be less than the experimental resolution, on the basis of the measured cross sections for $\rho^+$ and $\rho^0$. Ahmadzadeh and Sakmar have also suggested the possible connection of the Pomeranchuk particle with other experimental information for $t < 30\,m_\pi^2$.[4]

[16]The possibility of the Pomeranchuk trajectory "bending down" soon and never reaching 2 cannot be excluded. However, this is not likely to occur if the quantum numbers of the $f^0$ have been assigned correctly, which would imply that another trajectory with the same quantum numbers as the Pomeranchuk, but below it at $t=0$, is able to reach $\mathrm{Re}\,\alpha = 2$.

[17]After this work was completed, S. J. Lindenbaum et al. presented some high-energy $pp$ data at the 1963 Annual Meeting of the American Physical Society, New York, January 1963 (postdeadline paper), which support the latter alternative. See K. J. Foley et al., Phys. Rev. Letters 10, 376 (1963).

# CROSSING-SYMMETRIC WATSON-SOMMERFELD TRANSFORMATION*

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In all the work on Regge poles and their relation to high-energy behavior and other problems in the strong interactions,[1] one has had to deal with an expression which is not explicitly crossing-symmetric. Thus to get the asymptotic behavior of one channel, one has to look at the partial-wave expansion in the crossed channel. The problem is more acute when one wants to perform a calculation in which both the conditions of crossing symmetry and Regge behavior are imposed. In this note we shall briefly sketch the derivation of a crossing-symmetric Watson-Sommerfeld transformation which has the advantage that it simultaneously displays the contributions of the Regge poles in all three channels.

This result came out of an analysis which attempts to free Regge poles from their strict attachment to angular momentum and Legendre expansions, and shows that the same poles appear when one looks at the coefficients of other expansions. For example, we show that if a function $f(z) = \sum a(l) P_l(z)$ is such that $a(l)$ is meromorphic in $l$ for $\mathrm{Re}\,l > -\frac{1}{2}$, and as $|l| \to \infty$, $a(l) \sim l^{1/2} e^{-l\xi}$, $\xi > 0$, then if one expands $f(z)$ in a power series, $f(z) = \sum_\nu c(\nu) z^\nu$, $c(\nu)$ will again be meromorphic for $\mathrm{Re}\,\nu > -\frac{1}{2}$. It will have the same poles, $\alpha_j$, as $a(l)$ plus poles at $\alpha_j - 2, \alpha_j - 4, \cdots$, etc. A similar result holds if one expands the nonrelativistic scattering amplitude in powers of the momentum transfer $t$, $f(s,t) = \sum_\nu c'(\nu, s) t^\nu$. Now, however, $c'(\nu, s)$ will have the same poles as the partial-wave amplitude for $\mathrm{Re}\,\nu > -\frac{1}{2}$ plus poles at $\alpha_j - 1$, $\alpha_j - 2, \cdots$, etc. A detailed account of this analysis and a complete proof of the results given below will be published elsewhere.[2]

We consider the relativistic scattering of two spinless equal-mass particles and take the mass to be unity. We assume, for simplicity, that there are no bound states or single-particle poles and write the Mandelstam representation for the invariant amplitude

$$A(s,t,u) = L_{12}(s,t) + L_{23}(t,u) + L_{13}(s,u), \qquad (1)$$

where

$$L_{12}(s,t) = \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty dt' \frac{\rho_{12}(s',t')}{(s'-s)(t'-t)}, \qquad (2)$$

and similar expressions for the other two $L$'s. We do not write the subtractions explicitly in (2), as we shall not need them. Martin[3] recently showed, under assumptions weaker than those we take below, that the three functions $\rho_{ij}$ uniquely determine $A(s,t,u)$. Our final result gives an explicit demonstration of this fact.

Let us now assume that the partial-wave amplitudes of all three channels are each meromorphic in $l$ in the region $\mathrm{Re}\,l > -\frac{1}{2}$. Following Oehme,[4] we consider only moving poles in the $l$ planes. We further assume that the partial-wave amplitudes behave as in potential scattering for $|l| \to \infty$ in the right half-planes. These assumptions, of course, are far from proved but let us accept them for the present discussion.

