DOES THE f^0 PARTICLE LIE ON THE POMERANCHUK TRAJECTORY?*

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Two experimental groups have recently reported on the discovery of a $\pi\pi$ resonance at about 1250 MeV that is probably an $I=0, J=2$ state.[1,2] It was conjectured that this resonance, called f^0, is the "particle" predicted by Chew and Frautschi on the basis of the Regge-pole scheme.[3] In that case, it should lie on the Pomeranchuk trajectory. Ahmadzadeh and Sakmar have tried to test this conjecture, starting from a four-parameter expression for the imaginary part of the Pomeranchuk trajectory $\alpha(t)$.[4] They assume the usual properties for this trajectory and conclude that the f^0 cannot lie on it, if at the same time one accepts the published analysis of the high-energy pp scattering in terms of the Pomeranchuk trajectory. The purpose of this Letter is to prove that, as conjectured by Ahmadzadeh and Sakmar, this result is independent of the choice of an Ansatz for $\mathrm{Im}\alpha(t)$, provided some current ideas on Regge trajectories are correct. In the first part, some general bounds derived for boson Regge trajectories are used to relate values of $\alpha(t)$ for negative t to the slope of the trajectory at $t=0$. Then it is shown how this slope restricts the possible values for the energy and width of a resonance that lies on the same trajectory. When these results are applied to the Pomeranchuk particle, it is concluded that either the slope of the Pomeranchuk trajectory has been seriously overestimated from the pp data, or the Pomeranchuk particle is likely to have an energy less than or approximately equal to the ρ-meson energy. Experimental work directed towards testing the latter possibility is encouraged. In that case, the f^0 resonance still could fit into the Regge-pole scheme as possibly belonging to the trajectory proposed by Igi.[5]

We assume that a boson Regge trajectory as a function of the energy-squared satisfies the dispersion relation[6]

$$\alpha(t)=\alpha(\infty)+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}\alpha(t')}{t'-t}dt', \quad \text{for } t_0>0, \quad \text{(H1)}$$

with

$$\mathrm{Im}\alpha(t)\geqslant 0, \quad \text{for } t_0\leqslant t<\infty. \quad \text{(H2)}$$

As a consequence, all the derivatives of $\alpha(t)$ for $t<t_0$ are positive, and this implies a severe restriction on the behavior of α in that region. In particular, we expect to be able to put a lower bound to the slope of α at the origin, if $\alpha(\infty)$, $\alpha(0)$, and $\alpha(t_1)$ for any negative value of t_1 are given. In some instances, this bound will be twice the slope of a straight line through $\alpha(t_1)$ and $\alpha(0)$. In order to derive this and other related results, it will be useful to prove the following lemma: Let $\xi(t)$ satisfy the dispersion relation

$$\xi(t)=\frac{1}{\pi}\int_{a}^{\infty}\frac{\mathrm{Im}\xi(t')}{t'-t}dt', \quad (1)$$

with $\mathrm{Im}\xi(t)\geqslant 0$ for $a\leqslant t<\infty$. It follows that

$$\xi(t_3)\gtrless\frac{\xi(t_1)\xi(t_2)(t_2-t_1)}{(t_2-t_3)\xi(t_2)+(t_3-t_1)\xi(t_1)}, \quad (2)$$

for $t_1<t_2$, t_1, t_2, and $t_3\leqslant a$, where the $\geqslant$ sign holds for $t_3\leqslant t_1$ or $t_3\geqslant t_2$, and the $\leqslant$ sign for $t_1\leqslant t_3\leqslant t_2$.

<u>Proof:</u> — From Eq. (1) we have

$$\mathrm{Im}\xi(t)>0, \quad \text{for } \mathrm{Im}t>0,$$

and

$$\left.\begin{array}{l}\mathrm{Re}\xi(t)>0\\ \mathrm{Im}\xi(t)=0\end{array}\right\}, \quad \text{for } t \text{ real and } -\infty<t<a.$$

