A. Saladin for help with the electrostatic separators.

[1]The fitting program was written by A. G. Wilson of the Rutherford Laboratory, National Institute for Research in Nuclear Science, Harwell, England. We are greatly indebted to Mr. Wilson for his help.

[2]The number rejected for this reason was about 12 % of the number finally accepted as three-pion states.

[3]This description in terms of l and L is correct only in the nonrelativistic limit, and so the use of this representation in the present experiment where the ρ meson has a momentum of 780 MeV/c is an approximation.

[4]The lowest ${}^1P_1$ state leading to ρ^0 production is (0,1), $J=1^+$. It has been discussed in connection with the ω^0 spin analysis [M. L. Stevenson, L. W. Alvarez, B. C. Maglić, and A. H. Rosenfeld, Phys. Rev. 125, 687 (1962)].

[5]It has been suggested [S. Glashow, Phys. Rev. Letters 7, 469 (1961); J. Bernstein and G. Feinberg, Brookhaven National Laboratory report BNL 6122, 1962 (unpublished)] that the rate of decay of ω^0 into $\pi^+ + \pi^-$ by an electromagnetic interaction may be appreciable. The reaction $p+\bar{p} \to \omega^0 + \pi^-$ may proceed from S states only through the ${}^3S_1$ channel of isotopic spin 1, and so any contribution would be added coherently to the direct ${}^3S_1$ $\rho^0 + \pi^0$ amplitude. In this case equality of the rates of production of $\rho^0\rho^+\rho^-$ would in general imply some contribution of charged ρ's from, say, the ${}^1S_0$ state unless the interference in the ${}^3S_1$ channel was so arranged as to lead to the equality. The angular distribution of the charged ρ decay gives no evidence for charged ρ production from the ${}^1S_0$ state. Similarly it is not possible to exclude P-state contribution completely since interference between several (L,l) waves may occur in such a way as to allow the effect observed.

[6]C. Bouchiat and G. Flamand, Nuovo Cimento 23, 13 (1962).

[7]The relation used is $\gamma = \Gamma_0(q_\gamma^2 + m_\pi^2)/2q_\gamma^3$. Formula (5a) is a P-wave effective range approximation [B. Lee and M. T. Vaughn, Phys. Rev. Letters 4, 578 (1962)].

[8]D. D. Carmony and R. T. Van de Walle, Phys. Rev. Letters 8, 73 (1962).

[9]The contamination from the pseudo-three-pion events is included in this background

ASYMPTOTIC PROPERTIES OF FIELDS AND SPACE-TIMES*

Roger Penrose
Department of Mathematics, King's College, London, England
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This note outlines a new technique for studying asymptotic questions in (special or) general relativity whereby several new results are obtained. The questions dealt with here are the following: (1) a geometrical definition of asymptotic flatness, (2) covariant definitions of incoming and outgoing gravitational (and other) radiation fields, (3) simple deduction of detailed asymptotic behavior of the Riemann tensor (and other fields)—the "peeling off" property,[1-3] (4) definitions of total energy-momentum and its loss by radiation, with conservation laws, (5) unification of finite and asymptotic versions of the characteristic initial value problem,[2-7] and (6) geometrical derivation of the Bondi-Metzner-Sachs asymptotic symmetry group.[2,4,8] A longer term aim of this approach is for a covariant S-matrix theory incorporating gravitation.

The basic idea is as follows. Asymptotic questions are those relating to the "neighborhood of infinity." From the point of view of the metric structure of space-time, however, there is no such thing as a point at infinity, since such a point would be an infinite distance from its neighbors. But if we think only in terms of conformal structure of space-time (only ratios of neighboring infinitesimal distances are to have significance), then infinity can be treated as though it were simply an ordinary three-dimensional boundary $\mathscr{I}$ to a "finite" four-dimensional conformal region $\mathfrak{M}$. In fact, we may envisage a new "unphysical" metric $g_{\mu\nu}$ assigned (but perhaps only locally) to space-time, which is conformal to the original physical metric $\tilde{g}_{\mu\nu}$ with

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu},$$

and according to which "infinity" is now finite and in most places regular. The boundary $\mathscr{I}$ of $\mathfrak{M}$ is given by $\Omega = 0$, with $\Omega_{;\mu} \neq 0$. ("Infinity" is given finite coordinate values, so $\tilde{g}_{\mu\nu}$ becomes infinite there.)

