PHYSICAL REVIEW FLUIDS **9**, L032701 (2024)

Letter

Flow kinematics model for universal Strouhal number in the separated flow past a bluff body

A. Barrero-Gil and A. Velazquez
Department of Fluid Mechanics and Aerospace Propulsion, Universidad Politecnica de Madrid, Plaza del Cardenal Cisneros 3, 28040, Madrid, Spain

(Received 12 September 2023; accepted 5 February 2024; published 26 March 2024)

This Letter revisits the problem of the existence of a universal Strouhal number in the separated flow past a stationary bluff body. Physical hypotheses based on flow kinematics only are formulated on the phenomenom. They lead to a simple closed-form mathematical formulation. The model thus developped predicts a universal Strouhal number equal to $1/2\pi \approx 0.159$, whichmatches experimental observations. The model has been expanded to predict the Strouhal-Reynolds relationship in the supercritical regime. The formulation has a universal character since said relationship depends on the critical Strouhal and Reynolds numbers only. The circular, square, and triangular cross-section shapes have been used for validation purposes. The case on an accelerating incoming flow has also been considered. The theoretical model thus developed supports the idea, suggested experimentally by other researchers, that time-averaged flow speed during a vortex formation cycle governs vortex dynamics in this accelerating flow case. Lattice Boltzmann simulations were performed to validate the model.

DOI: 10.1103/PhysRevFluids.9.L032701

Introduction. The nondimensional vortex shedding frequency of the separated flow past a bluff body, also called the Strouhal number (St), is defined as $St = f_v D/U$. In this definition, f_v is the vortex shedding frequency, D is the body length normal to the flow, and U is the upstream unperturbed flow speed. This Strouhal number depends on both the actual body geometry and Reynolds number (Re), and it is determined either experimentally or numerically. In this context, Roshko [1] identified in his experiments specific features of the vortex streets that depend neither on geometry nor on Re. Thereby, he postulated a "universal" St^* based on a certain length scale L' and a flow velocity U', instead of D and U as defined above. In the universal St^* definition, L' represents the distance between free streamlines in the wake, and U' is the flow velocity at the point where the flow separates from the body's surface. With this definition, he observed that St^* remained approximately constant ($\approx$0.16) for a wide range of body shapes and Re. Bearman [2] pursued the same problem but used a different definition for L'. Later on, Simmons [3], taking L' as the distance between free streamlines measured at the end of the vortex formation region, found $St^* \approx 0.163$ for 2D body wakes with different locations of the separation points. Griffin [4] determined St^* for an extensive dataset of bluff geometries and Re, and found that $St^* = 0.16 \pm 0.02$. Along the same lines, Nakamura [5] measured St^* for a variety of bluff bodies, and found a mean value of $St^* = 0.156$.

Regarding theoretical explanations for the actual value of St^*, it is relevant to refer to the work of Levi [6]. The author, instead of pursuing a chain of reasoning based on fluid mechanics considerations, followed a purely mechanical argument. He assumed that a fluid body of some thickness excited by an external flow with velocity U can be modeled as a harmonic oscillator whose energy is equal to the kinetic energy of the incoming flow. In this way, Levi [6] found, remarkably, that $St^* = 1/2\pi = 0.159$. Recall, however, that since his model considered neither

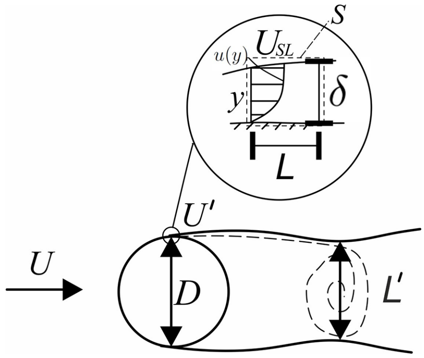

FIG. 1. Sketch of the flow configuration used for the model hypotheses.

the physical bases nor the equations of fluid mechanics, objections could be stated. For instance, Levi [6] implicitly assumed that all incoming flow kinetic energy is transformed into transverse oscillation energy of the wake, and this leaves the actual translational wake kinetic energy in the energy balance unaccounted for.

