EXCLUSION PRINCIPLE AND PHENOMENOLOGICAL OPTICAL-MODEL POTENTIALS*

Lee M. Frantz,† Robert L. Mills,
Roger G. Newton,‡ and Andrew M. Sessler
The Ohio State University,
Columbus, Ohio
(Received August 13, 1958)

In recent years the elastic scattering of particles by a nucleus has been successfully analyzed[1] in terms of a simple complex-potential model.[2] In such calculations the exclusion principle has been ignored although for not too energetic bombarding nucleons the effect can be considerable. A modification of the optical model is proposed here which takes into account the effect of the exclusion principle within the framework of a single-particle description of the many-body problem.

Suppose that the nucleus can be described by a single-particle real potential U, which could be momentum as well as coordinate dependent, let the eigenfunctions of $T+U$ be $u_n(r)$, and assume the first N levels are occupied. Suppose further that an incident nucleon in its interaction with the nucleus is to be described by a complex potential V. Note that it is in the nature of the optical model that, although the incident nucleon is identical with those in the nucleus, it is subject to a different complex potential to simulate the inelastic processes. Let the incident nucleon be described by the wave function ψ. We seek a one-body description of the scattering. Consequently, if we ignore the exclusion principle and let T be the kinetic energy, we are led to the Schrödinger equation

$$(T+V)\psi = E\psi. \tag{1}$$

It is clear that the solutions to Eq. (1) are in general not orthogonal to the wave functions $u_n(r)$, for $n=1, 2 \ldots N$, describing the nucleons in the nucleus. Let Λ be a projection operator onto the N occupied states. Then Eq. (1) should be replaced with

$$[T+U+(1-\Lambda)(V-U)]\,\psi = E\psi. \tag{2}$$

The solutions ψ of this equation are orthogonal to the occupied states. Note that the orthogonal complement of the space spanned by the occupied states is not invariant under the operator $T+V$. That is why it would not be correct simply to solve Eq. (1) and then to project out the occupied states. If the occupied states are eigenfunctions of $T+V$, then Eq. (2) reduces to Eq. (1). The effect of the exclusion principle is then taken into account simply by choosing orthogonal solutions to the Schrödinger equation.

Clearly Eq. (2) is a simple extension of the work of Bethe and Goldstone.[3] One of us[4] has been able to show rigorously, starting from the n-body problem, that the elastic scattering of a nucleon by a nucleus can be described by an equation of the form of Eq. (2), with V a nonlocal potential. Furthermore, an explicit expression was given for the potential in terms of the assumed forces between nucleons.

The point of this note is to suggest the use of Eq. (2) with V as a phenomenological potential. Actually, the exclusion principle will also have an explicit effect on V, arising for example from the requirement that in intermediate states the target nucleon as well as the incident nucleon must be in unoccupied states. We do not know the additional restrictions this may impose on V, but nevertheless suggest that Eq. (2) is clearly more nearly correct than Eq. (1).

If V is taken to be a local potential, the solutions to Eq. (2) are not only quantitatively but also qualitatively different from the solutions to Eq. (1). This may be seen by considering the one-dimensional problem in which a particle of mass m and energy E impinges on a potential hole

$$\begin{aligned} V(x) &= 0,\ x<0,\ x>t, \\ &= -V_0 - iV_1,\ 0<x<t, \end{aligned} \tag{3}$$

while a single identical particle is confined in the lowest state of an infinitely deep potential hole of the same extent. For this problem Eq. (2) can be solved exactly, and taking the limit of large t leads to the following simple expressions for the transmitted and reflected amplitudes if the incident wave has unit amplitude:

$$T = -\frac{e^{-ikt}}{(k+\alpha)^2}\left[4k\alpha e^{i\alpha t} + \frac{4\pi^2 ki}{t^3\,\alpha^2}\right],$$
$$R = \frac{1}{(k+\alpha)^2}\left[(k^2-\alpha^2) + \frac{4\pi^2 i}{t^3\,\alpha}\right], \tag{4}$$

where

$$k^2 = 2mE/\hbar^2,$$
$$\alpha^2 = (2m/\hbar^2)[E+V_0+iV_1],$$
$$\mathrm{Im}\,\alpha > 0.$$

The first term in each of these two expressions is the usual result while the second term reflects the effect of the exclusion principle. The transmission is thus greatly increased, since the first term in T corresponds to an exponential decay of ψ due to absorption. The relative change in R is much smaller. The t dependence of the exclusion-principle terms originates in part from the fact that the amount of momentum space occupied, and therefore excluded, by the bound nucleon decreases with increasing potential width.

