# SELF-FIELD LIMITING OF JOSEPHSON TUNNELING OF SUPERCONDUCTING ELECTRON PAIRS*

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Josephson[1] has pointed out that pairs of superconducting electrons can tunnel through a junction consisting of an insulating barrier separating two superconductors. A current is thereby produced which flows through the barrier at zero potential difference. Anderson (reference 1, footnote 8) has remarked that external magnetic fields are detrimental to this effect. The purpose of the present note is to extend Anderson's observation to the self magnetic fields which are necessarily set up by the Josephson current itself, even in the absence of an external laboratory field. The result of the inclusion of the self fields is that in a junction whose smallest diameter exceeds a certain penetration distance $\lambda_J$ (determined mainly by the thickness of the barrier), the Josephson current arises only from a fringe of width $\lambda_J$ at the edges of the junction. For a junction of such dimensions, the interior does not contribute to the Josephson effect. This reduces the magnitude of the maximum Josephson current, which is proportional to the circumference rather than the cross-sectional area for very wide junctions. For a typical junction, such as that reported by Anderson and Rowell,[2] the value of $\lambda_J$ can be estimated to be approximately one half a millimeter, or of roughly the same size as the junction.

It is useful first to present a simplified derivation of Josephson's equations[3] which is based on analogy with the tight-binding approximation to the electron energy bands of a one-dimensional periodic potential. It requires no energy to transfer $\nu$ electron pairs from a superconductor on one side of the barrier to the superconductor on the other side, just as an electron can be translated by $\nu$ lattice spacings with no expenditure of work. If the individual atomic wave functions are $\Phi_\nu$, then the degeneracy will be split by the combination

$$\Psi_\alpha = \sum_\nu e^{i\alpha\nu}\Phi_\nu, \tag{1}$$

where the crystal momentum $\hbar\alpha$ is canonically conjugate to the number variable $\nu$. In the tight-binding approximation, the energy is ($H^{(2)}$ is the Hamiltonian to 2nd order in electron transfer)

$$E(\alpha) = (\Psi_\alpha, H^{(2)}\Psi_\alpha)/(\Psi_\alpha, \Psi_\alpha) = -\tfrac{1}{2}\hbar J_1 \cos\alpha, \tag{2}$$

where the bandwidth $\hbar J_1$ is four times the off-diagonal matrix element of the Hamiltonian between states $\nu$ and $\nu+1$. Hamilton's equations of motion for the expectation values of the two canonically conjugate variables are

$$\frac{d\langle\nu\rangle}{dt} = \left\langle \frac{\delta E(\alpha)}{\delta(\hbar\alpha)} \right\rangle = \tfrac{1}{2}J_1\langle \sin\alpha\rangle, \tag{3}$$

$$d(\hbar\langle\alpha\rangle)/dt = 2eV(t). \tag{4}$$

$2eV(t)$ is the potential energy difference at time $t$ between states $\nu+1$ and $\nu$, corresponding to an actual scalar electric potential difference of $V(t)$ in the superconductor problem. If the expectation value of $\sin\alpha$ can be replaced by the sine of $\langle\alpha\rangle$, as may be justified for a wave packet in $\nu$ space, it follows that the electron current which flows across the barrier is twice the rate of the transfer of pairs, or

$$J(t) = 2d\langle\nu\rangle/dt = J_1 \sin[2e\hbar^{-1}\int^t V(t')dt']. \tag{5}$$

The self-field limiting of the Josephson current can be computed with the geometry shown in Fig. 1. Two identical semi-infinite superconductors are separated by a thin space perpendicular to the $z$ axis. The right half of this space ($x>0$) is completely impenetrable to electrons, while the left half ($x<0$) permits the tunneling of super-

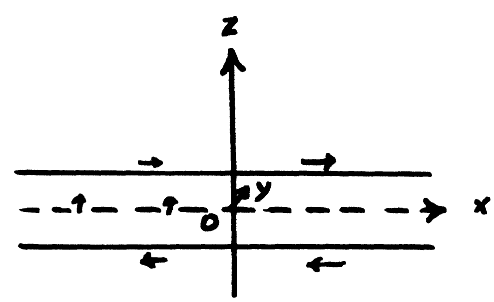


FIG. 1. Flow of Josephson tunneling current of superconducting electron pairs. The coherent tunneling in the $z$ direction is imagined to occur only for negative $x$ between the two semi-infinite superconductors (positive and negative $z$). By the Meissner effect, the accumulated current carried off and supplied by the bulk superconductors must be confined to their skin-depth regions. The resulting changes in the phase of the pair wave function limits the Josephson effect to the edge of the tunneling junction.

