INFRARED DIVERGENCE AND POSITRONIUM SPECTRUM*

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Recently, Lévy[1] pointed out that the exponential infrared factor which is involved in the cross section for electron scattering had exactly the form we would expect from the existence of Regge poles in the electron-positron channel. Furthermore, Lévy showed that this Regge trajectory gave to some extent the known positronium fine structure. However, in the infrared approximation it is not the elastic cross section but rather the total cross section including the emission of an arbitrary number of undetected photons that shows the characteristic energy dependence due to a Regge pole. Then, the identification of the Regge pole does not seem completely convincing. The purpose of this note is to give an alternative but less esoteric explanation as to why some information on the positronium spectrum can be derived from the infrared factor.

The amplitude for the elastic scattering of an electron by a potential involves a factor $\exp(\alpha B)$ due to the radiative correction by an arbitrary number of soft virtual photons,[2] where B is defined by

$$\alpha B=\frac{i\alpha}{4\pi^3}\int\frac{d^4k}{k^2-\lambda^2}\left[\frac{(2p-k)^2}{(k^2-2p\cdot k)^2}-\frac{(2p'-k)\cdot(2p-k)}{(k^2-2p'\cdot k)(k^2-2p\cdot k)}\right]. \tag{1}$$

Here λ is the photon mass, and p and p' denote the incident and outgoing electron momenta. αB is just the second-order radiative correction to a vertex, neglecting the so-called magnetic current of the electron, which does not give the infrared divergence. αB is a function of the invariant momentum transfer $t=(p'-p)^2$ and does not depend on the energy of scattering.[3] The factor $\exp(\alpha B)$ is related to the electron-positron scattering because αB has an imaginary part for $t\geqslant 4m^2$, corresponding to the scattering in the electron-positron channel. We obtain this from the second term of (1) where, replacing p' by $-p'$, the two electron propagators can be simultaneously singular for physical values of p and p'. Taking the center-of-mass system and writing $s=(p+p')^2=4E^2=4(p^2+m^2)$ instead of t, we obtain

$$\mathrm{Im}\,\alpha B=\frac{\alpha}{16\pi}\frac{p}{E}\int d\Omega_{p'}\frac{4E^2+(\vec{p}+\vec{p}\,')^2}{(\vec{p}\,'-\vec{p})^2+\lambda^2},\quad |\vec{p}\,'|=|\vec{p}|. \tag{2}$$

It will be noticed that the integrand is related to the one-photon exchange potential between two charged scalar particles on the energy shell,

$$V(\vec{p}\,',\vec{p})=-\frac{\alpha}{2\pi^2}\cdot\frac{1}{4E^2}\cdot\frac{4E^2+(\vec{p}+\vec{p}\,')^2}{(\vec{p}\,'-\vec{p})^2+\lambda^2}. \tag{3}$$

Performing the angular integration, we obtain, apart from a nonlogarithmic term,

$$\exp[i\,\mathrm{Im}\,\alpha B]=\exp[i\,\mathrm{Im}\gamma(s)\ln(2p/\lambda)], \tag{4}$$

where $\gamma(s)$ is the Regge trajectory given by Lévy[4] and is given by

$$\mathrm{Im}\gamma(s)=\alpha(s-2m^2)/[s(s-4m^2)]^{1/2}. \tag{5}$$

The logarithmic singularity in (4) comes from the singularity of the potential (3) for forward scattering.

Now we recognize that $\mathrm{Im}\,\alpha B$ gives an l-independent contribution to the phase shift for electron-positron scattering and that this large phase shift proportional to $\ln(2p/\lambda)$ is physically of the same origin as the familiar infinite Coulomb phase.[5] In fact, in the nonrelativistic limit, (4) reduces to the familiar Coulomb phase for a reduced mass $\frac{1}{2}m$,

$$\exp[i\alpha(m/2p)\ln(2p/\lambda)]. \tag{6}$$

Thus, whatever complicated interaction we may have between physical electrons and positrons, only the part of it which behaves like const/r at a large separation will be responsible for the factor (4). Conversely the factor (4) describes just the asymptotic behavior of the electron-positron system due to such a long-range force.