This example indicates that Eqs. (1) and (2), with the same potential V, will in general lead to different angular distributions and total cross sections. We want to emphasize the corollary: the phenomenological potential needed in order to fit given experimental data will depend on whether Eq. (1) or Eq. (2) is used. It might be felt that in a phenomenological calculation this is unimportant since V is adjusted to fit the experiments and is used only to summarize a great many data in a convenient form. We do not feel this is a valid view for two reasons.

First, a great deal of physical significance has always been ascribed to the optical parameters— such as a nuclear radius, and surface thickness. Including the exclusion principle will inevitably lead to different optical parameters. In principle, presumably, there always exists a "potential" which fits any given scattering data by means of Eq. (1); in practice only energy-dependent and spin-orbit, but otherwise local, potentials have been used. The effect of a local potential in Eq. (2) is quite different, since the exclusion principle already acts in a highly nonlocal manner. It is therefore to be expected that the use of a local potential in Eq. (2) has more physical significance than in Eq. (1).

Second, from a theoretical point of view, any fundamental theory attempting to describe the scattering of a nucleon by a nucleus in terms of nucleon-nucleon forces[4] will lead to an equation of the form of Eq. (2). If the analysis of data in terms of optical parameters is to prove useful to the theoretician, i.e., remove the necessity for theoretical calculations of cross sections or phase shifts, then the optical parameters must refer to Eq. (2).

Since Eq. (2) is nonlocal there may be some concern over the requirements of causality. We are unaware of any sufficiently general expression of this requirement to be applicable to this equation.[5] Although Eq. (2) does not lead to a local current continuity equation, it can be easily shown that the wave function normalization will decrease monotonically for a potential V whose imaginary part is everywhere negative.

In conclusion, we observe that the numerical calculations of scattering with Eq. (2) are not essentially more difficult than with the Schrödinger equation. One starts by calculating the first N states of U. Then recalling that

$$\Lambda(\vec{r}, \vec{r}') = \sum_{n=1}^{N} u_n(\vec{r}) u_n^*(\vec{r}'),$$

Eq. (2) may be written in the form[6]

$$(T+V-E)\psi(\vec{r}) = \sum_{n=1}^{N} \lambda_n u_n(\vec{r}), \qquad (5)$$

where the λ_n need only be chosen so that ψ is orthogonal to the functions u_n for $n=1, 2...N$. The solution of the inhomogeneous equation for ψ is straightforward for arbitrary λ_n, and finally the constants λ_n must be evaluated by the orthogonality requirements. An explicit expression for λ_n in terms of Green's function of the problem without the exclusion principle is easily derived, but does not seem very useful from a practical point of view.

---

* Supported in part by the National Science Foundation.

† Now at the Ramo-Wooldridge Corporation, Los Angeles, California.

‡ Permanent address: Indiana University, Bloomington, Indiana.

[1] Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949); Feshbach, Porter and Weisskopf, Phys. Rev. 96, 448 (1954).

[2] H. A. Bethe, Phys. Rev. 57, (1940).

[3] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 551 (1956); Gomes, Walecka, and Weisskopf, Ann. Phys. 3, 241 (1958); K. A. Brueckner and J. L. Gammel, Phys. Rev. 109, 1023 (1958).

[4] Lee M. Frantz, Bull. Am. Phys. Soc. Ser. II, 3, 49 (1958). A complete description is soon to be published.

[5] N. G. van Kampen [Phys. Rev. 91, 1267 (1953)] has considered the requirements imposed on the scattering by causality and found them satisfied by local potentials of finite range. However, even local potentials which decrease at infinity like exponentials may violate his causality criterion. Such potentials, which may lead to poles of the S matrix anywhere in the first quadrant of the complex k plane, can be easily constructed by the Bargmann procedure [V. Bargmann, Revs. Modern Phys. 21, 488 (1949)].

[6] This equation is mathematically equivalent to those used by Sachs [R. G. Sachs, Phys. Rev. 95, 1065 (1954)] for what he called orthogonality scattering. The physics involved, however, is quite different.