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THEORY OF SCATTERING WITH LARGE MOMENTUM TRANSFER

Robert Serber*

Columbia University, New York, New York

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It is surprising that it has not previously been observed that the measurements of the elastic proton-proton scattering for high momentum transfers follow a simple power law. This is shown by Fig. 1, in which all the experimental results of the CERN and Brookhaven groups,[1] for proton initial momentum greater than 8 BeV/c, are shown on a log-log plot as a function of the square of the momentum transfer, t. A relationship

$$(4\pi/k\sigma_{\rm tot})^2 d\sigma/d\Omega = \text{const}/t^5 \tag{1}$$

is seen to hold over a range of five decades. Here k is the wave number of the particle being scattered, $d\sigma/d\Omega$ is the differential elastic cross section, and $\sigma_{\rm tot}$ is the total cross section.

We shall show that this empirical law can be understood in terms of an optical model in which the medium is taken to be purely absorptive, and to be spatially distributed according to a Yukawa function. Thus the change in wave number in the scattering region is

$$k' - k = i\eta e^{-\Lambda r}/r. \tag{2}$$

The optical model gives for the scattering amplitude[2]

$$f(\theta) = ik\int_0^\infty \left[1 - e^{2i\delta(\rho)}\right] J_0(k\theta\rho)\rho d\rho, \tag{3}$$

and for the total and elastic cross sections

$$\sigma_{\rm tot} = 4\pi\int_0^\infty \left[1 - e^{2i\delta(\rho)}\right]\rho d\rho, \tag{4}$$

$$\sigma_{\rm el} = 2\pi\int_0^\infty \left[1 - e^{2i\delta(\rho)}\right]^2 \rho d\rho, \tag{5}$$

and the phase shift is

$$2\delta(\rho) = \int_{-\infty}^\infty (k' - k) ds, \tag{6}$$

with $r^2 = s^2 + \rho^2$. From (6) and (2) we find $\delta(\rho) = i\eta K_0(\Lambda\rho)$.

If we measure the momentum transfer in units of Λ, $y = k\theta/\Lambda$, Eqs. (3), (4), and (5) take the form

$$f(\theta) = i(k/\Lambda^2)F(\eta, y^2), \tag{7}$$

$$\sigma_{\rm tot} = (4\pi/\Lambda^2)S(\eta), \tag{8}$$

$$\sigma_{\rm el} = (2\pi/\Lambda^2)s(\eta), \tag{9}$$

with

$$F(\eta, y^2) = \int_0^\infty \{1 - \exp[-2\eta K_0(x)]\} J_0(yx) x dx, \tag{10}$$

$$S(\eta) = F(\eta, 0) = \int_0^\infty \{1 - \exp[-2\eta K_0(x)]\} x dx, \tag{11}$$

$$s(\eta) = \int_0^\infty \{1 - \exp[-2\eta K_0(x)]\}^2 x dx. \tag{12}$$

In Table I we give some values of $S(\eta)$ and $s(\eta)$, and in Fig. 2 we have plotted $(\Lambda^2/4\pi)\sigma_{\rm tot} = S(\eta)$ and $(\sigma_{\rm el}/\sigma_{\rm tot}) = s(\eta)/2S(\eta)$ as functions of η.

Eq. (10) for $F(\eta, y^2)$ is convenient for small y; to find a form good for large y, write $J_0(yx) = \frac{1}{2} \times [H_0^{(1)}(yx) + H_0^{(2)}(yx)]$ and rotate the two resulting integrals to the positive and negative imaginary