All covariant derivatives used here will be carried out according to the unphysical $g_{\mu\nu}$ metric so that properties of $\mathscr{I}$ and its neighborhood in $\mathfrak{M}$ may be studied. Any such property which is conformally invariant will then be a physically meaningful asymptotic property in the original space-time $\tilde{\mathfrak{M}}$. The basic reason for success with this approach is the conformal invariance of the zero rest-mass equations for each spin.[9,10] In particular, the spin-2 equation $K_{\mu\nu[\rho\sigma;\tau]}=0$ is preserved with

$$\tilde{K}^{\mu\nu}{}_{\rho\sigma}=\Omega^3 K^{\mu\nu}{}_{\rho\sigma}, \qquad (1)$$

$K_{\mu\nu\rho\sigma}$ having the symmetries and trace-free properties of an empty-space Riemann tensor (or Weyl tensor).

The physical curvature quantities are obtainable from the unphysical ones by[11]

$$\tilde{S}^{\mu}{}_{\nu}=\Omega^2 S^{\mu}{}_{\nu}+2\Omega\Omega^{;\mu}{}_{;\nu}-\Omega^{;\rho}\Omega_{;\rho}\delta^{\mu}{}_{\nu}, \qquad (2)$$

$$\tilde{C}^{\mu\nu}{}_{\rho\sigma}=\Omega^2 C^{\mu\nu}{}_{\rho\sigma}, \qquad (3)$$

where $S_{\mu\nu}=\frac{1}{6}Rg_{\mu\nu}-R_{\mu\nu}$, and $C_{\mu\nu\rho\sigma}$ is Weyl's tensor, comprising exactly the curvature information not contained in $S_{\mu\nu}$.

Assume there is no λ term in Einstein's equations, and that all fields except gravitation, electromagnetism, and neutrinos are restricted to a bounded region of space. Then $\tilde{R}=0$ in the neighborhood of $\mathscr{I}$, whence (2) implies $\Omega^{;\rho}\Omega_{;\rho}=0$ on $\mathscr{I}$. Thus, $\mathscr{I}$ is a null hypersurface. Consider, in particular, Minkowski space.[10] Here $\mathscr{I}$ can be separated into five distinguishable disjoint parts, namely, three points I^-, I^0, and I^+ representing, respectively, the past, spatial, and future infinities, and two null hypersurfaces $\mathscr{I}^-$ and $\mathscr{I}^+$ representing the past and future null infinities. Each of $\mathscr{I}^-$ and $\mathscr{I}^+$ has the topology of a three-dimensional cylinder ($S^2\times E^1$), bounded by I^- and I^0, and I^0 and I^+, respectively, at its "past" and "future" (see Fig. 1). Furthermore, any null geodesic in $\mathfrak{M}$,

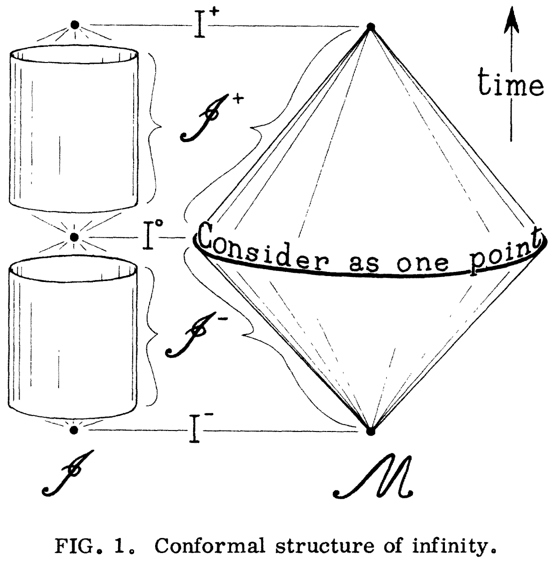

FIG. 1. Conformal structure of infinity.