Therefore the function

$$\eta(t)=-1/\xi(t) \quad (3)$$

has no singularities except for a right-hand cut and a pole at infinity, and can be represented as[7]

$$\eta(t)=C+At+\int_{a}^{\infty}\frac{1+t't}{t'-t}\phi(t')dt', \quad (4)$$

with C, A, and $\phi(t)$ real, and $\phi(t)\geqslant 0$. From Eq. (4) it follows for t real and $-\infty<t<a$ that

$$\eta''(t)\geqslant 0. \quad (5)$$

Therefore, because $\eta(t)$ is a concave function for $t<a$, we have

$$\eta(t_3)\gtrless\eta(t_2)\frac{t_3-t_1}{t_2-t_1}+\eta(t_1)\frac{t_3-t_2}{t_1-t_2}, \quad (6)$$

for $t_1<t_2$, t_1, t_2, and $t_3\leqslant a$. Here again the $\geqslant$ sign holds for $t_3\leqslant t_1$ or $t_3\geqslant t_2$, and the $\leqslant$ sign

for $t_1 \le t_3 \le t_2$. Equation (2) follows trivially from Eqs. (6) and (3). This completes the lemma.

We now apply the above lemma to the function $\alpha(t) - \alpha(\infty)$. In particular, if we let $t_1 \to t_2 = 0$ and $t_3 = t$ in Eq. (2), we obtain for $-\infty < t < 0$

$$\alpha(t) > \alpha(\infty) + \frac{[\alpha(0) - \alpha(\infty)]^2}{\alpha(0) - \alpha(\infty) + \alpha'(0)|t|} = L(t), \quad (7)$$

or

$$\alpha'(0) > \frac{1}{|t|}\left\{\frac{[\alpha(0) - \alpha(\infty)]^2}{\alpha(t) - \alpha(\infty)} - \alpha(0) + \alpha(\infty)\right\}. \quad (8)$$

In the case of the Pomeranchuk trajectory, relations (7) or (8) plus the values of $\alpha(t)$ obtained in the high-energy experiments provide a lower bound for $\alpha'(0)$.[8]

Our aim is now to study what restrictions on the energy and width of a resonance follow from a given value of $\alpha'(0)$. The idea is that the trajectory, which has a positive curvature up to threshold, has to stay above its tangent line at $t = 0$ until its imaginary part has become appreciably large, and that when this happens, the Regge pole is so far from physical values of l that it is likely to give rise either to a very broad resonance or to no resonance at all. The detailed argument is as follows: We want to find under what restrictions it is possible to have a resonance at a value t_R of the energy-squared such that

$$\mathrm{Re}\alpha(t_R) < \alpha(0) + t_R\alpha'(0), \quad (9)$$

i.e.,

$$t_R > [\mathrm{Re}\alpha(t_R) - \alpha(0)]/\alpha'(0). \quad (10)$$

We know that $\mathrm{Re}\alpha(t_R)$ has to be equal to the spin of the resonance in question, but that this condition is not sufficient to have a resonance. In addition, our experience with the nonrelativistic case shows that at $t = t_R$, $\mathrm{Re}\alpha$ must still be increasing steeply as a function of t, and $\mathrm{Im}\alpha$ must still be small. The width of the resonance in the energy variable is related to these two properties and is given approximately by the function

$$\Gamma(t_R) = \frac{1}{\sqrt{t}}\frac{\mathrm{Im}\alpha(t)}{(d/dt)\,\mathrm{Re}\alpha(t)}\bigg|_{t=t_R}. \quad (11)$$

We require that the trajectory considered present these two characteristics at $t = t_R$ by assuming

$\mathrm{Re}\alpha(t)$ has at most one inflection point for $t < t_R$; (H3)

$$\mathrm{Im}\alpha(t) \le \mathrm{Im}\alpha(t_R)\frac{t - t_0}{t_R - t_0}, \text{ for } t_0 < t < t_R.^{9} \quad (\mathrm{H4})$$

From assumption (H3) and the fact that $\alpha''(t) > 0$ for $t \le t_0$, it follows that

$$(d/dt)\,\mathrm{Re}\alpha(t)\Big|_{t=t_R} < [\mathrm{Re}\alpha(t_R) - \alpha(0)]/t_R, \quad (12)$$

for the values of t_R satisfying relation (10). Our next step is to put a lower bound to $\mathrm{Im}\alpha(t_R)$ such that, combined with Eqs. (11) and (12), it will provide a restriction on the permissible values for $\Gamma(t_R)$. It is clear that such a bound for $\mathrm{Im}\alpha(t_R)$ exists and is larger than zero, because if we had $\mathrm{Im}\alpha(t_R) = 0$, from assumptions (H1), (H2), and (H4) all the derivatives of $\mathrm{Re}\alpha$ would be positive up to $t = t_R$, and therefore $\mathrm{Re}\alpha$ would not satisfy relation (9). This suggests splitting α into two parts, β and γ, one of which has a zero imaginary part for $t < t_R$. Therefore, we define

