The energy gap parameters $C_{\vec{k}}$ are highly "anisotropic" in this theory: The index $\vec{k}$ breaks up into three indices (n, k_2, k_3), corresponding to motion in the x, y, and z directions. n is a discrete index, k_2 and k_3 are quasi-continuous with the usual periodic boundary conditions in the y and z directions. The energy $\epsilon_{\vec{k}}$ is related to η_n by

$$\epsilon_{\vec{k}} = \eta_n + (\hbar^2/2m)(k_2^2 + k_3^2). \quad (1)$$

The energy gap parameters $C_{\vec{k}}$ have the form

$$C_{\vec{k}} = C_n \text{ (independent of } k_2, k_3) \quad \text{for } |\epsilon_{\vec{k}} - \mu| < \hbar\omega_c, \quad (2a)$$

$$= 0 \quad \text{for } |\epsilon_{\vec{k}} - \mu| > \hbar\omega_c. \quad (2b)$$

The simple form (2), which greatly simplifies the numerical solution, is associated with the usual simplified model interaction, i.e., constant matrix elements for $|\epsilon_{\vec{k}} - \mu| < \hbar\omega_c$, zero outside that region.

For any given finite width a of the slab, only a finite number of eigenvalues η_n contribute; values of η_n in excess of $\mu + \hbar\omega_c$ give zero, because then all the $\epsilon_{\vec{k}}$, as given by Formula (1), lie outside the interaction region. It is this feature of the problem which permits a straightforward, though not terribly simple, machine calculation.

Returning to the results, it should be noted that the resonances are not symmetrical about the bulk value of C, which is also indicated on Fig. 1. The peaks are much higher than the bulk value, whereas the troughs are only slightly below the bulk value. It is possible, therefore, that some average effect remains even for films of nonuniform thickness.

The distance between resonances in Fig. 1 is one half of the deBroglie wavelength for an electron at the Fermi surface. The precise positions of the resonances are sensitive to the boundary conditions imposed, and should not be taken literally; but the distance between resonances is independent of boundary conditions at the surface. The transition region, near the surface, is only some 2Å thick for our self-consistent solutions.

We believe it to be highly likely that similar resonances occur also in other geometries; for example, there might be size resonances in the nuclear pairing energy.

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IMAGE OF THE PHONON SPECTRUM IN THE TUNNELING CHARACTERISTIC BETWEEN SUPERCONDUCTORS

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The equation for the energy gap function $\Delta(E)$ in a superconductor, using the Eliashberg[1] phonon interaction between electrons, predicts that $\Delta(E)$ will exhibit structure at energies determined by the phonon energies, specifically at the energy gap Δ plus multiples of the phonon frequencies[2] (assuming an Einstein phonon spectrum). It is evident that structure in the energy gap function $\Delta(E)$ will be reflected in the conductance vs voltage plot for a metal-insulator-superconductor tunnel junction, since the conductance is proportional to the density of states.[3] Structure in lead which can be related to the phonon spectrum has been reported by Giaever[4] and structure at phonon harmonics, also in Pb, by Rowell, Chynoweth, and Phillips.[5]

Considerable refinement in experimental technique has allowed us to make a much more detailed quantitative study of this phenomenon. This and the next Letter report the following develop-

ments:

(a) In Pb we have resolved the structure in detail and can assign much of it to specific Van Hove singularities expected from neutron measurements of the Pb phonon spectrum.[6]

(b) As a result we are able to propose a detailed model for Pb in terms of phonon-electron coupling strengths and phonon frequencies, which is consistent with the observed gap and which quantitatively predicts the over-all shape of the density of states vs energy curve.

(c) We have also found phonon structure in Al which correlates with the known phonon spectrum[7] of that metal, as well as in Sn for which no phonon data are available.