For simplicity we take only one pole in each channel which for some real value of $s$, $t$, or $u$ shows up in the region $\mathrm{Re}\,l > -\frac{1}{2}$. We assume that the trajectories turn back into the left half-plane for large values of their arguments. In other words, we take

$$\begin{aligned} &\mathrm{Re}\,\alpha_1(s) > -\tfrac{1}{2}, \quad s_0 < s < s_1; \\ &\mathrm{Re}\,\alpha_2(t) > -\tfrac{1}{2}, \quad t_0 < t < t_1; \\ &\mathrm{Re}\,\alpha_3(u) > -\tfrac{1}{2}, \quad u_0 < u < u_1. \end{aligned} \qquad (3)$$

Outside the intervals above we have $\mathrm{Re}\,\alpha_i < -\frac{1}{2}$. Obviously, if the $\alpha$'s satisfy the usual properties, we must have $s_1,\ t_1,\ u_1 > 4$.

Let us now expand each of the $L_{ij}$'s in a double power series and write

$$A(s,t,u) = \sum_{\nu,\mu} c_{12}(\nu,\mu)s^\nu t^\mu + \sum_{\nu,\mu} c_{23}(\nu,\mu)t^\nu u^\mu + \sum_{\nu,\mu} c_{13}(\nu,\mu)s^\nu u^\mu; \quad \nu,\mu = 0,1,\cdots. \tag{4}$$

These series converge absolutely for $s,t,u$ in the Euclidean region; i.e., $0<s,t,u<4$ and $s+t+u=4$. No matter how many subtractions we need in (2), if they are finite, then for large enough integers $\nu$ and $\mu$, we have the following expression for $c_{ij}(\nu,\mu)$:

$$c_{ij}(\nu,\mu) = \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy\, \rho_{ij}(x,y) x^{-\nu-1} y^{-\mu-1}; \quad i,j = 1,2,3. \tag{5}$$

The right-hand side of (5) can now be used to define a unique interpolation of $c_{ij}(\nu,\mu)$ for complex values of $\nu$ and $\mu$ which is clearly analytic in the region

$$\mathrm{Re}\nu > \sup_y[\mathrm{Re}\alpha_j(y)],$$
$$\mathrm{Re}\mu > \sup_x[\mathrm{Re}\alpha_i(x)]. \tag{6}$$

The domain given in (6) can be extended if we now use the assumptions we have made about the partial-wave amplitudes of each channel. By using the usual Regge type expressions for $\rho_{ij}$, one can factor out the singular terms in (5) and obtain the following result:

$$c_{ij}(\nu,\mu) = c_{ij}^{(0)}(\nu,\mu) + c_{ij}(\nu,\mu;\ \alpha_i) + c_{ij}(\nu,\mu;\ \alpha_j). \tag{7}$$

Here the function $c_{ij}^{(0)}(\nu,\mu)$ is regular in the region $\mathrm{Re}\nu > -\frac{1}{2}$ and $\mathrm{Re}\mu > -\frac{1}{2}$. For the other two terms, one can get an explicit representation of the form

$$c_{ij}(\nu,\mu;\ \alpha_i) = \sum_{r=0}^{n_i} \frac{1}{2\pi i}\int_4^\infty x^{-\nu-1}dx\left[\frac{4^a\gamma_{ir}(x)}{\mu-\alpha_i(x)+r} - \frac{4^{a'}\gamma_{ir}^*(x)}{\mu-\alpha_i^*(x)+r}\right], \tag{8}$$

where $a = -[\mu-\alpha_i(x)+r]$, $a' = -[\mu-\alpha_i^*(x)+r]$. The integers $n_i$ are determined by the conditions

$$\tfrac{1}{2} > \sup_x[\mathrm{Re}\alpha_i(x) - n_i] > -\tfrac{1}{2}; \quad i = 1,2,3. \tag{9}$$

The residues $\gamma_{ir}(x)$ are all proportional to $\beta_i(x)$, the residue of the partial-wave amplitude of the $i$th channel at $l = \alpha_i(x)$. The function $c_{ij}(\nu,\mu;\ \alpha_j)$ will be given by an expression like (8) with $i$ replaced by $j$ and the role of $\nu$ and $\mu$ interchanged.

