divergent contributions, we have

$$\sum_1 z\, \partial n(l)/\partial z = \sum_1 n^2(l)[1+\varphi_1(0) + \sum_2 \varphi_2(0,l')\, z\partial n(l')/\partial z + \varphi_2(0,0)\sum_1 z\partial n(l')/\partial z], \quad (12)$$

where $\sum_2$ implies summation over all l for which $|l|>\Delta$. Equation (12) is a contradiction under the hypotheses, since on the right we have a divergent quantity and on the left the same divergent quantity multiplied by another. In the case that $\sum_1 n^2(l)$ converges $\sum_1 z\partial n(l)/\partial z$, and therefore $z(\partial N/\partial z)$, converges a fortiori. By similar arguments all the second derivatives can be proved to converge.

In the case that $\sum_1 n^2(l)$ diverges, an even more surprising conclusion can be drawn from Eq. (12). The only way this equation can be satisfied self-consistently is to give the left-hand side a finite nonzero value determined by setting the bracket on the right-hand side equal to zero:

$$\sum_1 z\frac{\partial n(l)}{\partial z} = -[1+\varphi_1(0)+\sum_2\varphi_2(0,l') \times z\frac{\partial n(l')}{\partial z}]/\varphi_2(0,0). \quad (13)$$

The above statement means nothing else than that $z[\partial n(l)/\partial z]$ as a function of l has a high and narrow shape around $l = 0$ in the uncondensed state near the transition which becomes a δ-function at the transition. In the condensed state there is, of course, also a δ-function at $l = 0$. Further computation yields the result that the two δ-functions and, indeed, the complete functions approach the same limit from above and below the transition. This fact has the consequence that $z(\partial N/\partial z)$ (and the other second derivatives) are continuous. The transition is at least of third order. Equation (12) also shows that an infinite value of $z(\partial N/\partial z)$ would result if $\varphi_2(0,0)$ were zero at the transition. This could be expected to be true at most at one point of the line $z = z_0(\beta)$.

The experiments of Fairbank et al.[3] have shown that C_p is infinite on the λ-line at the saturated vapor pressure and other authors[6,7] have found supporting evidence for the infinite value of the other derivatives at the same pressure. It has generally been assumed that this conclusion will be found to be the case all along the λ-line. Tisza[8] has recently pointed out that an extension of the phase rule presents us with the following dilemma: either the domain of the λ-line where the second derivatives are infinite is a point or there is a hitherto unsuspected degeneracy in the ground state of liquid helium. Tisza himself takes the second alternative. The above considerations suggest that further experiments should be carried out all along the λ-line to confirm or decisively rule out the first alternative.

---

*This work was carried out at the Institute for Theoretical Physics of the University of Utrecht. The author gratefully acknowledges a Fulbright and a Guggenheim Fellowship which made his stay possible.

†On leave from the National Bureau of Standards, Washington, D. C.

[1]C. N. Yang, Lecture at Conference on the Quantum Mechanical Many-Body Problem, Stevens Institute, Hoboken, New Jersey, January, 1957 (unpublished).

[2]T. D. Lee and C. N. Yang, Phys. Rev. 105, 1119 (1957).

[3]Fairbank, Buckingham, and Kellers, Bull. Am. Phys. Soc. Ser II, 2, 183 (1957).

[4]Yang and Lee have reported in references 1 and 2 a summation procedure which, I believe, is different from the one described below, which has enabled them to compute the pressure and density at the λ-point.

[5]These equations are similar but not identical to expressions for the thermodynamic potentials derived by L. I. Schiff [Phys. Rev. 59, 751, 758 (1941)]. Dr. Yang and Dr. Yee have informed me that they have also obtained Eq. (1)-(4).

[6]M. H. Edwards, Can. J. Phys. 36, 885 (1958).

[7]E. Maxwell and C. E. Chase, Contributed Paper, Kamerlingh Onnes Conference on Low-Temperature Physics, Leyden, June, 1958 (unpublished). This paper also reports results at 1 atmosphere pressure.

[8]L. Tisza, Phys. Rev. 110, 587 (1958).

