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PREDICTION OF AN INTERACTION SYMMETRY FROM DISPERSION RELATIONS*

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Many authors have proposed that the masses and coupling constants of the fundamental particles may be determinable from some form of dispersion relations. It has also been proposed that, eventually, the symmetries of strong-interaction physics may occur as results of the dispersion theory rather than as separate assumptions.[1] The purpose of this note is to present an approximate set of dispersion relations from which an interaction symmetry is predicted. The PS (pseudoscalar) mesons π, K, and η (with isospin 0 and G-parity even) are considered, and their mass differences neglected. The V (vector) mesons ρ, ω, and K^*, with mass differences neglected, are considered as resonances or bound states of the various pairs of PS mesons, and the coupling constants $\gamma_{\rho\pi\pi}$, $\gamma_{\rho KK}$, $\gamma_{\omega KK}$, $\gamma_{K^*\pi K}$, and $\gamma_{K^*\eta K}$ are computed from a generalization of the "bootstrap" procedure of Zachariasen and Zemach.[2] The ratios of these computed constants are found to be in exact agreement with the predictions of the octet model of unitary symmetry.

We choose as a dimensionless energy variable $x = \frac{1}{4}W^2/m^2$, where W is the total energy in the center-of-mass system and m is the PS-meson mass. We define the P-wave amplitude for the production of a state j of two PS mesons from a similar state i in terms of the elements of the unitary S matrix by the equation

$$T_{ij} = \frac{x^{1/2}(S_{ij} - \delta_{ij})}{(x-1)^{3/2}2i}. \tag{1}$$

The only forces considered are those that arise from the exchanges of the various V mesons, with the energy widths of these mesons neglected. The form of the Born approximation resulting from these forces is well known,[2] and is given by

$$T_{ij} = N_{ij}F(x),$$

$$F(x) = \frac{(\mu + 2x - 1)}{4\pi}\left[-\frac{2}{(x-1)^2} + \frac{2\mu + x - 1}{(x-1)^3}\ln\frac{\mu + x - 1}{\mu}\right], \tag{2}$$

where μ is related to the mass M of the vector mesons by $\mu = \frac{1}{4}M^2/m^2$, and the N_{ij} are constants that depend on the coupling constants of the exchanged V mesons and on isospin-dependent factors that occur in the crossing matrices. As in reference 2, the amplitude T is set equal to $T(\text{Born})/D$, where the only singularity contained in D is the right-hand, "unitarity" branch cut, and only the contributions of P-wave states of two PS mesons are retained in the unitarity condition. A dispersion relation is written for D, with one subtraction chosen so that $D = 1$ at the edge of the left-hand cut, i.e., at the energy $x_l = 1 - \mu$. The V-meson mass and the various coupling constants are then determined from the consistency requirement that the parameters emerging from the dispersion relations agree with those used to specify the forces.

In our model the ω is regarded as a $K\bar{K}$ resonance, which physically is too light to decay into the $K\bar{K}$ state. The ρ and K^* are considered as resonances in the coupled $\pi\pi$ and $K\bar{K}$ states, and in the coupled πK and ηK states, respectively.[3] Hence, we write two-channel dispersion relations for the ρ and K^*, using the method of Bjorken.[4] Since our numerator matrix has the simple form of a constant matrix multiplied by the function of energy $F(x)$, it can be shown from the equations of reference 4 that the approximate scattering matrix obtained from our procedure is symmetric. The amplitudes have the form

$$T_{11} = F(x)D^{-1}(x)\{N_{11} + \alpha(x)[N_{12}{}^2 - N_{11}N_{22}]\}, \tag{3a}$$

$$T_{12} = F(x)D^{-1}(x)N_{12}, \tag{3b}$$

where

$$\alpha(x)=\frac{x-x_t}{\pi}\int_1^\infty \frac{dx'(x'-1)^{3/2}F(x')}{x'^{1/2}(x'-x_t)(x'-x-i\epsilon)}, \tag{3c}$$

and the denominator $D(x)$ is given by

$$D(x)=1-\alpha(x)(N_{11}+N_{22})-\alpha^2(x)(N_{12}^2-N_{11}N_{22}). \tag{3d}$$

The equation for T_{22} may be obtained by exchanging the subscripts 1 and 2 in Eq. (3a).

When two channels are open a resonance occurs if one of the eigenphase shifts increases through 90°. The two physical states (denoted by 1 and 2) are related to the two eigenstates of the scattering (a and b) by a real, orthogonal transformation matrix u, so that the scattering amplitudes may be written $T_{ij}=u_{ia}u_{ja}T_{aa}+u_{ib}u_{jb}T_{bb}$. It can be shown that if T_{ij} have the form of Eqs. (3), the transformation coefficients u_{ix} are energy-independent. If a resonance occurs in the eigenstate a at the energy x_0, then $\mathrm{Re}T_{aa}^{-1}(x=x_0)=0$, and the reduced width $(\gamma^2/4\pi)$ of the resonance is defined by the equation