$$\left.\begin{aligned}\mathrm{Im}\beta(t) &= \mathrm{Im}\alpha(t), && \text{for } t_0 < t < t_R\\ &= \mathrm{Im}\alpha(2t_R - t), && \text{for } t_R < t < 2t_R - t_0\end{aligned}\right\}, \quad (13\mathrm{a})$$

$$\beta(t) = \frac{1}{\pi}\int_{t_0}^{2t_R - t_0}\left[\frac{\mathrm{Im}\beta(t')}{t' - t}\right]dt', \quad (13\mathrm{b})$$

and

$$\gamma(t) = \alpha(t) - \beta(t). \quad (14)$$

From the above definitions it follows that $\beta(\infty) = 0$, $\beta(0) > 0$, $\mathrm{Re}\beta(t_R) = 0$, $\gamma(\infty) = \alpha(\infty)$, $\gamma(0) < \alpha(0)$, and $\gamma(t_R) = \mathrm{Re}\alpha(t_R)$. In order to have $\mathrm{Im}\gamma(t) \ge 0$ on the real axis, we assume further[10]

$$\mathrm{Im}\alpha(t) \ge \mathrm{Im}\alpha(2t_R - t), \text{ for } t_R < t < 2t_R - t_0, \quad (\mathrm{H5})$$

which is consistent with the properties discussed above. Thus we can apply our lemma to the function $\gamma(t) - \gamma(\infty)$. We let $t_1 \to t_2 = 0$ and $t_3 = t \le a = t_R$ in Eq. (2), and obtain

$$\gamma(t) - \gamma(\infty) \ge \frac{[\gamma(0) - \gamma(\infty)]^2}{\gamma(0) - \gamma(\infty) - t\gamma'(0)}.$$

The denominator in this expression is positive because it is so at $t = 0$, and $\gamma(t)$ is finite for

$t \leq t_R$. We can therefore write for $t = t_R$

$$\gamma'(0) \leq \frac{[\gamma(0)-\gamma(\infty)][\gamma(t_R)-\gamma(0)]}{t_R[\gamma(t_R)-\gamma(\infty)]}$$

$$< \frac{[\alpha(0)-\alpha(\infty)][\mathrm{Re}\,\alpha(t_R)-\gamma(0)]}{t_R[\mathrm{Re}\,\alpha(t_R)-\alpha(\infty)]}. \qquad (15)$$

Now, from assumption (H4), we have

$$\beta(0) \leq B_0(t_R)\,\mathrm{Im}\,\alpha(t_R)$$

and

$$\beta'(0) \leq B_1(t_R)\,\mathrm{Im}\,\alpha(t_R),$$

where

$$B_0(t_R) = \frac{1}{\pi(t_R-t_0)}\left[-t_0 \ln\left(\frac{2t_R-t_0}{t_0}\right) + 2t_R \ln\left(\frac{2t_R-t_0}{t_R}\right)\right]$$

and

$$B_1(t_R) = \frac{1}{\pi(t_R-t_0)} \ln\left[\frac{t_R^{\,2}}{t_0(2t_R-t_0)}\right].$$

It then follows that

$$\left.\begin{aligned}&\gamma(0) \geq \alpha(0) - \mathrm{Im}\,\alpha(t_R)B_0(t_R)\\ \text{and}\quad &\\ &\gamma'(0) \geq \alpha'(0) - \mathrm{Im}\,\alpha(t_R)B_1(t_R)\end{aligned}\right\}. \qquad (16)$$

From relations (15) and (16) we obtain a lower bound for $\mathrm{Im}\,\alpha(t_R)$. This result, combined with relations (11) and (12), gives the final bound

$$\Gamma(t_R) > \frac{(t_R)^{1/2}\{\alpha(\infty)-\alpha(0)+\alpha'(0)t_R[\mathrm{Re}\,\alpha(t_R)-\alpha(\infty)]/[\mathrm{Re}\,\alpha(t_R)-\alpha(0)]\}}{[\alpha(0)-\alpha(\infty)]B_0(t_R)+[\mathrm{Re}\,\alpha(t_R)-\alpha(\infty)]B_1(t_R)t_R}. \qquad (17)$$