One can easily see from (8) that $c_{ij}(\nu,\mu;\ \alpha_i)$ is analytic in $\nu$ in the region $\mathrm{Re}\nu > -\frac{1}{2}$. (The residues $\gamma_{ir}$ vanish as $x\to\infty$.) However, in the $\mu$ plane one has to exclude the curves $\mu = \alpha_i(x) - r$ traced as $x$ varies from 4 to $\infty$, and their complex conjugates $\mu = \alpha_i^* - r$. Thus a simple pole in the $l$ plane seems to have become a more complicated singularity when one goes to the $\mu$ or $\nu$ plane. However, fortunately when we substitute (8) into the double series (4) and carry out the summation, the situation simplifies tremendously and one obtains a form which is identical with the Regge pole contributions of a single power series expansion.

As $\mathrm{Re}\nu\to\infty$, $c_{ij}^{(0)}(\nu,\mu) \sim (4)^{-\nu}$ and as $\mathrm{Re}\mu\to\infty$, $c_{ij}^{(0)}(\nu,\mu)\sim(4)^{-\mu}$. Furthermore, one can show that $c_{ij}^{(0)}(\nu,\mu)$ vanishes if $|\mathrm{Im}\nu|\to\infty$ or $|\mathrm{Im}\mu|\to\infty$. Similar asymptotic properties hold for the other two functions in (7).

Keeping $s,t,u$ in the Euclidean region, we can now apply the Watson-Sommerfeld transformation twice to each series in (4). We obtain

$$L_{12}(s,t) = -\frac{1}{4}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\nu \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\mu \frac{c_{12}^{(0)}(\nu,\mu)}{\sin\pi\nu\,\sin\pi\mu} \times(-s)^\nu(-t)^\mu + \sum_{r=0}^{n_1} R(t;\ \alpha_1(s)-r) + \sum_{r=0}^{n_2} R(s;\ \alpha_2(t)-r). \tag{10}$$

The functions $R(t;\ \alpha_1(s)-r)$ are given by

$$R(t;\ \alpha_1(s)-r) = -\gamma_{ir}(s)\left[\int_0^4 \frac{x^{\alpha_1-r}}{x-t-i\epsilon}dx + \frac{\pi(-t)^{\alpha_1-r}}{\sin\pi(\alpha_1(s)-r)}\right],$$
$$\mathrm{Re}(\alpha_1(s)-r) > -\tfrac{1}{2}; \tag{11a}$$

and

$$R(t;\ \alpha_1(s)-r) = \gamma_{ir}(s)\int_4^\infty \frac{x^{\alpha_1-r}}{x-t-i\epsilon}dx,$$
$$\mathrm{Re}(\alpha_1(s)-r) < -\tfrac{1}{2}. \tag{11b}$$

These functions correspond to the full contributions of the usual Regge pole defined in reference 5. They represent the full contribution of a pole in the $\nu$ plane at $\nu = \alpha_1 - r$, which one would obtain if he considers a single power series expansion in powers $t^\nu$. The two expressions (11a) and (11b) are identical in the strip $-1 > \mathrm{Re}(\alpha_1 - r) < 0$. The bracket in (11a) has the correct cut starting at $t=4$, since one can easily check that the discontinuities of the two terms inside the bracket cancel in the interval $0 < t < 4$.

Representations analogous to (10) hold for $L_{23}$ and $L_{13}$. We can now continue in $s$, $t$, and $u$. The background term in (10) defines a function which is analytic in the cut $s$ and $t$ planes. The cuts seem to start at $s=0$ and $t=0$; however, we shall see below that using the asymptotic behavior of $c_{ij}^{(0)}$, one can show that the cuts start at $s=4$ and $t=4$. The functions $R$ are also analytic in the cut planes with the correct thresholds. As we vary $s$, $t$, or $u$, the different $R$'s take the form (11a) or (11b), depending on whether $\mathrm{Re}\alpha_i - r > -\frac{1}{2}$ or $< -\frac{1}{2}$. The change from (11a) to (11b) takes place in an analytic manner, since, if one considers $\alpha$ as a parameter, $R$ as defined in (11a) and (11b) is an entire function of $\alpha$. It is essential in the crossing-symmetric problem to factor out the background term of each pole as in (11a). Due to the condition $s+t+u=4$, one always varies at least two of the variables together and one must have an analytic mechanism for the appearance or disappearance of terms like the second term in (11a) as the $(\alpha_i - r)$ move from one half-plane to the other.