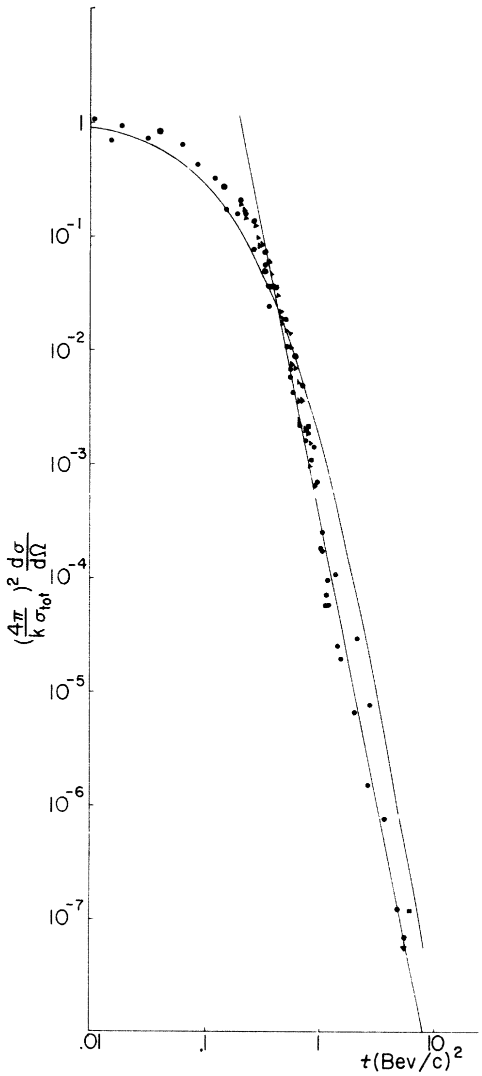

FIG. 1. $(4\pi/k\sigma_{tot})^2 d\sigma/d\Omega$ as a function of t. The straight line is a t^{-5} law. The curve is the prediction of the optical model. The circles give the experimental points of Diddens et al.,[1] the triangles those of Foley et al.,[1] the square that of Baker et al.[1]

axes. This gives

$$F(\eta, y^2) = (2/\pi)\int_0^\infty \{\sin[\pi\eta J_0(x)]\}\{\exp[\pi\eta N_0(x)]\} \times K_0(yx)x\,dx. \tag{13}$$

By changing the integration variable to $z = yx$, and suitably expanding the first two factors in the integral in powers of $(z/y)^2$, we obtain the asymptotic form of (13), valid for $y \gg 1$ and small η/y^2.[3] The leading terms in the asymptotic expansion are

$$F(\eta, t) = (2/\pi)t^{-1}(\gamma^2/t)^\eta[\Gamma(1+\eta)^2 \sin\pi\eta - \Gamma(2+\eta)^2\pi\eta\cos(\pi\eta)/t], \tag{14}$$

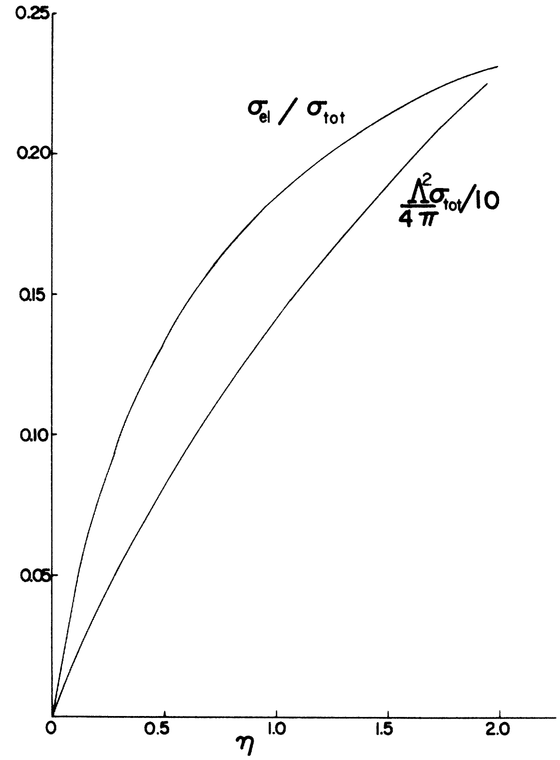

FIG. 2. $\sigma_{el}/\sigma_{tot} = s(\eta)/2S(\eta)$ and $(\Lambda^2/4\pi)\sigma_{tot} = S(\eta)$ as functions of η. The function plotted is $S(\eta)/10$.

where γ is Euler's constant, and $t = y^2$. For $\eta \to 0$, $F(\eta, t)$ is proportional to η, and (14) reduces to the Born approximation result, i.e., to the first two terms in the expansion of the Fourier transform of $k' - k$ in powers of $1/t$.

Eq. (14) gives, for large t,

$$\left(\frac{4\pi}{k\sigma_{tot}}\right)^2 \frac{d\sigma}{d\Omega} = \left[\frac{F(\eta, t)}{S(\eta)}\right]^2 = \left[\frac{2}{\pi}\frac{\Gamma(1+\eta)^2\gamma^{2\eta}}{S(\eta)}\sin\pi\eta\right]^2 t^{-2(1+\eta)}. \tag{15}$$

The observed t^{-5} law would require $\eta = \frac{3}{2}$. From

Table I. $(\Lambda^2/4\pi)\sigma_{tot} = S(\eta)$, $(\Lambda^2/2\pi)\sigma_{el} = s(\eta)$, and $\sigma_{el}/\sigma_{tot} = s(\eta)/2S(\eta)$ as functions of η.