$$\tfrac{1}{3}(\gamma^2/4\pi)=[(x_0-x)(\mathrm{Re}T_{aa}^{-1})^{-1}]_{x=x_0}.$$

The partial width in channel i is then $(\gamma_i^2/4\pi)=u_{ia}^2(\gamma^2/4\pi)$. We define α_r to be the real part of α, and define $T_{ij,r}$ and D_r to be the expressions for T_{ij} and D obtained if α is replaced by α_r. It can be shown from Eqs. (3) that if one of the eigenphase shifts increases through 90° at the energy x_0, then $D_r(x_0)=0$ and the partial widths of the resonance satisfy the equation[5]

$$\tfrac{1}{3}(\gamma_i\gamma_j/4\pi)=[(x_0-x)T_{ij,r}]_{x=x_0}. \tag{4}$$

This equation is also valid for a bound state $(x_0<1)$, if γ_i is interpreted simply as a coupling constant.

We now write the equations for the coupling constants and vector meson mass, considering first the states coupled to the K^*. The force from K^* exchange contributes to all three processes $\pi K\to\pi K$, $\eta K\to\eta K$, and $\pi K\to\eta K$, while ρ exchange contributes to $\pi K\to\pi K$. The coefficients N_{ij} may be computed by constructing the crossing matrices, taking proper account of the identitity of the particles in the $\pi\pi$ state. The condition of a resonance at $x=\mu$ implies

$$D_{K^*,r}(\mu)=0;$$

$$D_{K^*,r}(x)=1-\alpha_r(x)[\gamma_{K^*\eta K}{}^2-\tfrac{1}{3}\gamma_{K^*\pi K}{}^2+\sqrt{2}\gamma_{\rho\pi\pi}\gamma_{\rho KK}]-\alpha_r{}^2(x)H_{K^*}, \tag{5a}$$

$$H_{K^*}=\gamma_{K^*\eta K}{}^2(\tfrac{4}{3}\gamma_{K^*\pi K}{}^2-\sqrt{2}\gamma_{\rho\pi\pi}\gamma_{\rho KK}). \tag{5b}$$

By applying Eq. (4) to the expressions for the amplitudes $\pi K\to\pi K$, $\eta K\to\eta K$, and $\pi K\to\eta K$, respectively, we obtain the relations

$$\gamma_{K^*\pi K}{}^2=C_{K^*}[-\tfrac{1}{3}\gamma_{K^*\pi K}{}^2+\sqrt{2}\gamma_{\rho\pi\pi}\gamma_{\rho KK}+\alpha_r(\mu)H_{K^*}], \tag{5c}$$

$$\gamma_{K^*\eta K}{}^2=C_{K^*}[\gamma_{K^*\eta K}{}^2+\alpha_r(\mu)H_{K^*}], \tag{5d}$$

$$\gamma_{K^*\pi K}\gamma_{K^*\eta K}=C_{K^*}\gamma_{K^*\pi K}\gamma_{K^*\eta K}, \tag{5e}$$

$$C_{K^*}=-12\pi[F(x)(dD_{K^*,r}/dx)^{-1}]_{x=\mu}.$$

A similar procedure, applied to the $\rho-\pi\pi-K\bar{K}$ system, leads to the equations

$$D_{\rho,r}(\mu)=0;\quad D_{\rho,r}(x)=1-\alpha_r(x)[\gamma_{\rho\pi\pi}{}^2+\tfrac{1}{2}\gamma_{\omega KK}{}^2-\tfrac{1}{2}\gamma_{\rho KK}{}^2]-\alpha_r(x)H_\rho, \tag{6a}$$

$$H_\rho=\tfrac{8}{9}\gamma_{K^*\pi K}{}^4-\tfrac{1}{2}\gamma_{\rho\pi\pi}{}^2(\gamma_{\omega KK}{}^2-\gamma_{\rho KK}{}^2), \tag{6b}$$

$$\gamma_{\rho\pi\pi}{}^2=C_\rho[\gamma_{\rho\pi\pi}{}^2+\alpha_r(\mu)H_\rho], \tag{6c}$$

$$\gamma_{\rho KK}{}^2=C_\rho[\tfrac{1}{2}\gamma_{\omega KK}{}^2-\tfrac{1}{2}\gamma_{\rho KK}{}^2+\alpha_r(\mu)H_\rho], \tag{6d}$$

$$\gamma_{\rho\pi\pi}\gamma_{\rho KK}=C_\rho(\tfrac{8}{9})^{1/2}\gamma_{K^*\pi K}{}^2, \tag{6e}$$

$$C_\rho=-12\pi[F(x)(dD_{\rho,r}/dx)^{-1}]_{x=\mu}.$$

The corresponding equations for the ω are simpler, since only the one $(K\bar{K})$ channel is involved, i.e.,

$$D_{\omega,r}(\mu)=0;\quad D_{\omega,r}(x)=1-\alpha_r(x)[\tfrac{1}{2}\gamma_{\omega KK}{}^2+\tfrac{3}{2}\gamma_{\rho KK}{}^2], \tag{7a}$$

