the variation of position of maximum at $\omega = \omega_\pm$ with coupling strength g, $\Delta\omega_\pm = \omega_\pm - \omega_0$. Other parameters are as in Fig. 1.

approximately linearly with g (inset of Fig. 4) and resembles the vacuum Rabi splitting effect [5]. The shift $\Delta\omega_\pm$ explains two states $n_\pm$ of driven oscillator well described by $n_\pm = n_p\lambda^2/[4(\omega - \omega_0 - \Delta\omega_\pm)^2 + \lambda^2]$ [see dashed curves in Fig. 3 (left) traced with numerical values of $\Delta\omega_\pm$ from Fig. 4 inset]. To estimate $\Delta\omega_\pm$ we note that the frequency of effective Rabi oscillations between quasidegenerate levels is $\Omega_R \approx g\omega_0\sqrt{n_\pm + 1}$ [4,6] that gives $\Delta\omega_\pm \approx d\Omega_R/dn \approx \pm g\omega_0/2\sqrt{n_\pm + 1}$ in good agreement with data of Fig. 4 for moderate g.

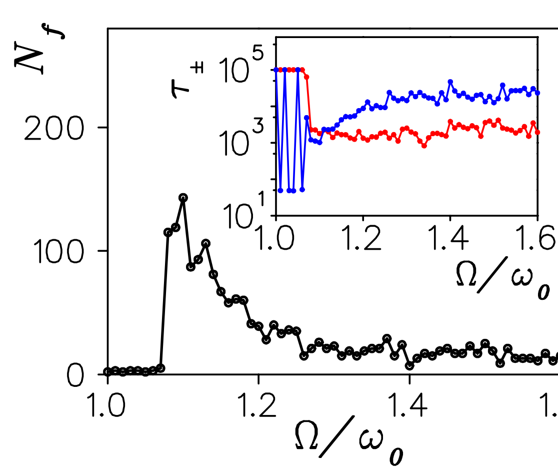

FIG. 5 (color online). Dependence of number of transitions N_f between metastable states on rescaled qubit frequency Ω/ω_0 for parameters of Fig. 1; N_f are computed along 2 QT of length 10^5 driving periods. Inset shows lifetime dependence on Ω/ω_0 for two metastable states [τ_+ for gray (red), τ_- for black (blue), $\tau_\pm$ are given in number of driving periods; color choice is as in Figs. 2 and 3].

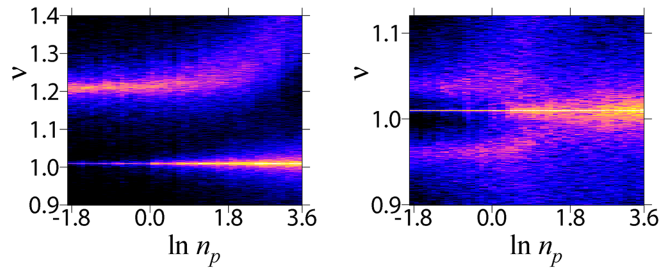

FIG. 6 (color online). Spectral density $S(\nu)$ of qubit radiation $\xi_z(t)$ as function of driving power n_p in the presence of phase noise in ϕ with diffusion rate $\eta = 0.004\omega_0$. Left: $\Omega/\omega_0 = 1.2$; right: $\Omega/\omega_0 = 1$. Other parameters are as in Fig. 1. Grayness (color) shows $S(\nu)$ in logarithmic scale [white (black) for maximal (zero)], ν is given in units of ω_0.

The number of transitions N_f between two metastable states is shown in Fig. 5. It has a pronounced peak at $\Omega \approx 1.1\omega_0$ that approximately corresponds to a resonance condition $\Omega - \omega \approx 2g\omega_0$. For $\Omega < 1.08\omega_0$ there is an abrupt drop of N_f and bistability becomes irregular, disappearing for certain Ω, but the synchronization still remains. The data show that for $\Omega > 1.1\omega_0$ the lifetimes of each state are rather different and enormously large, generally $\tau_- > \tau_+ \gg \omega_0/\lambda$.

The spectrum of qubit radiation $S(\nu)$, given by the spectrum of $\xi_z(t)$, in the presence of phase noise in ϕ is shown in Fig. 6 [$\xi_y(t)$ has a similar spectrum]. It confirms the main features discussed above: for $\Omega/\omega_0 = 1.2$ the growth of driving power n_p induces the synchronization of a qubit with radiation suppression at qubit frequency $\Omega = 1.2\omega_0$ and the appearance of a narrow line at $\nu = \omega$. The radiation field in the synchronized state is coherent and thus the system is lasing. For $\Omega = \omega_0$ the radiation spectrum $S(\nu)$ at $n_p < 1$ has two broad peaks at $\nu = \omega \pm g\omega_0$ corresponding to the vacuum Rabi splitting [5] (a narrow line from the driving source at $\nu = \omega$ is also visible in this case). At strong driving $n_p > 1$ the synchronization takes place with the appearance of one lasing line at $\nu = \omega$. For both values of Ω the transition to synchronization or lasing takes place at $n_p > n_{pl} \approx 2$. The spectrum $S(\nu)$ in Fig. 6 has close similarities with the spectrum observed recently in a single artificial-atom lasing [14] which appears at a similar threshold $n_{pl} \approx 1$. A shift related to splitting $\omega_\pm \approx g\omega_0/2\sqrt{n}$ is also seen experimentally, even if the vacuum Rabi splitting is not visible in Fig. 3(c) of [14]. Exact comparison requires much heavier numerical simulations since in [14] $Q \approx 10^4$ while we have $Q \sim 100$.

In summary, our studies show a nontrivial behavior of a rather simple model given by Eqs. (1) and (2). It is characterized by bistability and synchronization of a qubit induced by its coupling to a quantum driven dissipative oscillator. As for the vacuum Rabi splitting [5] it is important that the oscillator is quantum since the effect is absent for a classical dissipative oscillator with commuting $a, a^\dagger$ (1). A better understanding of the behavior discussed requires further studies [29]. Especially interesting is the analysis of long metastable lifetimes $\tau_\pm$ related to a macroscopic quantum tunneling. It is also important to study the effects of finite qubit lifetime and qubit pumping by quasiparticles injection in this regime.

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