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[4]The Columbia-Rutgers collaboration [C. Alff, D. Berley, D. Colley, N. Gelfand, U. Nauenberg, D. Miller, J. Schultz, J. Steinberger, T. H. Tan, H. Brugger, P. Kramer, and R. Plano, Phys. Rev. Letters 9, 325 (1962)] obtained $R < 0.02$ in the process $\pi^+ + p \to \pi^+ + p + \pi^+ + \pi^-$ at 2.3-2.9 BeV/c. They suggest that destructive interference may be responsible for the appearance of a dip rather than a peak at the ω mass.

[5]These branching-ratio limits rest on the assumption that some other resonance (so far unknown) decaying into two charged pions does not also peak near 790 MeV.

NEW UPPER BOUND FOR THE HIGH-ENERGY SCATTERING AMPLITUDE AT FIXED ANGLE

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(Received 26 March 1963)

The requirements of unitarity and analyticity seem to impose rather strong constraints on the growth of scattering amplitude at high energy. The first significant result in this respect was that of Froissart[1] who obtained upper bounds for the high-energy scattering amplitude from Mandelstam representation and unitarity. He found that the scattering amplitude $f(s, \cos\theta)$ (for scalar particles of equal mass) satisfies the inequalities

$$|f(s, \cos\theta)| < C_1 s (\log s)^2, \quad \text{for } \theta = 0 \text{ or } \pi, \qquad (1)$$

$$|f(s, \cos\theta)| < C_2 s^{3/4} (\log s)^{3/2}, \quad \text{for } \theta \neq 0 \text{ or } \pi, \qquad (2)$$

for very large s, where s and θ are the square of the total energy and the scattering angle in the center-of-mass system. f is normalized here in a relativistic way, so that

$$\sigma_{\text{tot}} \sim \frac{1}{s} \operatorname{Im} f(s, 1), \quad \frac{d\sigma_{\text{el}}}{d\Omega} \sim \frac{1}{s} |f|^2.$$

It was recognized later by one of us[2] that it is not necessary to make use of the full analyticity assumed in the Mandelstam representation to obtain the bounds (1) and (2). It is sufficient to assume that $f(s, \cos\theta)$ be analytic in an ellipse E_ρ in complex $\cos\theta$ plane, with foci at +1 and -1 and semimajor axis of length $\rho \equiv 1 + \alpha/k^2$ (α a positive constant, k the center-of-mass momentum), and that f be uniformly bounded in this ellipse by some power of s.[3]

We may then ask the following questions:

(i) Is it possible, with the weak assumptions just mentioned, to improve the bounds (1) and (2)?

(ii) Will it be possible to improve them if more analyticity is assumed?

As to the first question, it is easy to see that the answer is negative: It is, namely, possible to find counter examples.

The purpose of this note is to give a partial answer to the second question. We shall show that, if $f(s, \cos\theta)$ is analytic and uniformly bounded by a power of s in a domain D_s defined below, it is, in fact, possible to improve bound (2) and replace it by

$$|f(s, \cos\theta)| < C_3 (\log s)^{3/2}, \quad \text{for } \theta \neq 0 \text{ or } \pi. \qquad (3)$$

From this we obtain for the elastic differential cross section a bound

$$\frac{d\sigma_{\text{el}}}{d\Omega} < C_4 \frac{(\log s)^3}{s}, \qquad (4)$$

which decreases rapidly as s increases. [In contrast (2) gives a bound which increases with s.] So far we have not succeeded in improving the forward-backward bound (1). We do not know whether this improvement is possible under our assumptions.

In the following we put $\cos\theta = z$.

We assume that $f(s, z)$ is analytic and bounded by s^N (N is independent of s and z) in a domain D_s of the z plane [see Fig. 1(a)]. D_s is an intersection of a domain containing the ellipse E_ρ [shown by a dotted curve in Fig. 1(a)] and the z plane with cuts from $-\infty$ to $-\rho$ and ρ to ∞. The shape of D_s depends on s. Let us assume, however, that the distance between the boundaries of D_s and E_ρ is larger than some positive number

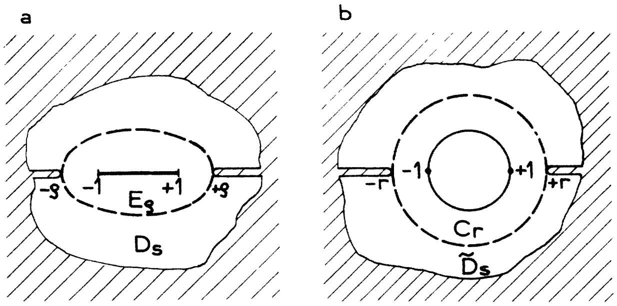

FIG. 1. The domains D_S and $\tilde{D}_S$ in the complex z plane.

for all values of s (except in the neighborhood of $z=\pm 1$ where the distance is at most of the order of $s^{-1/2}$). These conditions are, of course, satisfied if one assumes the validity of Mandelstam representation. However, one must notice that (i) we do not make use of analyticity with respect to s; (ii) the domain D_S in z may not be as big at the cut plane.