VLASOV INSTABILITY IN LONGITUDINAL PLASMA OSCILLATIONS

Peter L. Auer
Research Laboratory, General Electric Company, Schenectady, New York
(Received August 21, 1958)

In a recent communication Buneman[1] raised the point that the well-known two-stream instabilities observed in electronic devices may have important consequences in plasmas of thermonuclear interest. For fully ionized plasmas the analog of the instabilities discussed by Buneman is to be found in the Vlasov equation, which describes the collective motions of the plasma under the influence of externally applied plus internally generated coherent fields. The classic paper in this field is Landau's,[2] wherein it is shown that the evolution of plasma perturbations may be studied through an impulse response (or generalized impedance) function specified by the linearized version of the Vlasov equation. The spatial Fourier coefficient of this response function will be denoted here as $H_k(t)$, and the index k is always treated as real. In the present context the initial preparation of the plasma is stable or unstable according to whether there are any $H_k(t)$ which grow in time.

For an initially neutral plasma of uniform density, the $H_k(t)$ are given by the inverse Laplace transform of a quantity $1/D_k(p)$, where

$$D_k(p) = 1 + \frac{\omega_0^2}{ik}\int_{-\infty}^{\infty} \frac{f_0'(u)}{p+iku}\, du\,, \qquad (1)$$

$$\omega_0^2 f_0(u) = \sum \omega_\alpha^2 f_{0\alpha}(u)\,,$$

$$\omega_0^2 = \sum \omega_\alpha^2 = 4\pi \sum \frac{n_{0\alpha} e_\alpha^2}{m_\alpha}\,,$$

and $n_{0\alpha}$, e_α^2, and m_α are the partial initial density, charge, and mass of the α species in the plasma, and all $f_{0\alpha}$ are velocity distribution functions normalized to unity. The summations are over all charge species, subject to neutrality; the variable u denotes the velocity component in the direction of wave propagation. The important thing to observe is that $D_k(p)$ has a discontinuity along the entire $\mathrm{Im}(p)$ axis and its proper inversion requires that it be continued analytically from the positive into the negative real half-plane.[2,3] In the event the quantity $D_k(p)$ has no zeros in the complex plane, the above procedure assures that all $H_k(t)$ with finite k decay exponentially in time and the plasma remains stable. If zeros exist, they must come in conjugate pairs with positive and negative $\mathrm{Re}(p)$ of equal magnitude. The latter situation is unstable in that $H_k(t)$ can grow exponentially in time.

Since the nonlinear Vlasov equation conserves not only energy but also entropy,[4] it can be readily shown that all initial distributions Maxwellian in velocity are stable. Landau explicitly considers only these cases. More recently Berz[5] has shown that all bell-shaped distributions centered about the origin are stable. The analysis given by Berz is readily extended to show that any single-peaked composite distribution with or without an axis of symmetry is stable in the sense considered above. This follows from the lemma that $D_k(p)$ is analytic everywhere in the complex p plane (except for the discontinuity mentioned above) for real k and normalizable distribution functions with the property

$$f_0'(u) \gtrless 0 \text{ for } u \lessgtr u_0 \qquad (2)$$

for some u_0. The lemma is easily proved by the observation that when (2) applies, $\mathrm{Re}D_k(p)$ is always greater than zero for $\mathrm{Im}D_k(p)$ equal to zero.

Since the lemma given by (2) provides a sufficient criterion for stability, it immediately leads to a necessary condition for instability. Namely, it is necessary that the composite distribution function possess a minimum in order that unstable solutions in the sense considered here may arise. Although this requirement is not sufficient to assure that unstable solutions exist, it does allow us to compute lowest upper bounds on stability.

For example, an important consequence of (2) is the fact that an otherwise initially quiescent plasma containing a macroscopic current need not be unstable. Consider a plasma composed of singly charged positive ions of mass M and electrons at a uniform density n_0 and with initial current density J_0. If it is assumed that initially both ions and electrons have velocity distributions which are Maxwellian centered about their respective average stream velocities, the necessary condition for instability mentioned above leads to the conclusion that this plasma preparation is unconditionally stable in our sense provided J_0 satisfies the inequality

$$J_0 \leq e n_0 \gamma(\mu)(2kT/M)^{1/2}\,, \qquad (3)$$
$$\gamma(\mu) > 1\,;$$

where e is the magnitude of the electronic charge, k is Boltzmann's constant, T is the temperature, and μ is the ratio of ion to electron mass. For a deuterium plasma γ has the value of 2.8; with an initial density of 10^{15} particles per cc and temperature of 10 kv, the relation in (3) provides that the lowest upper bound on the current is 4000 amp/cm^2.