(a) Figure 1 shows a plot of essentially d^2I/dV^2 against voltage V (I is the current) for a Pb-Pb tunnel junction at 1.3°K. The "two-superconductor" junction is used to give a sharp peak in density of states with which to probe the phonon structure. The applied ac signal used to obtain d^2I/dV^2 was 42 μV rms ($\sim kT$) and no resultant smearing of the structure is expected. It can be shown theoretically[8] that at a Van Hove singularity $\bar{\omega}$ (a discontinuity in derivative of the phonon density of states, resulting from a stationary point in ω_{ph} vs k), the second derivative of the tunnel current should show a discontinuity. Predicted Van Hove singularities from reference 7 (these are only those along symmetry directions, and are certainly not a complete set) are denoted by arrows. The quoted errors in the neutron work as well as possible temperature dependence of the frequencies are large enough that discrepancies between discontinuities and observed points of steep slope are not significant.

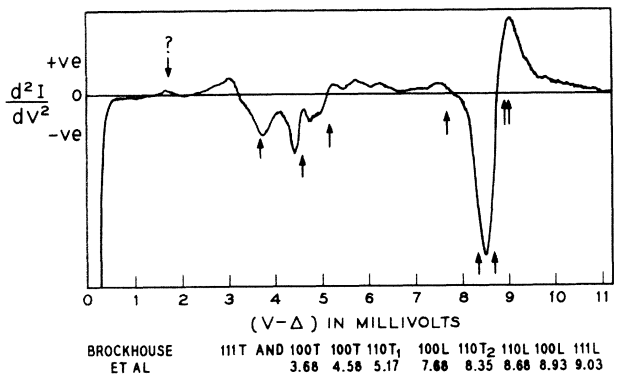

FIG. 1. d^2I/dV^2 vs V (measured from Δ) for a Pb-Pb junction at 1.3°K. Arrows indicate the bias for Van Hove singularities expected from the data of Brockhouse et al.

The finite widths of the discontinuities appear to be caused mainly by the fact that the gaps themselves are not sharp, presumably because of inhomogeneity in the films. This is demonstrated by the lack of discontinuity in I at a bias of 2Δ in the I - V characteristic. The smaller structure of Fig. 1 is reproducible from unit to unit; it is interesting that the peak marked "?" corresponds in energy to the structure observed in infrared absorption.[9] [This is also true for tin (see Fig. 2, curve C).]

From Fig. 1 it is evident that two prominent groups of phonons are involved, a transverse group near 4.5 mV and a longitudinal one near 8.5 mV. Harmonics and sum frequencies of these two groups are prominent in the structure at still higher biases,[5] and these are marked on curve B of Fig. 2. In reference 5 the longitudinal peak was assigned to a harmonic, accounting for the amplitude discrepancy noted there.

(b) The two groups of phonons are the basis of the model, simplified for machine calculation, which Schrieffer, Scalapino, and Wilkins (following Letter[3]) have used in the calculation of the function $\Delta(E)$ in Pb. The meaningful prediction of their calculation is the smoothed density of states which corresponds closely to the first derivative dI/dV; we show the experimental curve for an Al-Pb junction at 1.5°K (Al not superconducting), in Fig. 2 (curve A).

(c) In Fig. 3 we show the dI/dV vs V plot from 25 to 45 mV for an Al-Pb junction at 0.35°K.