To go back to the background term of (10), we introduce the functions

$$b_{ij}(x,y) = \frac{1}{(2\pi i)^2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\nu \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\mu\, x^{\nu} y^{\mu} c_{ij}^{(0)}(\nu,\mu). \quad (12)$$

Using the behavior of $c_{ij}^{(0)}$ for large $\nu$ and $\mu$, one can easily show that $b_{ij}(x,y)=0$ if either $x<4$ or $y<4$. Furthermore, $b_{ij}$ vanishes when either of its arguments becomes infinite. In terms of this function, (10) can be rewritten as

$$L_{12}(s,t) = \int_4^{\infty} dx \int_4^{\infty} dy \frac{b_{12}(x,y)}{(x-s)(y-t)} + \sum_{r=0}^{n_1} R(t;\ \alpha_1(s)-r) + \sum_{r=0}^{n_2} R(s;\ \alpha_2(t)-r). \quad (13)$$

The Mandelstam type double integral here always converges and needs no subtractions. The final result for the whole amplitude will be

$$\begin{aligned} A(s,t,u) = {} & \int_4^{\infty} dx \int_4^{\infty} dy \frac{b_{12}(x,y)}{(x-s)(y-t)} + \int_4^{\infty} dx \int_4^{\infty} dy \frac{b_{23}(x,y)}{(x-t)(y-u)} + \int_4^{\infty} dx \int_4^{\infty} dy \frac{b_{13}(x,y)}{(x-s)(y-u)} \\ & + \sum_{r=0}^{n_1} [R(t;\ \alpha_1(s)-r) + R(u;\ \alpha_1(s)-r)] \\ & + \sum_{r=0}^{n_2} [R(u;\ \alpha_2(t)-r) + R(s;\ \alpha_2(t)-r)] \\ & + \sum_{r=0}^{n_3} [R(s;\ \alpha_3(u)-r) + R(t;\ \alpha_3(u)-r)]. \end{aligned} \quad (14)$$

Thus starting from a subtracted Mandelstam representation for $A(s,t,u)$, we have obtained an expression for it in terms of an unsubtracted Mandelstam representation plus the contributions of all the Regge poles that for some value of $s$, $t$, or $u$ show up in region $\mathrm{Re}\,l > -\frac{1}{2}$. The $R$ functions in (14) can themselves be represented by double dispersion type integrals which, however, will need subtractions.

Needless to say, it is easy to check from (11) and (3) that (14) will lead to the same asymptotic behavior in $s$, $t$, or $u$ as the usual Regge representation.[6] In writing down (14), we have for simplicity taken all three poles $\alpha_i$ to have even signature. The $\gamma_{ir}$'s are all proportional to $\beta_i$, the residue of the partial-wave amplitude of the $i$th channel at $l=\alpha_i$. Their dependence on the $\alpha_i$'s is such that near a resonance in any channel ($\mathrm{Re}\alpha_i \approx n$, $n$ integral), the second terms of (11a) just combine to give us the usual Regge term proportional to $\beta_i P_{\alpha_i}(z_i)/\sin\pi\alpha_i$.

Finally, we note that even if we have bound states or single-particle poles to start with, the final result will still be as in (14). This is, of course, under the assumption that the partial-wave amplitudes have no fixed poles in $l$. The bound states then will be on a trajectory and the pole on the physical sheet of the $s$, $t$, or $u$ plane will appear in the $R$ functions. The derivation of

(14) will differ slightly in this case and the details will be given in reference 2.

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[1]For a complete list of references, see the review talk by S. D. Drell, in Proceedings of the International Conference on High-Energy Nuclear Physics, Geneva, 1962 (CERN Scientific Information Service, Geneva, Switzerland, 1962), pp. 897-911.

[2]N. N. Khuri (to be published).

[3]A. Martin, Phys. Rev. Letters 9, 410 (1962).

[4]R. Oehme, Phys. Rev. Letters 9, 358 (1962).

[5]N. N. Khuri, Phys. Rev. 130, 429 (1963).

[6]In all the present discussions, one assumes that $\beta_i(x)$ vanish faster than $x^{-1/2}$ as $x \to \infty$.