not on $\mathscr{I}$, originates at a point of $\mathscr{I}^-$ and terminates on $\mathscr{I}^+$. An examination of Schwarzschild's solution and of the more general radiative metrics[2,4,5] suggests as a suitable global definition of asymptotic flatness[12] for $\tilde{\mathfrak{M}}$, the fact that $\mathfrak{M}$ exists, with the structure as defined above, which is regular everywhere (say C^5) up to and including its boundary $\mathscr{I}$ except at I^-, I^0, and I^+.

Many metrics are consistent with a given such conformal structure for $\mathfrak{M}$, and a choice for which $\Omega_{;\mu}=-n_\mu$ with $n_\mu n^\mu=0$ is always locally possible. For simplicity, such a choice will be made here together with a physically reasonable (but probably unnecessary) assumption that $\tilde{R}^{\mu}{}_{\nu}=O(\Omega^4)$ in the neighborhood of $\mathscr{I}^+$ (say). Then (2) implies $\Omega_{;\mu;\nu}=0$ on $\mathscr{I}^+$, whence the shear and divergence[1,3] of $\mathscr{I}^+$ vanish, this vanishing of shear being a conformally invariant property. Differentiating, $\Omega_{;\mu;\nu;\rho}=\frac{1}{2}S_{\mu\nu}n_\rho$, $\Omega_{;\mu;\nu;\rho;\sigma}=S_{\mu\nu;(\rho}n_{\sigma)}$, $n^\mu S_{\mu\nu}=0$, $n^\mu S_{\mu\nu;\rho}=0$ on $\mathscr{I}^+$, whence Ricci identities give $C_{\mu\nu\rho\sigma}n^\sigma=0$, $C_{\mu\nu\rho\sigma}{}^{;\sigma}=0$ on $\mathscr{I}^+$. This and the topology of $\mathscr{I}$ imply $C_{\mu\nu\rho\sigma}=0$ on $\mathscr{I}$. Hence the gravitational spin-2 field $K_{\mu\nu\rho\sigma}$ [see (1), (3)] can be defined continuously throughout $\mathfrak{M}$ by $\Omega K_{\mu\nu\rho\sigma}=C_{\mu\nu\rho\sigma}$ with $-K_{\mu\nu\rho\sigma}n_\tau=C_{\mu\nu\rho\sigma;\tau}$ on $\mathscr{I}$. Completing a null tetrad with m_μ and $\bar{m}^\mu$ complex, l_μ real satisfying $m_\mu m^\mu=l_\mu l^\mu=l_\mu m^\mu=n_\mu m^\mu=1-l_\mu n^\mu=1+m_\mu\bar{m}^\mu=0$, we can define the outgoing gravitational radiation field as the complex tetrad component[3] $\Psi_4=K_{\mu\nu\rho\sigma}n^\mu\times m^\nu n^\rho m^\sigma$ on $\mathscr{I}^+$. This is uniquely determined, except for scaling, by the conformal geometry of $\mathfrak{M}$. The incoming field is defined similarly on $\mathscr{I}^-$. (The electromagnetic radiation field is $F_{\mu\nu}n^\mu m^\nu$; the neutrino radiation field is also simply defined.)

