ticularly for the states which are expected to belong to the system whose ground state is the Σ. According to Kycia and Riley,[6] the Y_0^*(1815 MeV) may form the top level of the Λ system. This interpretation is made plausible by the fact that the Y_0^* and Λ can lie on the same Regge trajectory. On this assumption, the K^* particle would constitute the link between the nucleon and the top level of the Λ system. Similarly, the K meson provides the link between the nucleon system and the Σ, and that between the Y_0^* isobar and the Ξ particle.

In analogy to the existence of the two K-meson links, there may exist a second K^* link going from the Σ to a state of strangeness $S=-2$, as shown in Fig. 1. This state Ξ_1^* would have a mass

$$m_{\Xi_1^*}=m_\Sigma+m_{K^*}=2077 \text{ MeV}. \tag{4}$$

In view of Eqs. (1), (3), and (4), one obtains

$$m_{\Xi_1^*}-m_\Xi=m_\rho. \tag{5}$$

The mass difference between Ξ_1^* and the $(\Xi\pi)$ resonance Ξ^* at 1530 MeV is 547 MeV, which is approximately equal to the mass of the η meson, $m_\eta=548$ MeV. Thus Ξ_1^* may belong to a level system whose ground state is the Ξ^* resonance. It should be emphasized that the assumption of the second K^* link is to be considered as hypothetical, in contrast to the three K and K^* links discussed above, which are directly based on empirical relationships [Eqs. (1)-(3)].

I wish to thank Dr. G. B. Collins, Dr. T. F. Kycia, Dr. S. Pasternack, and Dr. K. F. Riley for several helpful discussions.

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*Work performed under the auspices of the U. S. Atomic Energy Commission.

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[5]The fact that the πN resonance at 900 MeV occurs at the threshold for $\Sigma+K$ production has been pointed out previously. After the present paper was completed, it was noticed that the fact that the Y_0^*(1815 MeV) corresponds to the threshold for $K^-+p\rightarrow\bar{K}^*+N$ has been used to explain the mechanism of this resonance [J. S. Ball and W. R. Frazer, Phys. Rev. Letters 7, 204 (1961); S. Minami, Progr. Theoret. Phys. (Kyoto) 25, 863 (1961); W. Krolikowski, Nuovo Cimento 22, 872 (1961)]. It should be noted that there are many thresholds which do not give rise to a resonance maximum in the πN or $\bar{K}N$ scattering, e.g., the thresholds for $\Lambda+K$, $\Lambda+K^*$, and N_1+K production.

[6]T. F. Kycia (private communication).

REDUCTION FORMULA FOR THE FIVE-POINT FUNCTION

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(Received 23 January 1963)

In analyzing the singularities of the triangle diagram in a two-dimensional space-time (coordinates x_0x_1), Källén and Wightman[1] found the following interesting formula:

$$\int_0^1 d\alpha_1 \int_0^1 \alpha_2 \int_0^1 d\alpha_3 \delta(1-\alpha)/D^2$$

$$=-\frac{1}{2\Phi}\sum_i P_i \int_0^1 d\alpha_j \int_0^1 d\alpha_k [\delta(1-\alpha)/D]_{\alpha_i=0}, \tag{1}$$

where $\alpha=\alpha_1+\alpha_2+\alpha_3$, $D=\alpha_2\alpha_3z_1+\alpha_3\alpha_1z_2+\alpha_1\alpha_2z_3-(\alpha_1+\alpha_2+\alpha_3)(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3)$, the sum is taken from $i=1$ to $i=3$, and j and k are the two numbers different from i. The left-hand side is the integral for the triangle diagram and the right-hand side is the sum of three self-energy diagrams with coefficients $-P_i/2\Phi$, where $P_i=\partial\Phi/\partial a_i$, and Φ is the Landau determinant appropriate to the integral. The triangle in two dimensions has poles for the same reason that the pentagon does in four dimensions[2]; hence one might be led to conjecture that the integral for the pentagon can be written as the sum of several four-point functions. That this is true will be shown.

The triple integral on the left-hand side of (1) is with the δ-function condition an integral over a triangle with vertices at the unit points of each of the three coordinate axes in α space. Each piece on the right-hand side represents an integration along one of three edges of the triangle. Thus Stokes' theorem is strongly suggested. It is possible to write $1/D^2$ as $\vec{n}\cdot\mathrm{curl}\vec{V}$, where $\vec{n}$ is the unit normal to the plane $\vec{\alpha}=1$.