The work being presented in this Letter consists of the development of a closed-form simple mathematical model of vortex formation and shedding in the wake of bluff bodies based on flow kinematics only. It is based on some hypotheses about the topology of the wake itself. As it will be shown in the next sections, the model predicts (1) a reasonably accurate value for the universal Strouhal number St^*; (2) a valid $St-Re$ relationship above the critical Re for a circular cylinder and other bluff geometries; and (3) a quantitative insight on the problem of vortex shedding under unsteady incoming flow conditions.

Vortex shedding model: Development and validation: Background. The model is based on two main hypotheses. The first one is about description of vortex growth stage. The second addresses completion of the vortex growth, and further convection downstream into the wake. Starting from an initial time, $t = 0$, the vortex-shedding process begins with circulation, Γ, from the boundary layer (or shear layer) at the flow-separation point feeding a growing vortex. This circulation can be approximated as

$$\Gamma(t) \cong \int_S \omega dS, \tag{1}$$

where ω is the flow vorticity and S is a region that surrounds the separation point at the boundary layer, as illustrated in Fig. 1. Since the flow in the boundary layer is essentially 1D, vorticity can be approximated by $\omega = \partial u(y)/\partial y$, where u is the tangential flow velocity in the boundary layer and y is the transverse coordinate inside the boundary layer whose height is δ (see Fig. 1).

From the fact that $dS = dy/dl$, it follows that

$$\Gamma(t) = \int_S \omega dS = \int_0^L \int_0^\delta \frac{\partial u}{\partial y} dydl. \tag{2}$$

Now, a linear profile is assumed for the velocity at the boundary layer $u(y) = U_{SL}(y/\delta)$, where U_{SL} is the velocity outside the boundary layer at the separation point (shear layer velocity). Introducing the approximation $dl \approx \bar{u}dt$ where $\bar{u} = U_{SL}/2$ is the spatially averaged value of the tangential flow velocity in the boundary layer, it follows that

$$\Gamma(t) = \int_0^t \frac{U_{SL}^2}{2} dt. \tag{3}$$

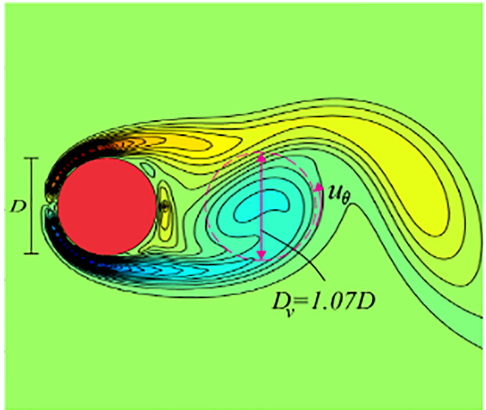

FIG. 2. Computed near-wake vorticity from a circular cylinder at $Re = 200$.

Then, circulation during a vortex-shedding period (from $t = 0$ to $t = t_v$) is

$$\Gamma(t = t_v) = \int_0^{t_v} \frac{U_{SL}^2}{2} dt = \frac{\overline{U_{SL}^2} t_v}{2} = \frac{\overline{U_{SL}^2}}{2 f_v}, \tag{4}$$

where $f_v = t_v^{-1}$ is the vortex-shedding frequency, and the overline stands for the time-averaged value between 0 and t_v. With these preliminaries, the two model hypotheses are stated in the next two subsections.

Hypothesis No. 1. Vorticity generated in the shear layer is concentrated into a single circular vortex with circulation $\Gamma_v = 2\pi R_v u_\theta$ where u_θ is the circumferential flow speed at the circular vortex of radius R_v (Rankine vortex model).