In view of the sizeable currents computed above one may be led to conclude that Vlasov instabilities of the simple type considered here will not play a major role in the problem of magnetic confinement and attendant diamagnetic currents. The nature of the problem, however, is appreciably altered by the presence of inhomogeneous magnetic fields and must be investigated separately. On the other hand, the situation investigated here may have a valid application to the tenuous plasma of the sun's corona and possibly these simple types of Vlasov instabilities have some connection with solar prominences.

It is a pleasure to acknowledge a stimulating conversation on these matters with E. A. Frieman, of Project Matterhorn, and a number of helpful discussions with H. Hurwitz, Jr., and S. Tamor at the Research Laboratory.

---

[1] O. Buneman, Phys. Rev. Lett. 1, 8 (1958).

[2] L. Landau, J. Phys. U.S.S.R. 10, 25 (1946).

[3] N. G. Van Kampen, Physica 23, 641 (1957). This reference contains a comprehensive review of previous literature.

[4] Appendix by W. Newcomb in I. Bernstein, Phys. Rev. 109, 20 (1958).

[5] F. Berz, Proc. Phys. Soc. (London) B69, 939 (1956).

PROTON GYROMAGNETIC RATIO

R. L. Driscoll and P. L. Bender
National Bureau of Standards,
Washington, D. C.
(Received November 12, 1958)

The proton precession frequency has been measured in a 12-gauss magnetic field supplied by a precision solenoid. The technique used was a variation of the free precession methods of Hahn[1] and of Packard and Varian.[2] The sample consisted of a 2-cm diameter sealed glass sphere containing distilled water. It was polarized in a 5000 gauss field and then shot through a forty-foot pneumatic tube into the center of the solenoid. After arrival about two seconds later the magnetization was still large and lay along the direction of the solenoid field. A short pulse at the resonance frequency of about 52.5 kilocycles/sec was then applied perpendicular to the solenoid axis. This pulse left the magnetization perpendicular to the field direction and precessing about it. The precession frequency was obtained by measuring the period for a given number of cycles of the signal induced in a pickup coil surrounding the sample. The Q of the pickup coil was kept low to prevent radiation damping and possible associated frequency shifts.[3,4] The average period over a three second time interval could be measured to $1:10^7$ with this method.

The solenoid[5] used in this experiment consisted of a single-layer helical winding in a lapped groove on a fused silica form. The value of the magnetic field was calculated from the carefully measured geometry of the solenoid and the current through it. The only critical dimension was the pitch, which could be measured to about 1 ppm. Extensive efforts were made to prevent errors due to slightly magnetic materials used in or near the solenoid. The components of the earth's field perpendicular to the solenoid axis were cancelled out and the solenoid current was reversed between measurements to average out the component of the earth's field along the axis. In order to avoid magnetic field gradients and man-made perturbations, the experiment was carried out at the Fredericksburg Magnetic Observatory of the U. S. Coast and Geodetic Survey.

In the final measurements with the apparatus the scatter was usually less than 1 ppm. Most of this was due to short-term variations in the earth's field and in the current through the solenoid. For protons in water the preliminary result that has been obtained in terms of the NBS electrical standards is $\gamma_p = (2.67515 \pm 0.00001) \times 10^4$ gauss^{-1} sec^{-1}. Benzene was found to give a result 1.9 ± 0.2 ppm higher. The accuracy of the result will be increased somewhat when further measurements on the solenoid pitch are available and when more checks on other possible sources of small systematic errors have been made. Measurements with a second precision solenoid on a Pyrex form are also under way.

When the above preliminary value is combined with the results of recent determinations of the NBS ampere in absolute measure,[5,6] the result for water, uncorrected for diamagnetism,[7] is $\gamma_p = (2.67513 \pm 0.00002)\times 10^4$ gauss^{-1} sec^{-1}. This value differs somewhat from the widely used value of $\gamma_p = (2.67523 \pm 0.00006)\times 10^4$ gauss^{-1} sec^{-1} obtained by Thomas, Driscoll, and Hipple[8] using the value for the NBS ampere as given at the time of their experiment. Both measurements are in disagreement with a value of $\gamma_p = (2.67549 \pm 0.00008)\times 10^4$ gauss^{-1} sec^{-1} published recently by Wilhelmy.[9]

Recently Cohen and DuMond[10] have given corrections to their 1955 set of adjusted values of the atomic constants[11] for a revised theoretical estimate of the anomalous moment of the electron. However, if the present value of the proton gyromagnetic ratio were used instead of the value obtained by Thomas, Driscoll, and Hipple, the values of a number of the constants in the 1955 adjustments of Cohen, DuMond, Layton,