We want to apply the above results to the Pomeranchuk trajectory, which controls the high-energy behavior of total cross sections and therefore satisfies the condition[3,11]

$$\alpha(0) = 1. \qquad (\mathrm{H6})$$

We also take

$$\alpha(\infty) \geq -1, \qquad (\mathrm{H7})$$

as suggested by the work of Gribov and Pomeranchuk.[12] It can be easily verified that the bounds given by relations (7), (8), and (17) are increasing functions of $\alpha(\infty)$. Therefore we replace in these relations $\alpha(\infty) = -1$, and our results are valid a fortiori if $\alpha(\infty) > -1$. Finally we put $t_0 = 4m_\pi^{\,2}$ and $\mathrm{Re}\,\alpha(t_R) = 2$, as is the case for the Pomeranchuk particle. We then obtain from Eq. (17)

$$\Gamma(t_R) > \frac{(t_R)^{1/2}[3t_R\alpha'(0)-2]}{2B_0(t_R)+3t_R B_1(t_R)} = \Phi(t_R),$$

$$\text{for } t_R > 1/\alpha'(0). \qquad (18)$$

In Fig. 1 the function $L(t)$ from relations (7), (H6), and (H7) is plotted for $\alpha'(0) = 1/80$, 1/50, and 1/35. The values of $\alpha(t)$ obtained by the

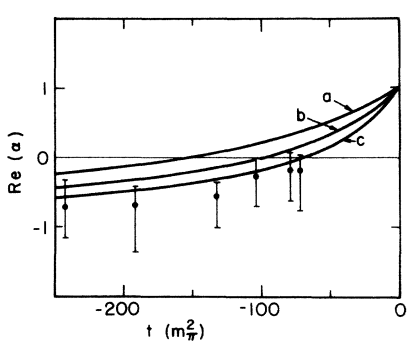

FIG. 1. Curves a, b, and c are lower bounds for $\alpha(t)$ given by Eqs. (7), (H6), and (H7) for $\alpha'(0) \leq 1/80$, 1/50, and 1/35, respectively. The experimental data are from reference 13.

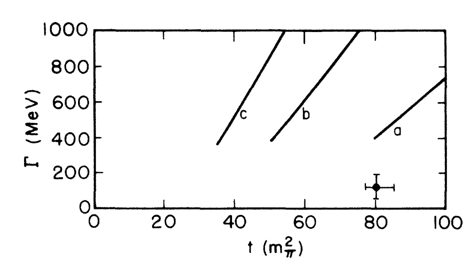

FIG. 2. Curves a, b, and c show lower bounds for $\Gamma(t_R)$ given by Eq. (18) for $\alpha'(0) \geq 1/80$, 1/50, and 1/35, respectively. The experimental values for the energy squared and width of the f^0 are indicated (references 1 and 2).

Brookhaven and Cornell groups from the analysis of the high-energy data are also indicated.[13,14] If we apply the inequality (8) to the three values for lower $|t|$, we obtain $\alpha'(0) > 1/29$ with 90% confidence. Eventually one may expect this bound to be lowered by the presence of other than statistical errors in these data. However, the values of $\Phi(t_R)$ obtained from Eq. (18) and plotted in Fig. 2 show that unless $\alpha'(0)$ is less than 1/80, then $\Gamma(80)$ has to be much larger than the experimental width of the f^0 resonance. Therefore, within the scheme presented here, and provided the analysis of the high-energy pp data is accepted, this resonance cannot lie on the Pomeranchuk trajectory. The Pomeranchuk particle might be found at a value of t smaller than or approximately equal to $30\,m_\pi^2$.[15,16] Alternatively one may conclude that other l-plane singularities for pp scattering (e.g., the P' and ω Regge poles) have conspired to simulate a falsely large slope for the Pomeranchuk trajectory.[17]

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[1]W. Selove, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters 9, 272 (1962).

[2]J. J. Veillet, J. Hennessy, H. Bingham, M. Block, D. Dryard, A. Lagarrigue, P. Mittner, and A. Rousset (to be published); G. Bellini, M. di Corato, E. Fiorini, and P. Negri (to be published).

[3]G. F. Chew and S. Frautschi, Phys. Rev. Letters 7, 394 (1961). We will call this particle the Pomeranchuk particle.