For the pentagon diagram the integral is over five $\vec{\alpha}$'s, and the denominator which will still be called D is a homogeneous quadratic form in the $\vec{\alpha}$'s. There is a generalized form of Stokes' theorem[3] available, that when applied to the region in question gives the following result:

$$5^{-1/2}\int_S \frac{\partial A_i}{\partial \vec{\alpha}_i} = 4^{1/2}\sum_j \int_{B_j} A_j, \quad (2)$$

where A_i are five quantities subject to $\sum A_i = 0$, S is the "surface" on which $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$, and $0 \leq \alpha_i \leq 1$. The boundary of S consists of the B_j specified as those parts of S on which $\alpha_j = 0$. To state the general version of Stokes' theorem, it would be necessary to introduce a tensor having 10 independent components, but because of the simple nature of the region S, only the five A's are required. In the more familiar three-dimensional case, $\vec{n}\cdot\mathrm{curl}\vec{V}$ may be written as

$$(3)^{-1/2}[\partial/\partial\alpha_1(V_2 - V_3) + \partial/\partial\alpha_2(V_3 - V_1) + \partial/\partial\alpha_3(V_1 - V_2)],$$

and, if the objects in parentheses are identified with the A_i, it is easy to see the origin of the restriction $\sum A_i = 0$. The construction of the A_i proceeds in two steps; first, functions Λ_i are found such that $\sum_i \partial\Lambda_i/\partial\alpha_i = 1/D^n$ (the integrand) not subject to the restriction $\sum\Lambda_i = 0$, and, secondly, from these the A_i are constructed. The general case when the exponent of the denominator D is n and it is a homogeneous quadratic function of m α's is treated below.

Since D is a homogeneous quadratic form in the α's, $\partial D/\partial\alpha_i$ may be written $\partial D/\partial\alpha_i = C_{ij}\alpha_j$, where the C's are independent of α. These may be regarded as m linear equations satisfied by the m α's. An $(m+1)$st linear relation is furnished by the equation defining the surface of interest $\alpha = 1$. Consequently, the $(m+1)\times(m+1)$ determinant Δ,

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ C_{11} & C_{12} & \cdots & C_{1m} & \partial D/\partial\alpha_1 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ C_{m1} & C_{m2} & \cdots & C_{mm} & \partial D/\partial\alpha_m \end{vmatrix},$$

must vanish on the surface $\alpha = 1$. This determinant may be expanded as $\Delta = P_i\partial D/\partial\alpha_i - \Phi = 0$. Care must be taken in the ensuing arguments not to put $\alpha = 1$ before differentiating. This leads to the result

$$-[(n-1)\Phi]^{-1}(\partial/\partial\alpha_i)[P_i/D^{n-1}] = 1/D^n,$$

and the Λ_i are found to be $-[(n-1)\Phi]^{-1}P_i/D^{n-1}$. If $\sum\Lambda_i = \Lambda$ and $\sum P_i = P$, then a satisfactory choice of A_i will be $A_i = (\Lambda_i\alpha - \Lambda\alpha_i)$. Clearly, $\sum A_i = 0$ and $\partial A_i/\partial\alpha_i = [(n-1)\Phi D^n]^{-1}\{(n-1)\alpha P_i\partial D/\partial\alpha_i + PD(m-1) - (n-1)P\alpha_i\partial P/\partial\alpha_i\}$, where the summation convention is implied. Euler's theorem is used to simplify $\alpha_i\partial P/\partial\alpha_i$, and the result about $P_i\partial D/\partial\alpha_i$ is used to simplify the first term in braces. With these changes,

$$\partial A_i/\partial\alpha_i = 1/D^n + [(n-1)\Phi D^n]^{-1}PD(m+1-2n).$$

If $m+1-2n$ vanishes, the A_i are satisfactory. Allowable choices are $m=3, n=2$; $m=5, n=3$, etc. The first gives the result of Källén and Wightman, and the second gives

$$F_5 = -\left(\frac{5}{4}\right)^{1/2}\left(\frac{1}{4\Phi}\right)$$

$$\times\sum_i P_i\int_0^1 d\alpha_j\int_0^1 d\alpha_k\int_0^1 d\alpha_l\int_0^1 d\alpha_m\left[\frac{\delta(1-\alpha)}{D^{n-1}}\right]_{\alpha_i=0},$$

where F_5 is the integral for the pentagon diagram. This formula expresses the five-point function as the sum of five four-point functions and explicitly exhibits the five-point poles in the factor $1/\Phi$.