We want to derive a bound for $f(s,z)$ for z real, $|z|\leq 1$. Inside of the ellipse E_ρ, we can represent $f(s,z)$ by a Legendre series

$$f(s,z)=\frac{\sqrt{s}}{2k}\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(z). \tag{5}$$

We have

$$\limsup_{l\to\infty}|a_l|^{1/l}=\rho-(\rho^2-1)^{1/2}. \tag{6}$$

The unitarity condition $\mathrm{Im}\,a_l\geq |a_l|^2$, or somewhat weaker inequality

$$|a_l|\leq 1,\quad l=0,1,2,\cdots, \tag{7}$$

is more easily applicable to the function $g(s,z)$, defined by the power series

$$g(s,z)=\frac{\sqrt{s}}{2k}\sum_{l=0}^{\infty}(2l+1)a_l(s)z^l \tag{8}$$

in the circle C_r: $|z|<r=\rho+(\rho^2-1)^{1/2}(>1)$. Namely, (7) implies that (8) converges in $|z|<1$, and that

$$\frac{2k}{\sqrt{s}}|g(s,z)|\leq\frac{1+|z|}{(1-|z|)^2}<\frac{2}{(1-|z|)^2},\quad \text{for } |z|<1. \tag{9}$$

As s goes to infinity the ellipse E_ρ collapses to the real segment $(-1,+1)$ as is shown in Fig. 1(a), whereas C_r converges to a unit circle [see Fig. 1(b)]. Thus the analyticity domain for $g(s,z)$ has a finite size for all values of s. This makes $g(s,z)$ convenient for the study of high-energy behavior.

Making use of the formula[4]

$$P_l(z)=\frac{1}{\pi}\int_0^{\pi}[z+(z^2-1)^{1/2}\cos t]^l dt$$

and of the generating function of P_l's, together with their orthogonality properties, we find the following relations between f and g:

$$f(s,z)$$

$$\left.\begin{aligned}&=\frac{1}{\pi}\int_0^{\pi}g[s,z+(z^2-1)^{1/2}\cos t]dt &(10a)\\ &=\frac{1}{\pi i}\int_{z-(z^2-1)^{1/2}}^{z+(z^2-1)^{1/2}}\frac{g(s,u)du}{(1-2uz+u^2)^{1/2}} &(10b)\end{aligned}\right\},\quad z\in E_\rho,$$

$$g(s,z)=\int_{-1}^{+1}\frac{(1-z^2)f(s,x)dx}{(1-2xz+z^2)^{3/2}},\qquad |z|<1. \tag{10c}$$

These formulas can be extended to other values of z by deforming the integration path. One can then show that g can be analytically continued into a certain domain $\tilde{D}_S$ ($\tilde{D}_S$ is the union of the circle $|z|\leq 1$ and of the domain $\{z|z=w+(w^2-1)^{1/2}, w\in D_S\}$) [see Fig. 1(b)], and that

$$|g(s,z)|<s^{N'} \tag{11}$$

($z\in\tilde{D}_S$; $s>s_0$, where s_0 is independent of z).

We shall now show that, making use of (9), we can improve the bound (11) on the unit circle $|z|=1$. The general technique consists in using properties of subharmonic functions. However, we shall use here a particular theorem which is the following:

Let $\phi(z)$ be an analytic function defined in a domain limited by two arcs of circles C_1 and C_3 which intersect at the points A and B. Let $|\phi(z)|$ be less than M_1 on C_1, M_3 on C_3. Then the upper bound of $|\phi(z)|$ on an intermediate arc of circle C_2, which connects the points A and B, is less than M_2, where

$$M_2=M_1^{\beta/(\alpha+\beta)}M_3^{\alpha/(\alpha+\beta)}. \tag{12}$$

Here α and β are the intersecting angles of C_1, C_2 and C_2, C_3, respectively.

To prove this, one just has to apply with some caution the maximum modulus principle to the function $\phi(z)\exp\{-i\lambda\log[(z-A)/(z-B)]\}$, where λ is a conveniently chosen constant.

This theorem may be applied to our problem as follows (see Fig. 2): In the domain $\tilde{D}_S$ we draw a circle of radius $1-\epsilon$ centered at the origin (ϵ will be adjusted later). It plays the role of C_1.