[4]A. Ahmadzadeh and I. Sakmar, Lawrence Radiation Laboratory Report UCRL-10584, December 1962 (unpublished).

[5]K. Igi, Phys. Rev. Letters 9, 76 (1962). We call P' the trajectory proposed in this paper.

[6]This and other properties are discussed by G. F. Chew, Phys. Rev. 129, 2363 (1963); further references are given here.

[7]J. A. Shohat and J. D. Tamarkin, The Problem of Moments, Mathematical Surveys, No. 1 (American Mathematical Society, New York, 1943), p. 23.

[8]We will present here some other results that follow from our lemma applied to a boson Regge trajectory. We call t_g the value of t at which α vanishes. In Eq. (2) let $t_1 = t_g$, $t_2 = 0$, and $t_3 = t$; then for $t_g < t < 0$, we have

$$\alpha(t) < \alpha(\infty) + \frac{t_g\alpha(\infty)[\alpha(0) - \alpha(\infty)]}{-t\alpha(0) + t_g\alpha(\infty)}. \qquad \text{(F1)}$$

Let $t_1 \to t_2 = t$, $t_3 = 0$; if follows that, for $-\infty < t < 0$,

$$\alpha'(t) < [\alpha(t) - \alpha(\infty)][\alpha(t) - \alpha(0)]/t[\alpha(0) - \alpha(\infty)].$$

Finally, from relations (3) and (5) we obtain for $t < t_0$

$$\alpha''(t) > \frac{2[\alpha'(t)]^2}{\alpha(t) - \alpha(\infty)}.$$

[9]This last hypothesis is generous near threshold for the Pomeranchuk trajectory because there we know $\mathrm{Im}\alpha(t) \sim (t - t_0)^b$, with $b = \mathrm{Re}\alpha(t_0) + 1/2$, and $\mathrm{Re}\alpha(t_0) > 1$ [see A. O. Barut and D. E. Zwanziger, Phys. Rev. 127, 974 (1962)]. Moreover, the maximum for $\mathrm{Re}\alpha(t)$ is expected to occur at approximately the same energy as the inflection point for $\mathrm{Im}\alpha(t)$. Therefore, as $\mathrm{Re}\alpha(t)$ has not yet reached its maximum at the resonance energy, $\mathrm{Im}\alpha(t_R)$ is presumably still concave, which is stronger than required by the linear assumption (H4). Obviously this hypothesis has to be modified for trajectories such that $\mathrm{Re}\alpha(t_0) < 0.5$.

[10]A weaker bound may be chosen without modifying the essence of our results. However, such a change is hardly expected to be necessary, and it would make our proof more complicated.

[11]G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 8, 41 (1962).

[12]V. N. Gribov and I. Ya. Pomeranchuk, in Proceedings of the International Conference on High-Energy Nuclear Physics, Geneva, 1962 (CERN Scientific Information Service, Geneva, Switzerland, 1962).

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[14]The reader is referred also to the data of A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters 9, 111 (1962). If one considers all their data, the result $\alpha(-70) < 0$ is definitely established. However, their results for $t > -50$ are too linear and therefore not consistent with Eq. (F1) in footnote 8, and the values of $\alpha(\infty)$ and $\alpha(0)$ given by relations (H6) and (H7). If one takes only their data for $(s/2M^2) > 10.5$, their results agree with those of Baker et al.,[13] although with larger errors.

[15]Both groups reporting on the f^0 have observed some anomalies in the angular distribution around the ρ peak that can be attributed to an $I = 0$ state because they were not present in the analogous experiment with $\pi^+\pi^0$ or $\pi^-\pi^0$. On the other hand, Dr. Takeda has pointed out to me that if the Pomeranchuk particle has about the same energy as the ρ meson, its width is likely to

be less than the experimental resolution, on the basis of the measured cross sections for ρ^+ and ρ^0. Ahmadzadeh and Sakmar have also suggested the possible connection of the Pomeranchuk particle with other experimental information for $t < 30 m_\pi^2$.[4]

[16]The possibility of the Pomeranchuk trajectory "bending down" soon and never reaching 2 cannot be excluded. However, this is not likely to occur if the quantum numbers of the f^0 have been assigned correctly, which would imply that another trajectory with the same quantum numbers as the Pomeranchuk, but below it at $t = 0$, is able to reach $\mathrm{Re}\alpha = 2$.