More generally, one may inquire when such reduction formulas are possible. There is another similar example due to Brown[4] reducing the hexagon to a sum of pentagons. The hexagon and pentagon have the same singularities. Thus it is not surprising they can be simply related. Aside from its poles, which are simple singularities that do not affect the topology of the Riemann sheets, the pentagon and square have the same singularities; thus it is not surprising to find the pentagon expressed in terms of squares. It should be very interesting to attempt to find such reduction formulas for diagrams with internal structure. It is not clear whether such formulas may be useful to study analyticity or whether a knowledge of analytic properties will help to invent further formulas. The present formula should also be useful in looking for representations of the five-point function.

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PREDICTION OF AN INTERACTION SYMMETRY FROM DISPERSION RELATIONS*

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(Received 14 January 1963)

Many authors have proposed that the masses and coupling constants of the fundamental particles may be determinable from some form of dispersion relations. It has also been proposed that, eventually, the symmetries of strong-interaction physics may occur as results of the dispersion theory rather than as separate assumptions.[1] The purpose of this note is to present an approximate set of dispersion relations from which an interaction symmetry is predicted. The PS (pseudoscalar) mesons π, K, and η (with isospin 0 and G-parity even) are considered, and their mass differences neglected. The V (vector) mesons ρ, ω, and K^*, with mass differences neglected, are considered as resonances or bound states of the various pairs of PS mesons, and the coupling constants $\gamma_{\rho\pi\pi}$, $\gamma_{\rho KK}$, $\gamma_{\omega KK}$, $\gamma_{K^*\pi K}$, and $\gamma_{K^*\eta K}$ are computed from a generalization of the "bootstrap" procedure of Zachariasen and Zemach.[2] The ratios of these computed constants are found to be in exact agreement with the predictions of the octet model of unitary symmetry.

We choose as a dimensionless energy variable $x=\frac{1}{4}W^2/m^2$, where W is the total energy in the center-of-mass system and m is the PS-meson mass. We define the P-wave amplitude for the production of a state j of two PS mesons from a similar state i in terms of the elements of the unitary S matrix by the equation

$$T_{ij}=\frac{x^{1/2}(S_{ij}-\delta_{ij})}{(x-1)^{3/2}2i}. \qquad (1)$$

The only forces considered are those that arise from the exchanges of the various V mesons, with the energy widths of these mesons neglected. The form of the Born approximation resulting from these forces is well known,[2] and is given by

$$T_{ij}=N_{ij}F(x),$$

$F(x)$

$$=\frac{(\mu+2x-1)}{4\pi}\left[-\frac{2}{(x-1)^2}+\frac{2\mu+x-1}{(x-1)^3}\ln\frac{\mu+x-1}{\mu}\right], \qquad (2)$$

where μ is related to the mass M of the vector mesons by $\mu=\frac{1}{4}M^2/m^2$, and the N_{ij} are constants that depend on the coupling constants of the exchanged V mesons and on isospin-dependent factors that occur in the crossing matrices. As in reference 2, the amplitude T is set equal to $T(\text{Born})/D$, where the only singularity contained in D is the right-hand, "unitarity" branch cut, and only the contributions of P-wave states of two PS mesons are retained in the unitarity condition. A dispersion relation is written for D, with one subtraction chosen so that $D=1$ at the edge of the left-hand cut, i.e., at the energy $x_t=1-\mu$. The V-meson mass and the various coupling constants are then determined from the consistency requirement that the parameters emerging from the dispersion relations agree with those used to specify the forces.

In our model the ω is regarded as a $K\bar{K}$ resonance, which physically is too light to decay into the $K\bar{K}$ state. The ρ and K^* are considered as resonances in the coupled $\pi\pi$ and $K\bar{K}$ states, and in the coupled πK and ηK states, respectively.[3] Hence, we write two-channel dispersion relations for the ρ and K^*, using the method of Bjorken.[4] Since our numerator matrix has the simple form of a constant matrix multiplied by the function of energy $F(x)$, it can be shown from the equations of reference 4 that the approximate scattering matrix obtained from our procedure is symmetric. The amplitudes have the form

$$T_{11}=F(x)D^{-1}(x)\{N_{11}+\alpha(x)[N_{12}{}^2-N_{11}N_{22}]\}, \qquad (3a)$$

$$T_{12}=F(x)D^{-1}(x)N_{12}, \qquad (3b)$$