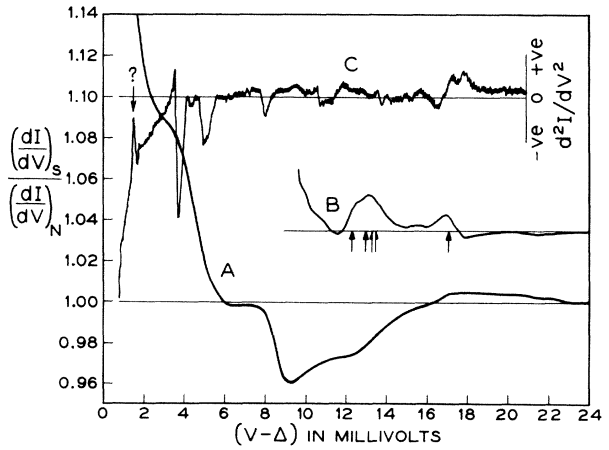

FIG. 2. (a) $(dI/dV)_S/(dI/dV)_N$ vs V (measured from Δ) for an Al-Pb junction at 1.5°K. (b) d^2I/dV^2 vs V (measured from Δ) for a Pb-Pb junction at 1.3°K. Arrows mark the sum and harmonic positions of the main structures of Fig. 1. (c) d^2I/dV^2 vs V (measured from Δ) for a Sn-Sn junction at 1.3°K.

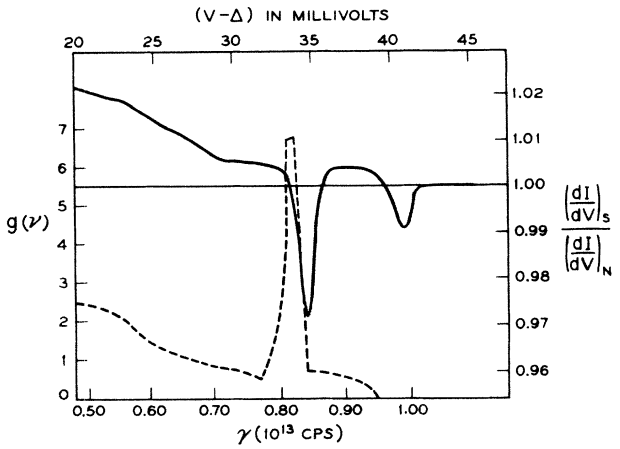

FIG. 3. $(dI/dV)_S/(dI/dV)_N$ vs V (measured from Δ) for an Al-Pb junction at 0.40°K compared with the phonon spectrum for aluminum.

Also shown in Walker's[7] Al phonon spectrum (recalculated by Phillips[7]). It will be seen that the prominent longitudinal peak at 34 mV, as well as the end point of the spectrum at 39 mV, are reflected in the tunnel characteristic. Structure at lower biases was masked by the Pb but an Al-Al sandwich will be investigated.

Figure 2 (curve C) also shows d^2I/dV^2 vs V for a Sn-Sn sandwich and a large amount of structure is observed, some near the Debye energy of 17 mV. The rather surprising low-energy structures were at first thought to be due to Pb impurity in the Sn films but were exactly reproduced after thorough cleaning of the evaporation system and use of a high-purity tin source. The fundamental structure in tin is only as strong as the sum and harmonic structure in lead (curve B) and therefore about $\frac{1}{10}$ the magnitude of the lead fundamental peaks.

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EFFECTIVE TUNNELING DENSITY OF STATES IN SUPERCONDUCTORS*

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Recent tunneling experiments[1,2] involving superconducting metals exhibit structure in the I - V characteristic which has been interpreted in terms of electron-phonon processes. In the preceding Letter[3] Rowell, Anderson, and Thomas present the results of improved experiments which more clearly resolve this structure. Below we summarize the results of a theoretical determination of the tunneling characteristic which is in good agreement with these experiments.

To include dynamic interactions between phonons and electrons in a consistent manner, it is necessary to extend the conventional expression for the tunneling current.[4,5] We take the point of view of Bardeen and of Cohen, Falicov, and Phillips who characterize the tunneling process by an effective one-body Hamiltonian

$$H_T = \sum_{kk's} \{T_{kk'} c_{ks}^{\ b\dagger} c_{k's}^{\ a} + \text{H.c.}\}. \qquad (1)$$

Here $c_{ks}^{\ a}$ and $c_{ks}^{\ b\dagger}$ destroy and create electrons in Bloch states of momentum k, energy ϵ_k measured relative to the chemical potential μ, and