η	$S(\eta)$	$s(\eta)$	$s(\eta)/2S(\eta)$
0.318	0.5528	0.1127	0.1020
0.637	0.9927	0.3004	0.1513
1.000	1.413	0.5216	0.1846
1.273	1.685	0.6825	0.2025
1.500	1.893	0.8104	0.2140
1.910	2.231	1.028	0.2304

Fig 2, we see that $\eta=\frac{3}{2}$ corresponds to $\sigma_{el}/\sigma_{tot}=0.214$, in good agreement with the observed ratio, $\sigma_{el}/\sigma_{tot}=9$ mb/39.5 mb = 0.23.

However, there is a difficulty with this explanation. Because of the factor $\sin\pi\eta$, $F(\eta,t)$ is negative for $\eta=\frac{3}{2}$, and this means that the scattering amplitude, and the elastic cross section, must vanish for a value of t smaller than those at which (14) is valid. Since such a diffraction zero is not observed, we are led to seek another explanation. The largest value of η for which $F(\eta,t)$ does not change sign is $\eta=1$. At this critical value (14) has a remarkable behavior: The first term vanishes and the second term becomes dominant. Instead of the t^{-4} law indicated by (15), we obtain t^{-6}, (15) being replaced by

$$(4\pi/k\sigma_{tot})^2 d\sigma/d\Omega=[8\gamma^2/S(1)]^2 t^{-6}. \tag{16}$$

The difference between t^{-6} and t^{-5} is not of too great concern, since we shall see shortly that the values of t (in units of Λ^2) covered in the experiments are not so large that the asymptotic form is literally correct. In fact, we may anticipate, since the curve of Fig. 1 must bend to the left for smaller t, that for intermediate t the true curve will be a little less steep than indicated by (16).

The choice $\eta=1$ leads to the unique prediction $\sigma_{el}/\sigma_{tot}=0.185$. This is somewhat lower than the experimental ratio, 0.23, which has an error of about 10%, but is in the right neighborhood. The interesting point is not the exact value of the predicted ratio, which could be altered somewhat by a more complicated choice of the radial dependence in (2), but rather the notion that a particular value of σ_{el}/σ_{tot} is a significant parameter. This leads us to examine the ratio for $\pi^{\pm}+p$ scattering, which, according to Jones et al.,[4] is 0.24 for π^-+p and 0.22 for π^++p at 5 BeV (again with a 10% error), strikingly close to the p-p value. Our thought is that a condition such as $\eta=1$ should be interpreted as being the value approached at very high energies. It would follow that the shrinking of the diffraction pattern in p-p scattering, observed by Lindenbaum and Yuan[1] between 8 and 20 BeV/c, is a transitory phenomenon, and that ultimately the width, and σ_{el}/σ_{tot}, decline to a limiting value.

For $\eta=1$, the shape of the diffraction curve is completely determined, and can be calculated numerically, from (10) for small y, and from (13) for large y. The results are given in Table II.

Table II. $(\Lambda^2/ik)f=F(1,y^2)$ and $(4\pi/k\sigma_{tot})^2 d\sigma/d\Omega=[F(1,y^2)/S(1)]^2$ as functions of y; $y=t^{1/2}$.

y	$F(1,y^2)$	$[F(1,y^2)/S(1)]^2$
0	1.413	1.000
0.25	1.330	0.8858
0.5	1.053	0.5575
1	0.5300	0.1407
2	0.1082	5.861×10^{-3}
3	2.389×10^{-2}	2.858×10^{-4}
4	6.264×10^{-3}	1.965×10^{-5}
5	1.934×10^{-3}	1.873×10^{-6}
6	6.666×10^{-4}	2.226×10^{-7}

In order to compare with the experimental results, it remains only to determine the scale factor, Λ. Eq. (8), with $\sigma_{tot}=39.5$ mb, $S(1)=1.413$, gives $\Lambda^2=0.1750$ (BeV/c)^2, $\Lambda=0.4172$ BeV/$c=2.996\,m_\pi$. The calculated curve, with this scale factor, is shown in Fig. 1. It imitates a t^{-5} law remarkably well between values of the ordinate from 10^{-3} to 10^{-7}, and also gives a fair description for smaller t.