$$\gamma_{\omega KK}{}^2=C_\omega(\tfrac{1}{2}\gamma_{\omega KK}{}^2+\tfrac{3}{2}\gamma_{\rho KK}{}^2), \tag{7b}$$

$$C_\omega=-12\pi[F(x)(dD_{\omega,r}/dx)^{-1}]_{x=\mu}.$$

We take from experiment the fact that $\gamma_{K^*\pi K}{}^2\neq 0$, and assume that each particle is coupled to at least one other, so that $\gamma_{K^*\eta K}{}^2\neq 0$. Equations (5d) and (5e) then imply that $C_{K^*}=1$ and $H_{K^*}=0$, so that the $\alpha_r{}^2$ term of $D_{K^*,r}$ vanishes. The relation $D_{K^*}=D_\omega$ then follows from this fact, the assumption of equal K^* and ω masses, and the requirement that each resonance must correspond to the lowest energy at which $D_r=0$. (There is a second root at which the eigenphase shift decreases back down through 90°.) A continuation of this line of reasoning yields the results that

$H_\rho=0$, $D_\rho=D_\omega$, the equations are consistent, and the five coupling constants are in the ratios

$$\gamma_{\rho\pi\pi}^2:\gamma_{\rho KK}^2:\gamma_{\omega KK}^2:\gamma_{K^*\pi K}^2:\gamma_{K^*\eta K}^2 = \tfrac{4}{3}:\tfrac{2}{3}:2:1:1, \qquad (8)$$

$$\gamma_{\rho\pi\pi}\gamma_{\rho KK}>0.$$

(Only the relative sign of $\gamma_{\rho\pi\pi}$ and $\gamma_{\rho KK}$ is measurable since the other signs may be changed by changing the phase associated with one or more of the particles.) If the integral for $\alpha(x)$ is carried out, one also finds $\mu\sim1.56$ and $\gamma_{\rho\pi\pi}^2/4\pi\sim1.7$.[6]

It can be shown that the ratios of Eq. (8) correspond exactly to the predictions of the octet model of unitary symmetry,[7] so this symmetry is predicted by the bootstrap technique. Of course, these dispersion relations do not predict the existence of the PS- and V-meson octets. There are other conceivable sets of particles for which consistent solutions to the bootstrap equations exist, for example, the set of the π and ρ alone.

Since we have neglected mass differences, a comparison of μ and γ^2 with experiment is not very meaningful. However, $\frac{1}{4}$ the ratio of the average square of the masses of the eight V mesons to the corresponding average for the PS mesons is 1.02, and the experimental reduced width for the $\rho\rightarrow\pi\pi$ is $\gamma_{\rho\pi\pi}^2/4\pi=3M_\rho^2/(M_\rho^2-4m_\pi^2)^{3/2}\sim0.6$.

The solutions given here are only the first approximation to the N/D method, as discussed in reference 2. Furthermore, there is no good reason to think that all the forces left out are less important than those included. However, one can hope that the inclusion of other forces may help lead to an understanding of the mass differences.

A detailed calculation of the $K^*-\pi K-\eta K$ system, with mass differences included, is in progress. We note here that if mass differences are included in the present model of the ω particle, the ωKK and ρKK forces must be very strong, since they produce the ω below the $K\bar{K}$ threshold. If the ωKK coupling is strong enough, it may produce a large negative scattering amplitude in the $I=0$, P-wave, $K\bar{K}$ state at energies shortly above threshold.[8]

Finally, we remark that in unitary symmetry, there are 28 independent P-wave states of two PS mesons, corresponding to the representations of dimensions 8, 10, and 10. In the model given here, it can be shown that the forces existing in the 20 nonresonating states are zero.

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[1]See G. F. Chew, Proceedings of the International Conference on High-Energy Nuclear Physics, Geneva, 1962 (CERN Scientific Information Service, Geneva, Switzerland, 1962), pp. 525-528.

[2]Fredrik Zachariasen and Charles Zemach, Phys. Rev. 128, 849 (1962).

[3]The $\pi\eta$ state is not coupled to the $\pi\pi$ and $K\bar{K}$ states. This follows from G-parity invariance, but may be derived in our model from the crossing relations with no use being made of G invariance.

[4]J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).

[5]We have assumed that the two eigenstates are not resonant simultaneously, an assumption that is clearly valid for the examples considered later.

[6]The value of μ is identical to that occurring in the one-channel $\pi-\rho$ problem discussed in reference 2, while $\gamma_{\rho\pi\pi}^2$ in our model is $\frac{2}{3}$ the value in the one-channel model. However, the value of $\gamma_{\rho\pi\pi}^2$ given in reference 2 is in error, and is too low by a factor of four [F. Zachariasen (private communication)].

[7]These ratios may be derived by using the formalism of M. Gell-Mann, Phys. Rev. 125, 1067 (1962), and California Institute of Technology Synchrotron Laboratory Report CTSL-20, 1961 (unpublished). There is no arbitrariness in the predictions; the behavior of the meson octets under charge conjugation requires that only the representation denoted by F by Gell-Mann is allowed for the V-PS-PS meson interactions.

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