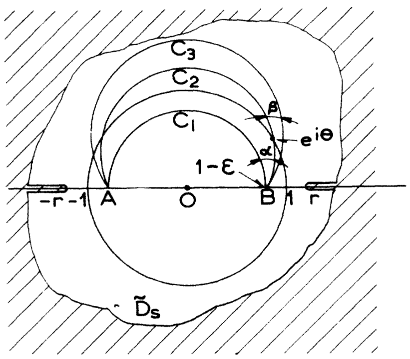

FIG. 2. Arcs of circles C_1, C_2, C_3 in the domain $\tilde{D}_S$.

C_2 cuts the real axis at the same points as C_1 and intersects the unit circle at $z = e^{i\theta}$. C_3 can be chosen as the largest arc inside D_S. Because of our assumption on the domain D_S and hence $\tilde{D}_S$, the angle $\alpha + \beta$ between C_1 and C_3 has a finite minimum m for any ϵ $(0 < \epsilon < 1)$. The angle α is given by

$$\tan\alpha = \epsilon(1 - \tfrac{1}{2}\epsilon)/\sin\theta(1-\epsilon). \qquad (13)$$

Now, we have $M_1 \leq 2/\epsilon^2$ and $M_3 < s^{N'}$ from (9) and (11), respectively. From these bounds of M_1 and M_3 we find that M_2 of (12) becomes smallest for

$$\epsilon = |\sin\theta|/\log s. \qquad (14)$$

For this choice of ϵ, we get

$$M_2 < (2/\epsilon^2)e^{N'/m} = 2C/\epsilon^2 \qquad (15)$$

along C_2 and, in particular, at $z = e^{i\theta}$. Obviously, this is also the maximum of g between C_1 and C_2.

Let us now exploit the above results to obtain an upper bound of $|f(s, \cos\theta)|$. We shall use Formula (10b) and integrate along two radii, $e^{-i\theta} \to 0$, $0 \to e^{i\theta}$. Then we find from (9) and (15) that

$$|f(s,\cos\theta)| < \frac{2}{\pi}\int_0^1 \frac{|g(s, re^{i\theta})| dr}{(1-r)^{1/2}(|\sin\theta|)^{1/2}} \leq \frac{4}{\pi(|\sin\theta|)^{1/2}}$$

$$\times\left[\int_0^{1-\epsilon}\frac{dr}{(1-r)^{3/2}} + \frac{C}{\epsilon^2}\int_{1-\epsilon}^1 \frac{dr}{(1-r)^{1/2}}\right].$$

Carrying out the integration and inserting (14), we finally obtain

$$|f(s,\cos\theta)| < C'(\log s)^{3/2}/\sin^2\theta, \quad \text{for } s > s_0, \qquad (16)$$

where C' is a constant independent of s and θ. This inequality implies inequality (3) and has the further merit of being uniform in θ. The derivation of (16) exhibits clearly that the distance of the branch points to $\cos\theta = \pm 1$ plays no role, although it is essential to derive the forward-backward bound.

Inequality (16) is better than inequality (1) as long as $|\cos\theta|$ is less than $1 - \text{const}/s(\log s)^{1/2}$. In particular, for fixed momentum transfer, it gives

$$|f(s, 1 - |t|/2k^2)| \leq (C''/|t|)s(\log s)^{3/2}, \qquad (17)$$

a result which is implicitly contained in reference 2.

Finally, let us indicate that if D_S consists of the whole cut plane, an alternative method may be used. It consists in studying the upper bound of the partial-wave amplitude a_l in the complex l plane for $\mathrm{Re}\,l > N''$. From this one can derive a bound very close to the bound (3) by making use of the Watson-Sommerfeld transformation. In this manner one can also understand the deep reason why (2) should be replaced by (3). This is because of the fact that for l big enough, but much smaller than $s^{1/2}\log s$, a_l is a very slowly varying function of l. Details will be published elsewhere.

We wish to thank Professor M. Zerner (Marseilles) for useful discussions on mathematical points.

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[1]M. Froissart, Phys. Rev. 123, 1053 (1961).

[2]A. Martin, Phys. Rev. 129, 1432 (1963). See also Proceedings of the International Conference on High-Energy Nuclear Physics, Geneva, 1962 (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 566.

[3]The more restrictive assumption that $f(s, \cos\theta)$ be analytic in a Lehmann ellipse (semimajor axis $\approx 1 + \alpha/k^4$) gives a weaker bound than (1). See O. W. Greenberg and F. E. Low, Phys. Rev. 124, 2047 (1961).

[4]See, for example, Higher Transcendental Functions, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 157.