conducting electron pairs. A surface current $J_x(x)$ flows toward $x=+\infty$ along the upper superconductor and in the opposite direction along the surface of the lower superconductor. By conservation of particles, the tunneling current per unit area is

$$J_z(x)=J_x'(x), \tag{6}$$

where the prime indicates differentiation. By Ampere's law we have a magnetic field at $z=0$ (between the superconductors) of

$$B_y(x)=-(4\pi e/c)J_x(x), \tag{7}$$

where the $y$ direction of the Cartesian coordinates is into the paper. If we choose a gauge for the vector potential such that its $x$ component vanishes at $z=0$, then we have in the interior of the upper superconductor ("Meissner region" beyond the skin depth)

$$A_x(x)=-(4\pi\lambda e/c)J_x(x), \tag{8}$$

where $\lambda$ is the static penetration depth of the superconductor. Now if the phase of the electron pairs in the upper superconductor relative to those in the lower superconductor is $\alpha(x)$, we require for zero current in the Meissner region

$$0=\tfrac{1}{2}\hbar\alpha'(x)+(2e/c)A_x(x), \tag{9}$$

which gives, by substituting from Eq. (8),

$$\alpha'(x)=(16\pi\lambda e^2/\hbar c^2)J_x(x). \tag{10}$$

Differentiating and substituting from Eqs. (6) and (3) yield

$$\alpha''(x)=\lambda_J^{-2}\sin\alpha(x), \tag{11}$$

where the "Josephson penetration depth" is defined by

$$\lambda_J=(16\pi\lambda e^2J_1/\hbar c^2)^{-1/2}. \tag{12}$$

As $J_1$ is proportional to the probability density for finding an electron pair in the middle of the barrier, Eq. (12) is reminiscent of the dependence of the ordinary London penetration depth on the reciprocal square root of the superfluid density. It should be noted that here we are treating the Josephson effect as a current per unit area which depends upon the local phase difference at any point across the barrier. It may also be remarked that Eq. (12) may be inverted to express the Josephson tunneling parameter in terms of $\lambda$ and $\lambda_J$ which may be regarded as the basic parameters which characterize any given junction.[4]

Equation (11) can be solved for a junction of infinite extent with the boundary condition $\alpha(-\infty)=0$ to yield (for $x<0$)

$$\alpha(x)=2\sin^{-1}\operatorname{sech}[(x_0-x)/\lambda_J] \tag{13}$$

and

$$\alpha'(x)=2\lambda_J^{-1}\operatorname{sech}[(x_0-x)/\lambda_J]. \tag{14}$$

The constant of integration $x_0$ fixes the total tunneling current of the junction. Maximum current is obtained by setting $x_0=0$, giving

$$J_x^{\max}(0)=2\lambda_J J_1. \tag{15}$$

Thus the magnetic field set up by the Josephson current renders ineffective all of the junction except essentially that within a distance of $2\lambda_J$ of the edge.

By Eq. (7), fixing the surface current and fixing the magnetic field are completely equivalent. Hence the critical field of the junction is

$$H_c'=\phi_0/(2\pi\lambda\lambda_J), \tag{16}$$

which corresponds to the introduction of one half of the fundamental flux unit $\phi_0=2\times10^{-7}$ gauss cm$^2$ into the tunneling region. This equation can alternatively be written in the form

$$\tfrac{1}{2}\hbar J_1=\lambda H_c'^2/8\pi. \tag{17}$$

Thus the magnetic pressure, acting over an effective breadth of $\lambda$, cannot be held in check if it exceeds the coherence energy per unit area of the junction in its ground state.[5]

The numerical values of the parameters can be estimated for the junction studied by Anderson and Rowell.[2] $J_1$ is of the order[6] of 4 mA for the entire junction; per unit area, $J_1\approx10^{19}$ sec$^{-1}$ cm$^{-2}$, from which Eq. (17) yields $H_c'\approx0.2$ gauss. (Here we have taken $\lambda\approx400$ Å as a reasonable average for Pb and Sn.) Equation (16) then gives $\lambda_J\approx0.05$ cm, which is twice the width of their junction. The maximum surface current $J_x^{\max}$ flowing across the edges of an infinite ribbon junction of finite width $2w$ can be computed from Eq. (11). The problem is mathematically equivalent to finding the optimal initial position of a pendulum such that upon being released in a gravitational field it acquires the maximum amount of kinetic energy in a given amount of time. The result is plotted in Fig. 2 as $J_x^{\max}/wJ_1$ vs $w/\lambda_J$. It will be noted