[17]After this work was completed, S. J. Lindenbaum et al. presented some high-energy pp data at the 1963 Annual Meeting of the American Physical Society, New York, January 1963 (postdeadline paper), which support the latter alternative. See K. J. Foley et al., Phys. Rev. Letters 10, 376 (1963).

CROSSING-SYMMETRIC WATSON-SOMMERFELD TRANSFORMATION*

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In all the work on Regge poles and their relation to high-energy behavior and other problems in the strong interactions,[1] one has had to deal with an expression which is not explicitly crossing-symmetric. Thus to get the asymptotic behavior of one channel, one has to look at the partial-wave expansion in the crossed channel. The problem is more acute when one wants to perform a calculation in which both the conditions of crossing symmetry and Regge behavior are imposed. In this note we shall briefly sketch the derivation of a crossing-symmetric Watson-Sommerfeld transformation which has the advantage that it simultaneously displays the contributions of the Regge poles in all three channels.

This result came out of an analysis which attempts to free Regge poles from their strict attachment to angular momentum and Legendre expansions, and shows that the same poles appear when one looks at the coefficients of other expansions. For example, we show that if a function $f(z) = \sum a(l) P_l(z)$ is such that $a(l)$ is meromorphic in l for $\mathrm{Re}\, l > -\frac{1}{2}$, and as $|l| \to \infty$, $a(l) \sim l^{1/2} e^{-l\xi}$, $\xi > 0$, then if one expands $f(z)$ in a power series, $f(z) = \sum_\nu c(\nu) z^\nu$, $c(\nu)$ will again be meromorphic for $\mathrm{Re}\,\nu > -\frac{1}{2}$. It will have the same poles, α_j, as $a(l)$ plus poles at $\alpha_j - 2, \alpha_j - 4, \cdots$, etc. A similar result holds if one expands the nonrelativistic scattering amplitude in powers of the momentum transfer t, $f(s,t) = \sum_\nu c'(\nu, s) t^\nu$. Now, however, $c'(\nu, s)$ will have the same poles as the partial-wave amplitude for $\mathrm{Re}\,\nu > -\frac{1}{2}$ plus poles at $\alpha_j - 1$, $\alpha_j - 2, \cdots$, etc. A detailed account of this analysis and a complete proof of the results given below will be published elsewhere.[2]

We consider the relativistic scattering of two spinless equal-mass particles and take the mass to be unity. We assume, for simplicity, that there are no bound states or single-particle poles and write the Mandelstam representation for the invariant amplitude

$$A(s,t,u) = L_{12}(s,t) + L_{23}(t,u) + L_{13}(s,u), \tag{1}$$

where

$$L_{12}(s,t) = \frac{1}{\pi^2} \int_4^\infty ds' \int_4^\infty dt' \frac{\rho_{12}(s',t')}{(s'-s)(t'-t)}, \tag{2}$$

and similar expressions for the other two L's. We do not write the subtractions explicitly in (2), as we shall not need them. Martin[3] recently showed, under assumptions weaker than those we take below, that the three functions ρ_{ij} uniquely determine $A(s,t,u)$. Our final result gives an explicit demonstration of this fact.

Let us now assume that the partial-wave amplitudes of all three channels are each meromorphic in l in the region $\mathrm{Re}\, l > -\frac{1}{2}$. Following Oehme,[4] we consider only moving poles in the l planes. We further assume that the partial-wave amplitudes behave as in potential scattering for $|l| \to \infty$ in the right half-planes. These assumptions, of course, are far from proved but let us accept them for the present discussion.

For simplicity we take only one pole in each channel which for some real value of s, t, or u shows up in the region $\mathrm{Re}\, l > -\frac{1}{2}$. We assume that the trajectories turn back into the left half-plane for large values of their arguments. In other words, we take

$$\begin{aligned} &\mathrm{Re}\,\alpha_1(s) > -\tfrac{1}{2}, \quad s_0 < s < s_1; \\ &\mathrm{Re}\,\alpha_2(t) > -\tfrac{1}{2}, \quad t_0 < t < t_1; \\ &\mathrm{Re}\,\alpha_3(u) > -\tfrac{1}{2}, \quad u_0 < u < u_1. \end{aligned} \tag{3}$$

Outside the intervals above we have $\mathrm{Re}\,\alpha_i < -\frac{1}{2}$. Obviously, if the α's satisfy the usual properties,