A simple and physical way to understand the relation between this Coulomb phase and the spectrum is to study a wave equation which reproduces the universal phase factor (4). As such an equation, we take

$$\left[\frac{E^2-m^2}{m}+\frac{\nabla^2}{m}+\frac{A}{r}+\tilde{V}(r)\right]\psi(\vec{r})=0, \tag{7}$$

where we assume $r\tilde{V}(r)\to 0$ as $r\to\infty$. As indicated later, this form of equation results from a relativistic two-body equation by an "asymptotic" reduction and differs from the nonrelativistic reduction by relativistic kinematical factors which we keep here. Equation (7) is equal to the Schrödinger equation for $r\to\infty$ if we replace A by α, and we should obtain (6) for this case. Hence to obtain (4), we have to choose

$$A=(2p/m)\,\mathrm{Im}\gamma(s)=\alpha(2E^2-m^2)/Em. \qquad (8)$$

A differs from α because of the inclusion of the transverse photon exchange as is clearly seen in (3) and because of the kinematical factor mentioned above. Corresponding to (6), the scattered wave behaves asymptotically like $r^{-1}\exp[ipr+i\alpha\times(m/2p)\ln(2pr)]$ in the nonrelativistic case. Then, replacing α by A, our Eq. (7) gives as the asymptotic scattered wave

$$r^{-1}\exp[ipr+i\,\mathrm{Im}\gamma(s)\ln(2pr)]. \qquad (9)$$

This factor multiplies the angular function, and therefore any partial wave behaves asymptotically in the same way. Now from (5), $i\,\mathrm{Im}\gamma(s)$ is itself a real analytic function of s. Then we can continue (9) into the positronium region $2m^2<s<4m^2$, where $i\,\mathrm{Im}\gamma(s)$ has a real positive value and the wave function behaves like

$$r^{i\,\mathrm{Im}\gamma(s)-1}\exp[-|p|r]. \qquad (10)$$

This must give the asymptotic radial wave function for the bound state when s is equal to a bound-state energy. However, the asymptotic behavior of a bound state cannot be determined from the $1/r$ potential only. It does depend on the short-range potential $\tilde{V}(r)$, which we do not know. Only when we neglect $\tilde{V}(r)$ do we know that a bound state behaves like $r^{n-1}\exp[-|p|r]$ for a principal quantum number n. The neglect of $\tilde{V}(r)$ may be justified if for large l the centrifugal barrier is much bigger than $\tilde{V}(r)$ near the origin. We then have

$$i\,\mathrm{Im}\gamma(s_n)=n. \qquad (11)$$

The spectrum we obtain from this equation must be a part of the real spectrum which we would get neglecting all short-range forces.[6] Equation (11) differs from the equation derived by Lévy,[7] $\gamma(s)=n$, but the difference in the resulting spectrum appears only from α^5 order.

Finally we mention that the wave Eq. (7) can be derived from the relativistic two-body equation with the potential (3),

$$2[E-(p^2+m^2)^{1/2}]\psi(\vec{p})=\int V(\vec{p},\vec{p}\,')\psi(\vec{p}\,')d^3p'$$

by a consistent reduction neglecting terms smaller than $1/r$ in the asymptotic region.

In summary we have shown that the infrared phase proportional to $\ln\lambda$ gives the asymptotic radial dependence of the electron-positron scattering, and that by matching it with an approximate asymptotic radial function of a bound state we obtain a relation which determines the spectrum.

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[3]The total cross section including the emission of real photons with the total maximum energy ϵ has a factor $\exp(2\alpha B+2\alpha\tilde{B})$, where $\tilde{B}$ comes from the real photon emission. αB contains a term $\gamma(t)\ln(m/\lambda)$, while $\alpha\tilde{B}$ depends on the reference system in which ϵ is defined, the laboratory system in this case, and hence on the energy of the system E and has a term $-\gamma(t)\ln(m\epsilon/\lambda E)$. The factor $\exp(2\alpha B+2\alpha\tilde{B})$ now gives $\exp[2\gamma(t)\ln(E/\epsilon)]$, which Lévy discussed.

[4]Apart from the radial quantum number plus one which Lévy subtracted rather arbitrarily, $\gamma(s)$ is defined by

$$\gamma(s)=\frac{\alpha}{\pi}s\int_0^1\frac{1+x^2}{4m^2-(1-x^2)s}dx,$$
$$=\frac{s}{\pi}\int_{4m^2}^{\infty}ds'\frac{\mathrm{Im}\gamma(s')}{s'(s'-s-i\epsilon)}.$$

[5]This was pointed out in reference 2.

[6]H. A. Bethe and E. E. Salpeter, Handbook of Physics (Academic Press, Inc., New York, 1957), Vol. 35, p. 204. The positronium spectrum up to α^4 order is given by

$$2E=2m-\frac{\alpha^2m}{4n^2}+\left[\frac{11}{64}\frac{1}{n^4}+\left(\epsilon_{lsJ}-\frac{1}{2l+1}\right)\frac{1}{2n^3}\right]\alpha^4m,$$

which is obtained from the Breit potential. Equation (11) gives only the $11/64n^4$ term of the fine structure, which we obtain by neglecting spin-orbit interaction and also short-range parts of the Breit potential.

[7]$i\,\mathrm{Im}\gamma(s)$, as a real analytic function of s, has the same imaginary part as $\gamma(s)$ defined in reference 4 on the cut $s=[4m^2,+\infty]$. However, $i\,\mathrm{Im}\gamma(s)$ has an additional branch cut due to $s^{1/2}$ in (5).