tribution as the Fourier transform of the characteristic function can be generalized in the present case, using probability functions and characteristic functionals.[9] The methods developed here are thus adequate to determine the quantum mechanical density matrix, provided all the correlation functions are given.[10]

It is a pleasure to thank Professor Emil Wolf for introducing me to the subject and for his interest in this work. Mr. C. L. Mehta and Mr. N. Mukunda made several helpful suggestions.

---

[1]For a comprehensive review, see M. Born and E. Wolf, Principles of Optics (Pergamon Press, New York, 1959), Chap. X.

[2]See L. Mandel (to be published), for a systematic account.

[3]R. J. Glauber, Phys. Rev. Letters 10, 84 (1963).

[4]V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); I. E. Segal (to be published); J. R. Klauder, Ann. Phys. (N.Y.) 11, 123 (1960); S. S. Schweber, J. Math. Phys. 3, 831 (1962).

[5]These states are easily represented in a Schrödinger coordinate representation

$$|0\rangle \equiv \psi(0) \to \pi^{-1/4}\exp(-\tfrac{1}{2}x^2),$$

$$|z\rangle = \exp(za^\dagger)\psi(0) \to \pi^{-1/4}\exp(-\tfrac{1}{2}[x+z]^2),$$

as pointed out by Dr. C. Ryan.

[6]For a general theory of representation by an overcomplete family of states, see J. R. Klauder (to be published).

[7]All strange representations of the infinite canonical ring are deliberately ignored here.

[8]E. M. Purcell, Nature 178, 1449 (1956). L. Mandel, Proc. Phys. Soc. (London) 71, 1037 (1958); 74, 233 (1959). See also reference 2.

[9]I. E. Segal, Com. J. Math. 13, 1 (1961). See also E. Hopf, J. Rat. Mech. Anal. 2, 587 (1953). These have to be generalized here by admitting indefinite linear functionals.

[10]The only case where all correlation functions are known is for the important but familiar example of the blackbody radiation. We hope that this circumstance is not time independent!

TWO-STREAM PLASMA INSTABILITY AS A SOURCE OF IRREGULARITIES IN THE IONOSPHERE

D. T. Farley, Jr.
Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colorado
(Received 3 January 1963)

The purpose of this note is to describe an extension of the theory of the two-stream plasma ion wave instability and an application of the theory to the physics of the ionosphere. We shall include in the theory the effect of a magnetic field and also the effect of collisions with neutral particles. Both of these effects can be important in the ionosphere. We find that the qualitative and quantitative predictions of the theory are in agreement with the observed characteristics of a certain type of irregularity found in the equatorial ionosphere. These are often referred to as "equatorial sporadic-E" irregularities. Similar irregularities often appear in the polar ionosphere during auroral displays; it seems very likely that these too are caused by the two-stream instability.

The theory of the so-called two-stream plasma wave instability has been studied by a number of authors in the past few years.[1] A frequently considered case is that of a collisionless plasma with no imposed magnetic field in which the ions and electrons have Maxwellian velocity distributions. The velocity distribution of the electrons is taken to be shifted with respect to that of the ions by an amount $\vec{V}_d$, the mean relative drift velocity. For equal ion and electron temperatures, one finds that longitudinal ion waves will grow in the plasma if V_d is greater than $0.926\ V_{th}$, where $V_{th} = (2KT/m_e)^{1/2}$.

During the day at the magnetic equator, a strong Hall current, called the "electrojet," flows in the direction perpendicular to the earth's magnetic field at an altitude of about 100 km in the ionosphere.[2] It has long been known from radio soundings of the ionosphere that there are irregularities of ionization density associated with this current.[3] Recently these irregularities have been studied in much more detail using VHF radio scattering measurements.[4] The principal motivation for the present study was to see if a plasma microinstability could be the source of these irregularities.

In this note we can only outline the derivation of the results. The full details and more numerical results will be given in a later paper. We assume the unperturbed electron and ion velocity distributions to be Maxwellian. The mean electron velocity is taken to be perpendicular to the magnetic field and to have a magnitude V_d, whereas the mean ion and neutral-particle velocities are assumed to be zero. The perturbation in the charged particle distribution functions is taken to be wavelike, of the form $\exp[i(\omega t - \vec{k}\cdot\vec{r})]$, where the wave vector $\vec{k}$ is real, but ω may be complex. We define α to be the angle between $\vec{k}$ and the magnetic field $\vec{B}$ and β to be the angle between $\vec{k}$ and $\vec{V}_d$.