Let l be a null geodesic meeting $\mathscr{I}^+$ at P. Then, in the neighborhood of P, $g_{\mu\nu}$ may be chosen so that $-\Omega$ and $\tilde{r}=\Omega^{-1}$ are affine parameters on l according to $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, respectively. Choosing l^μ tangent to l, and m^μ suitably, the "tetrads" (l^μ, m^μ, n^μ), $(\tilde{l}^\mu=\tilde{r}^2 l^\mu, \tilde{m}^\mu=\tilde{r}m^\mu, \tilde{n}^\mu=n^\mu)$ are transported parallelly along l according to $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, respectively. It follows at once from continuity of $K_{\mu\nu\rho\sigma}$ that $\tilde{\Psi}_4=O(\tilde{r}^{-1})$ and that the remaining complex tetrad components $\tilde{\Psi}_i$ of $\tilde{K}_{\mu\nu\rho\sigma}$ $(i=0,\cdots,3)$, appropriately ordered, are $O(\tilde{r}^{-5-i})$. This is the "peeling off" property of the Riemann tensor.[1-3] A similar result holds for the electromagnetic and neutrino fields.

Further specialize $g_{\mu\nu}$ so that $R=12$ on $\mathscr{I}^+$. Consider any spanning hypersurface $\mathcal{S}$ which meets $\mathscr{I}$ in a spherelike region S, containing one point of each generator of $\mathscr{I}^+$. (This implies $\mathcal{S}$ is asymptotically null in $\tilde{\mathfrak{M}}$.) Then the total energy-momentum intercepted by $\mathcal{S}$ is (compare references 2, 4, and 5)

$$P(W) = (8\pi)^{-1/2}\int(\sigma N - \Psi_2)WdS,$$

where σ is the shear[1,3] of the null hypersurface meeting $\mathcal{I}$ in S and whose tangent vector l_μ serves to define[3] m_μ, Ψ_2 and where

$$N = -\tfrac{1}{2}R_{\mu\nu}\bar{m}^\mu\bar{m}^\mu$$

is essentially Bondi's "news function." It is also the derivative in the l_μ direction at $\mathcal{I}^+$ of the shear of $\Omega = \text{const}$ and it has a conformally invariant interpretation. W is a real weighting factor satisfying $1 - W^2 = W_{;\mu;\nu}m^\mu\bar{m}^\nu W - |W_{;\mu}m^\mu|^2$ and has four degrees of freedom (corresponding to different possible "time axes"), one choice ("energy") being $W=1$. The others are generated by the different permissible choices of metric $g_{\mu\nu}$, and with appropriate interpretations, $P(W)$ behaves as a 4-vector. Taking the difference between $P(W)$ and the corresponding value for a "later" hypersurface leads to a conservation law with the definition of gravitational energy-momentum flux across $\mathcal{I}^+$ as $N\bar{N}$. (The electromagnetic energy-momentum flux is $|F_{\mu\nu}m^\mu n^\nu|^2$, and correspondingly for the neutrino field.)

Initial data for gravitation may be specified on $\mathcal{S}$, and N specified[2,4,5] on the part of $\mathcal{I}^+$ "below" S. Essentially equivalent, however, is to use $\Psi_4 = N_{;\mu}n^\mu$ on $\mathcal{I}^+$ and, if $\mathcal{S}$ is null, to use Ψ_0 on $\mathcal{S}$ (and complete the data by giving certain quantities on S). The exact analogy between Ψ_4 on $\mathcal{I}^+$ and Ψ_0 on $\mathcal{S}$ leads to a unification of the finite[6,7] and asymptotic[2,4,5] versions of the characteristic initial value problem.

The Bondi-Metzner-Sachs[2,4,8] asymptotic symmetry group for general relativity can be interpreted as a group of conformal self-transformations of the three-dimensional manifold $\mathcal{I}^+$ (or $\mathcal{I}^-$). Conformal transformations always preserve finite angles, but there are also the null angles on $\mathcal{I}^+$ between tangent vectors of which n^μ is a linear combination. Parallel transport establishes an equivalence relation between null angles which turns out to be independent of the choice of $g_{\mu\nu}$. The required group now consists of the self-transformations of $\mathcal{I}^+$ (or $\mathcal{I}^-$) which preserve both angles and null angles (and do not reverse time-sense). If any of I^-, I^0, and I^+ were nonsingular, then the inhomogeneous Lorentz subgroup could be singled out, but this is not generally the case.[13]

A full account of these results will be published elsewhere.

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