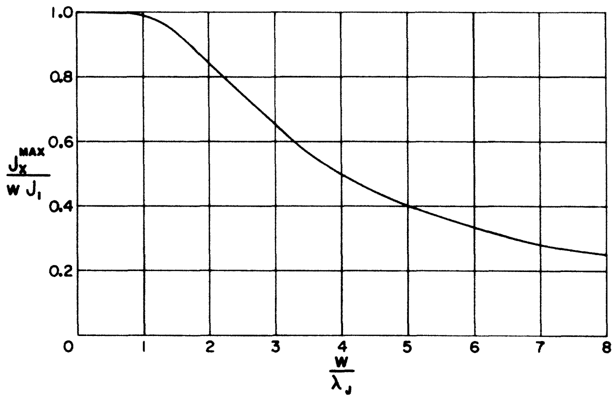


FIG. 2. Reduction factor for the self-field limiting of the Josephson effect. $wJ_1$ is the uncorrected Josephson current flowing off the edge of a ribbon-shaped junction of half-width $w$. The actual maximum current $J_x^{\max}$ is smaller and approaches the constant $2\lambda_J J_1$ when $w$ becomes larger than the Josephson penetration depth $\lambda_J$.

that although not much self-field limiting of the Josephson current would be expected for the Anderson-Rowell junction ($w/\lambda_J \approx \frac{1}{4}$), it should not be difficult to make junctions which would exhibit self-field limiting.

Finally we remark that the limit $w/\lambda_J \to \infty$ studied above makes it clear that there can be no one-dimensional model for the Josephson effect. For in such a case there would be no magnetic fields but only longitudinal electric fields and currents. Just as in the gauge-invariant treatment of the wave-number dependent dielectric constant of a bulk superconductor,[7] when collective effects are adequately taken into account, longitudinal electric fields cannot produce persistent supercurrents flowing across the barrier, but only transient effects resulting eventually in static screening. For the actual dc Josephson supercurrent to flow, magnetic fields must be present. These would be carried off by the edges and become infinitely far removed in a one-dimensional model.

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[1]B. D. Josephson, Phys. Letters 1, 251 (1962).

[2]P. W. Anderson and J. M. Rowell, Phys. Rev. Letters 10, 230 (1963).

[3]This derivation was obtained by one of the authors (R.A.F.) on summer leave at the University of California and supported in part by the National Science Foundation and by the Union Carbide Corporation.

[4]The self-field limiting has also been studied by Josephson and he has obtained Eq. (12) for the characteristic penetration distance of the junction [Ph.D. thesis, Trinity College, Cambridge University, Cambridge, Massachusetts (unpublished)].

[5]This result is a quantitative form of a remark by Anderson and Rowell (reference 2).

[6]For identical superconductors, we find $J_1 = \pi\Delta/2R$, where $\Delta$ is one half of the energy gap and $R$ is the resistance of a square centimeter of the junction in the normal state. This result for $J_1$ is a factor of four smaller than reported by Josephson (reference 1). It is interesting to note that the Josephson coherence energy $-J_{1/2}$ of the ground state of the junction is exactly canceled by a nonphase-dependent suppression resulting from the energy gap of the (negative) tunneling self-energy.

[7]R. E. Prange, Phys. Rev. 129, 2495 (1963).

## ULTRASONIC AMPLIFICATION IN BISMUTH

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Theoretical papers by Dumke and Haering,[1] Hopfield,[2] and Eckstein[3] indicate that under certain conditions, ultrasonic amplification in bismuth might be obtained similar to that observed in CdS.[4] We should like to report preliminary results which indicate that we have experimentally observed a substantial increase in the amplitude of sound waves propagated through bismuth in the presence of applied electric and magnetic fields. We have also applied the electric and magnetic fields, with no input signal, and observed a build-up of acoustic oscillations, indicating a net acoustic gain in the sample.

In the present experiment a sound pulse is propagated in the direction normal to the plane of the bismuth sample. Simultaneously, a current pulse is applied to the sample. A constant magnetic field is applied in the plane of the sample (i.e., perpendicular to the sound propagation direction), making an angle $\theta$ with the current direction. For the results to date, the sound propagation direction is bisectrix, the sound frequency is about 15 Mc/sec, the current is in the binary direction, and the magnetic field direction is varied between