The agreement could undoubtedly be improved by a more complicated choice of (2), since the values for small and large t depend on different features of the assumed radial dependence. For example, if (2) is generalized to

$$k'-k=i\left[\eta_1 e^{-\Lambda r}+\eta_2 e^{-\beta\Lambda r}\right]\Big/r,\quad \beta>1, \tag{17}$$

we have $\delta(\rho)=i[\eta_1 K_0(\Lambda\rho)+\eta_2 K_0(\Lambda\beta e)]$, and in (13) we replace $\pi\eta C_0(x)$ by $\pi[\eta_1 C_0(x)+\eta_2 C_0(\beta x)]$, with $C_0=J_0$ or N_0. Eq. (14) is replaced by

$$\begin{aligned}F(\eta,t)&\\ =\frac{2}{\pi}\frac{\beta^{2\eta_2}}{t}\left(\frac{\gamma^2}{t}\right)^{\eta}&\Bigg\{\Gamma(1+\eta)^2\sin(\pi\eta)\Bigg[1+\frac{2(1+\eta)^2}{t}\\ &\times\left\{(\eta_1+\eta_2\beta^2)\left[\tfrac{1}{2}\ln\frac{t}{\gamma^2}-\psi(2+\eta)+1\right]-\eta_2\beta^2\ln\beta\right\}\Bigg]\\ &-\frac{\Gamma(2+\eta)^2\pi(\eta_1+\eta_2\beta^2)\cos\pi\eta}{t}\Bigg[1+\frac{2(2+\eta)^2}{t}\\ &\times\left\{(\eta_1+\eta_2\beta^2)\left[\tfrac{1}{2}\ln\frac{t}{\gamma^2}-\psi(3+\eta)+1\right]-\eta_2\beta^2\ln\beta\right\}\Bigg]\Bigg\},\end{aligned} \tag{18}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, and $\eta=\eta_1+\eta_2$. In (18) we have included the next terms in the asymptotic expansion.

Aside from a scale factor, the dominant terms in (18) again depend solely on η. This illustrates the fact that for large t we are looking in the region near $r=0$, where (17) becomes

$$k' - k = i\eta/r. \tag{19}$$

The characteristic feature of (14) and (18), the occurrence of $t^{-\eta}$ and of the $\sin\pi\eta$ and $\cos\pi\eta$ factors, disappears if we consider a radial dependence in (17) which does not have a $1/r$ dependence at the origin ($\eta=0$). In such a case, the asymptotic expansion is a power series in $1/t$, beginning with $1/t^2$ or a higher power. It should also be noted that the asymptotic formula becomes good only when t is large compared to the square of the largest mass in (17); this is shown by the factor $(\eta_1+\eta_2\beta^2)$ in the correction terms.

While the result for large t depends primarily on η and $(\eta_1+\eta_2\beta^2)$, the integrals appearing in $\sigma_{\rm el}$ and $\sigma_{\rm tot}$ depend on different features of the radial function. In a crude approximation these may be expected to be related to the Fourier transform of (17) for small t, and thus on the combination $(\eta_1+\eta_2/\beta^2)$, giving $\sigma_{\rm el}/\sigma_{\rm tot} \sim s(\eta_1+\eta_2/\beta^2)/2S(\eta_1+\eta_2/\beta^2)$. In this way it would undoubtedly be possible to somewhat alter the predicted $\sigma_{\rm el}/\sigma_{\rm tot}$ without much changing the scattering for large t.

We turn now to the meaning of the unique condition $\eta=1$. Let us suppose that in the center-of-mass system our scattering problem is the solution of a Klein-Gordon equation with an imaginary potential. If the potential is taken to be a scalar function, it will lead to a $\sigma_{\rm tot}$ which vanishes as the energy goes to infinity. We therefore introduce the potential as the fourth component of a four-vector, i.e., like the electrostatic potential. The wave equation is then

$$\{p^2+m^2-[E+iV(r)]^2/c^2\}\psi=0, \tag{20}$$

and, for V small compared to E, the change in wave number in the scattering region is given by $k'^2-k^2=2iEV/\hbar^2c^2$, or, for large k,

$$k'-k=i(E/pc)V(r)/\hbar c=i(c/v)V(r)/\hbar c. \tag{21}$$

To give the correct scattering, $V(r)$ must have the property

$$V(r)\to g^2/r, \quad \text{as } r\to 0. \tag{22}$$

Comparing this with (19), in the high-energy limit ($v/c=1$), we find

$$g^2/\hbar c=\eta=1. \tag{23}$$

The features of the elastic p-p scattering, the ratio of elastic to inelastic scattering and the rapid fall-off without diffraction oscillations, require for their explanation an absorption which is concentrated towards the center, as in (22), and a unique coupling strength, given by (23). The final statement, of course, goes beyond what can be strictly inferred from experiments covering only a limited range of t, but is strongly suggested by the existing results.

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