Starting with these assumptions, one obtains the dispersion equation relating $\vec{k}$ and ω from Boltzmann's equation and Maxwell's equations. When collisions are neglected, this can be done in a fairly straightforward way using the method of Bernstein.[5] We are interested in situations in which collisions with neutral particles are not negligible, however. In many problems one can adequately include the effects of collisions by replacing the collision term in Boltzmann's equation by a simple relaxation term which causes the distribution function to decay toward a uniform Maxwellian distribution (the mean-free-path approach). However, this approximation has a defect which is serious in our problem; i.e., the number of particles is not conserved locally, but only on the average.[6,7] To avoid this difficulty it is necessary to modify the relaxation term so that it affects only the velocity distribution of the particles and not their distribution in space.

The resulting approximation appears to be sufficiently accurate for our purposes. The procedure for deriving the dispersion equation is considerably complicated by the collision terms. However, following the method of Dougherty,[7] one finally obtains the following result:

$$F_e(\theta_e - \zeta_e) + (T_e/T_i)F_i(\theta_i) + ih^2k^2 = 0, \qquad (1)$$

where T_e and T_i are the electron and ion temperatures, and h is the Debye shielding length $(KT_e/4\pi Ne^2)^{1/2}$. Here we have assumed only a single type of ion to be present. For a mixture of ions, one simply replaces the second term in (1) by a weighted sum. The function $F(X)$ is defined (suppressing the e and i subscripts) by

$$F(X) = [(X - i\psi)J + i]/(1 - \psi J), \qquad (2)$$

where J is the Gordeyev integral of the complex argument $X - i\psi$, i.e.,

$$J = \int_0^\infty \exp[-i(X - i\psi)t - \phi^{-2}\sin^2\alpha \sin^2(\tfrac{1}{2}\phi t) - (\tfrac{1}{4}t^2)\cos^2\alpha]dt. \qquad (3)$$

In (1)-(3), the functions θ, ψ, ϕ, and ζ_e are the normalized oscillation frequency (ω), collision frequency (ν), gyrofrequency (Ω), and electron drift velocity component in the $\vec{k}$ direction ($V_d \times \cos\beta$), respectively. These are defined by

$$\theta_{e,i} = \frac{\omega}{k}\left(\frac{m_{e,i}}{2KT_{e,i}}\right)^{1/2} \sim \frac{\text{wave phase velocity}}{\text{particle thermal velocity}}, \qquad (4)$$

$$\psi_{e,i} = \frac{\nu_{e,i}}{k}\left(\frac{m_{e,i}}{2KT_{e,i}}\right)^{1/2} \sim \frac{\text{wavelength}}{2\pi \times \text{mean free path}}, \qquad (5)$$

$$\phi_{e,i} = \frac{\Omega_{e,i}}{k}\left(\frac{m_{e,i}}{2KT_{e,i}}\right)^{1/2} \sim \frac{\text{wavelength}}{2\pi \times \text{gyroradius}}, \qquad (6)$$

$$\zeta_e = V_d \cos\beta \left(\frac{m_e}{2KT_e}\right)^{1/2} \sim \frac{\text{drift velocity}}{\text{electron thermal velocity}}. \qquad (7)$$

When ζ_e is 0, the above results for the dispersion equation, apart from notation, agree with those recently obtained by Lewis and Keller.[8]

In general, of course, it is difficult to solve (1) for ω as a function of $\vec{k}$. Therefore, in this note we shall consider only the special case appropriate to the ionosphere in the altitude range 100-110 km. For comparison with experiment, we are interested in wavelengths of 1 and 3 meters.[4] For the other parameters we shall take $T_e = T_i = 210°K$, $\nu_e = 3\times10^4\ \text{sec}^{-1}$, $\nu_i = 10^3\ \text{sec}^{-1}$, $\Omega_e = 5\times10^6\ \text{sec}^{-1}$, $\Omega_i = 80\ \text{sec}^{-1}$, and $N = 10^5$ electrons/cc, and we shall assume the ions to be mostly NO^+.[9] The electron thermal velocity $(2KT/m_e)^{1/2}$ is 80 km/sec, and the ion thermal velocity $(2KT/m_i)^{1/2}$ is 340 m/sec. From these values we obtain the following useful inequalities: $h^2k^2 \ll 1$, $\phi_i \ll \psi_i$, and $\phi_e \gg 1, \psi_e$. Using these various inequalities, (1) can be simplified to

$$\frac{1 + P_iZ(p_i)}{1 + i\psi_iZ(p_i)} + \frac{1 + (1-\delta)p_eZ(p_e)}{1 + i(1-\delta)(\psi_e/\cos\alpha)Z(p_e)} = 0, \qquad (8)$$

where $p_i = -\theta_i + i\psi_i$, $p_e = (\zeta_e - \theta_e + i\psi_e)/\cos\alpha$, and δ is the small quantity $1/2\phi_e^2$. Z is the plasma dispersion function tabulated by Fried and Conte,[10] i.e.,

$$Z(w) = 2i \exp(-w^2)\int_{\infty}^{iw} \exp(-t^2)dt \qquad (9)$$

for all values of w in the complex plane. In the limit as $\cos\alpha \to 0$, (8) can be simplified still further, yielding

$$\frac{1+p_i Z(p_i)}{1+i\psi_i Z(p_i)} + \frac{\delta(\zeta_e - \theta_e + i\psi_e)}{\zeta_e - \theta_e + i\delta\psi_e} = 0. \qquad (10)$$