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# RELATIVE WEIGHTS OF THE DECAYS OF CERTAIN RESONANCES IN THEORIES WITH BROKEN SYMMETRY*

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(I) The principal aim of this work is to derive, in illustration of a general procedure,[1] relationships that obtain among the relative weights[2] of the decays of various resonances into baryon-meson states in the Ne'eman–Gell-Mann theory[3] of strong interactions, when exact invariance under $SU_3$ suffers a first-order perturbation by interactions invariant under only isospin and strangeness transformations. In particular, we deal with those resonances[4] which Glashow and Sakurai[5] have associated with the irreducible representation (IR) of $SU_3$ with highest weight[6] (3, 0).

(II) We may best explain our procedure by considering first a much simpler situation involving the decay of a particle (of isospin $I$ with $z$ component $\nu$) into two particles (of isospins $I_1$ and $I_2$ with $z$ components $\nu_1$ and $\nu_2$) in a theory whose charge independence suffers a first-order perturbation by an interaction (e.g., the electromagnetic interaction) which commutes with the operator $I_z$ but not with $\vec{I}^2$ and hence may be taken to transform under $R_3$ like the $I_z = 0$ component of a vector operator. We denote the matrix element for the decay by $\langle I_1\nu_1 I_2\nu_2 | T | I\nu\rangle$ and consider the sums

$$P_1(\nu) = \sum_{\nu_2} |\langle I_1\nu - \nu_2 I_2\nu_2 | T | I\nu\rangle|^2, \tag{1}$$

$$P_2(\nu_1) = \sum_{\nu_2} |\langle I_1\nu_1 I_2\nu_2 | T | I\nu_1 + \nu_2\rangle|^2. \tag{2}$$

We easily prove that $P_1(\nu)$ is ($A_1$) independent of $\nu$ to zeroth order and ($B_1$) of the form $(\alpha + \beta\nu)$ to first order in the perturbation, and similarly that $P_2$ is ($A_2$) independent of $\nu_1$ and ($B_2$) of the form $(\gamma + \delta\nu_1)$. The result ($A_1$) states the equality of the total weight for all decays for different charge states of the decaying particle; result ($A_2$) is the Shmushkevich theorem[7] for the decay. Results ($B_1$) and ($B_2$) are new. The proof of these results involves only simple facts regarding $R_3$ including properties of Clebsch-Gordan coefficients, the Wigner-Eckart theorem, and the fact that $C(j1j, m0m)$ is proportional to $m$ for fixed $j$.

We illustrate using the decays of the well-known 3-3 nucleon resonance $N^*$ into nucleon plus pion states. As an aid to the application of the above results, we draw up a Shmushkevich table (Table I) for the allowed processes. We see that results ($A_1$) and ($B_1$) give

$$\Gamma_1 = \Gamma_2 + \Gamma_3 = \Gamma_4 + \Gamma_5 = \Gamma_6, \tag{3.A$_1$}$$

$$\Gamma_1 + \Gamma_2 + \Gamma_4 = \Gamma_3 + \Gamma_5 + \Gamma_6, \tag{3.A$_2$)$N$}$$

$$\Gamma_1 + \Gamma_3 = \Gamma_2 + \Gamma_5 = \Gamma_4 + \Gamma_6, \tag{3.A$_2$)$\pi$}$$

and hence we have the complete solution

$$2\Gamma_1 = 3\Gamma_2 = 6\Gamma_3 = 6\Gamma_4 = 3\Gamma_5 = 2\Gamma_6, \tag{4}$$

in the zeroth order of the perturbation. In the first order of the perturbation, we fail to get a complete solution, but only the identities

$$\Gamma_1 - (\Gamma_2 + \Gamma_3) = (\Gamma_2 + \Gamma_3) - (\Gamma_4 + \Gamma_5) = (\Gamma_4 + \Gamma_5) - \Gamma_6, \tag{5.B$_1$}$$

$$(\Gamma_1 + \Gamma_3) - (\Gamma_2 + \Gamma_5) = (\Gamma_2 + \Gamma_5) - (\Gamma_4 + \Gamma_6). \tag{5.B$_2$)$\pi$}$$

(III) We now generalize the discussion of para-