Using the data given above, we have solved (10) graphically for k equal to 2π m^{-1} and $2\pi/3$ m^{-1}. In the first case, $\psi_e = 0.06$, $\psi_i = 0.5$, and $\delta = 5 \times 10^{-3}$. One finds that the condition for critical stability is $\delta_e - \theta_e = 2.55\,\delta\psi_e = 7.6\times10^{-4}$ and $\theta_i = (m_i/m_e)^{1/2}\theta_e = 1.19$ (purely real). Thus $\zeta_e = 5.85 \times 10^{-3}$, which corresponds to a component of drift velocity in the direction of $\vec{k}$ of 470 m/sec. The above value of θ_i corresponds to a phase velocity of 405 m/sec. A drift velocity greater than 470 m/sec will cause waves with this wave vector $\vec{k}$ to grow (ω will have a negative imaginary part).

For a wavelength of 3 meters, $\psi_e = 0.18$, $\psi_i = 1.5$, and $\delta = 5.5\times10^{-4}$. The critical conditions are $\theta_i = 1.10$ and $\zeta_e = 5.12\times10^{-3}$, corresponding to a phase velocity of 375 m/sec and a drift velocity of 410 m/sec.

For a collisionless plasma with no magnetic field, the conditions for critical stability are $\zeta_e = \theta_i = 0.926$ (assuming $h^2k^2 \approx 0$). It can be seen that the magnetic field and the collisions do not greatly alter the frequency of oscillation (i.e., the value of θ_i), but they drastically alter the magnitude of the relative drift velocity required for instability. It is clearly quite easy to excite waves that move in directions nearly perpendicular to the magnetic field. The drift velocity need only be somewhat greater than the ion thermal velocity (i.e., the velocity of sound in the plasma), not the electron thermal velocity as in the collisionless, field-free case.

One can show that (10) is applicable as long as $|\cos\alpha|$ is somewhat less than $\psi_e/\phi_e = \nu_e/\Omega_e \approx 6 \times 10^{-3}$. For departures from orthogonality greater than this, one must use (8). The results of some calculations for propagation directions departing slightly from orthogonality are shown in Fig. 1. In this figure we have plotted the critical drift velocity component ($V_d \cos\alpha$) as a function of the angular departure from orthogonality measured in degrees.

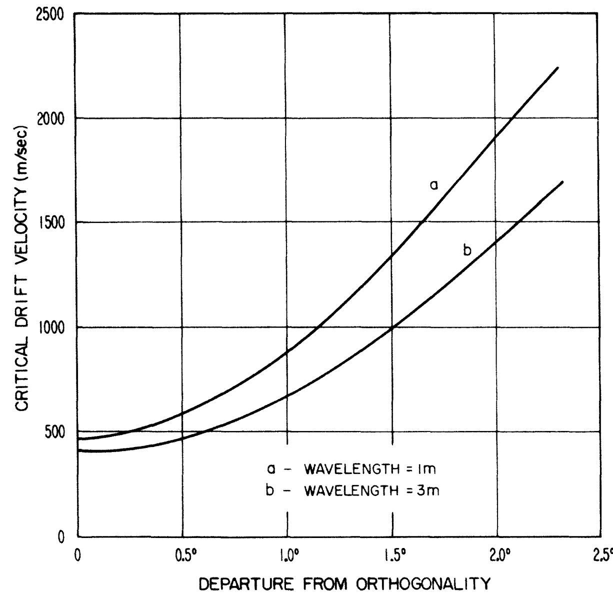

FIG. 1. The critical drift velocity $V_d \cos\beta$ (the velocity required for vanishingly small growth) as a function of the angular departure from orthogonality of $\vec{k}$ (the wave propagation vector) and $\vec{B}$ (the magnetic field).

The electron drift velocities in the equatorial electrojet are thought to reach a maximum value of the order of 500-600 m/sec, but this is only a fairly rough estimate. The ions move only slightly with respect to the neutral particles. On the basis of the theory described here, one would expect plane wave irregularities with wavelengths of 1 and 3 meters to be formed in the electrojet, since the minimum critical velocities are less than 500 m/sec. These waves should propagate only in directions which are within a degree or so of being orthogonal to the magnetic field. The angular spread should be somewhat less for the shorter wavelength; the spread for both wavelengths should increase slightly as V_d increases. The propagation vector should always have a component in the direction of V_d (i.e., $\beta < \frac{1}{2}\pi$). The phase velocity of the waves should be approximately the speed of sound in the plasma (about 340 m/sec). Further, as the drift velocity increases from zero, the 3-meter waves traveling in a particular direction should be excited somewhat before the 1-meter waves, because of the different threshold velocities.

The radar backscatter measurements at the equator at 50 and 150 Mc/sec are in complete accord with these predictions of the theory, and almost all of the above characteristics have in fact been observed.[4] There would seem, then, to be little doubt that at least one type of sporadic-E irregularity at the equator is caused by the two-stream mechanism described here.

Radar observations indicate that much the same sort of irregularity is also formed in the polar ionosphere during auroral displays.[11] It is known that a current which may be even stronger than the equatorial electrojet flows normal to the lines of force at these times,[12] and so one would expect the same instability mechanism to be effective. With a sufficiently strong current, irregularities with quite small wavelengths could be excited, and the deviation from orthogonality of these waves could be somewhat greater than at the equator. It is tempting to think that many of the observed types of irregularities in the ionosphere whose origins have long been explained may be caused by various plasma microinstabilities.

I would like to thank Dr. K. L. Bowles and Dr. R. Cohen for stimulating my interest in this problem and for many helpful comments. It is a pleasure to acknowledge many useful discussions with Dr. J. P. Dougherty concerning the method of including the collision effects. I am also grateful to C. Romero for help with the numerical calculations.

---

[1]For example, O. Buneman, Phys. Rev. 115, 503 (1959); J. D. Jackson, J. Nucl. Energy: Pt. C 1, 171 (1959); I. B. Bernstein and R. M. Kulsrud, Phys. Fluids 3, 937 (1960); B. D. Fried and R. W. Gould, Phys. Fluids 4, 139 (1961); D. Pines and J. R. Schrieffer, Phys. Rev. 5, 1387 (1961).

[2]See, for instance, C. A. Onwumechilli, J. Atmospheric Terrest. Phys. 13, 222,235 (1959); A. J. Zmuda, J. Geophys. Res. 65, 2247 (1960).

[3]For a review of this subject, see Ionospheric Sporadic-E, edited by E. K. Smith and S. Matsushita (Pergamon Press, New York, 1962).

[4]K. L. Bowles, R. Cohen, G. R. Ochs, and B. B. Balsley, J. Geophys. Res. 65, 1853 (1960); R. Cohen and K. L. Bowles (to be published); K. L. Bowles, B. B. Balsley, and R. Cohen (to be published).

[5]I. B. Bernstein, Phys. Rev. 109, 10 (1958).

[6]P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. 94, 511 (1954); E. P. Gross and M. Krook, Phys. Rev. 102, 593 (1956).

[7]J. P. Dougherty (to be published).

[8]R. M. Lewis and J. B. Keller, Phys. Fluids 5, 1248 (1962).

[9]Satellite Environment Handbook, edited by F. S. Johnson (Stanford University Press, Stanford, California, 1961).

[10]B. D. Fried and S. D. Conte, The Plasma Dispersion Function (Academic Press, Inc., New York, 1961).

[11]This subject is reviewed by H. G. Booker in Physics of the Ionosphere, edited by J. A. Ratcliffe (Academic Press, Inc., New York, 1960).

[12]S. Akasofu, J. Atomospheric Terrest. Phys. 19, 10 (1960).

EXPERIMENTAL CONFIRMATION OF THE LANDAU LAW IN COUETTE FLOW

R. J. Donnelly
Institute for the Study of Metals and Department of Physics, The University of Chicago, Chicago, Illinois
(Received 6 February 1963)

The theory of hydrodynamic stability has been most successful in the linearized approximation where the aim is to discover the conditions under which a flow will first become unstable.[1] The origin of the instability is considered to result from the amplification of an initially infinitesimal disturbance which grows with time as $\exp\sigma t$. The criterion for instability is that σ should have a positive real part, otherwise the flow is considered to be stable. On the other hand, it is known experimentally that in some cases the disturbances do not amplify continuously as $\exp\sigma t$, but approach a steady, finite amplitude after a passage of time. The calculation of this steady amplitude, involving as it does the full nonlinear set of hydrodynamical equations, presents a formidable mathematical problem. Landau first suggested[2,3] from very general considerations that the equilibrium amplitude of such a disturbance must increase as $(R-R_c)^{1/2}$, where R is some suitably defined "Reynolds number," beyond the onset of instability at R_c. Quantitative visual observation of a three-dimensional vortex is very difficult, and as a result the experimental consequences of this law have been explored by torque measurements on flow between rotating cylinders (Couette flow). This method suffers from the necessity to make measurements[4-6] which integrate the effects over many wavelengths of the disturbance. In this Letter a new technique is described which allows direct measurements of the amplitude of the disturbances envisioned in the theory; and the measurements confirm Landau's equation for the variation of the equilibrium amplitude.