Hypothesis No. 2. The relation $u_\theta = \overline{U_{SL}}$ holds at the end of the vortex formation phase. In other words, when the the vortex is fully formed at $t = t_v$, its contour is enclosed by a streakline whose average flow speed is $\overline{U_{SL}}$. Then,

$$\Gamma_v(t = t_v) = 2\pi R_v(t_v)\overline{U_{SL}}. \tag{5}$$

That leads to

$$f_v = \frac{\overline{U_{SL}^2}}{2\pi(2R_v)\overline{U_{SL}}}. \tag{6}$$

This relation can be further simplified if the shear-layer flow-velocity variation between $t = 0$ and $t = t_v$ is small, $\overline{U_{SL}^2} \approx \overline{U_{SL}}^2$, as follows:

$$f_v = \frac{\overline{U_{SL}}}{2\pi(2R_v)} = \frac{\overline{U_{SL}}}{2\pi D_v}, \tag{7}$$

where $D_v = 2R_v$. This indicates that the frequency of vortex shedding is proportional to the time-averaged value of the flow velocity at separation divided by the vortex diameter D_v.

2.3 Validation. The model thus developed was validated via computation of a series of cases using a lattice Boltzmann method; see Refs. [7,8] for details of the numerical algorithm. The case $Re = 200$ is presented for illustration purposes in this subsection. Figure 2 shows vorticity isolines. Incoming flow velocity was $U = 0.05$ (in lattice dimensionless units). The computed shear velocity at separation was $U_{SL} = 0.07$. The vortex diameter was estimated to be $D_v \approx 1.07\, D$. Substitution of $\frac{U_{SL}}{U} = \frac{0.07}{0.05} = 1.4$, and $\frac{D}{D_v} = \frac{1}{1.07} = 0.935$ into Eq. (7) leads to the following Strouhal number: $St = f_v D/U = \overline{U_{SL}} D/2\pi U D_v = 1.4 \times 0.935/2\pi = 0.208$. Actual St obtained from the numerical simulation was 0.211. That is, the discrepancy between computed and predicted values of St was less than 2%.

Model results: Derivation of the universal St^*. Equation (7) could be rewritten as follows:

$$\frac{f_v D_v}{\overline{U_{SL}}} = \frac{1}{2\pi} = St^*, \tag{8}$$

which is nothing more than the universal Strouhal number, St^*, with $L' = D_v$ and $U' = \overline{U_{SL}}$ being the characteristic length and velocity scales of the wake, respectively. Equation (8) predicts a theoretical value of $St^* = 1/(2\pi) = 0.159$. The range reported in the literature [1–5] is from 0.156 to 0.163. The present St^* model does not depend on the specifics of the body geometry. It depends on D_v and $\overline{U_{SL}}$ only. This explains why experiments reported in the literature exhibit similarities regarding wake behavior.

Strouhal-Reynolds relationship above the critical Re. The classical St number, $St = f_v D/U$, is related to St^*, as follows:

$$St = \frac{f_v D}{U} = \frac{f_v D_v}{\overline{U_{SL}}}\left(\frac{D}{D_v}\right)\left(\frac{\overline{U_{SL}}}{U}\right) = St^*\left(\frac{D}{D_v}\right)\left(\frac{\overline{U_{SL}}}{U}\right). \tag{9}$$

It might be assumed that to a first approximation, $\overline{U_{SL}}$ depends linearly on the incoming flow velocity, $\overline{U_{SL}} = AU$, where A is a constant that depends on the particular body geometry. Since the larger the Re the smaller the D_v (for high Re, large flow structures are resolved into finer ones) a linear relationship between D_v and $1/Re$ could be proposed that leads to $D_v/D = B + C/Re$, where B and C are constants. Then, from Eq. (9) it follows that

$$St = St^* A \frac{1}{B + C/Re} = \frac{1}{A' + B'/Re}, \tag{10}$$

where $A' = B/(ASt^*) = 2\pi B/A$, $B' = C/(ASt^*) = 2\pi C/A$, and $St^* = 1/2\pi$. This type of $St-Re$ relationship has been proposed previously by Roushan and Wu [9] and Kim and Wu [10] based on experimental observations, and by Jiang and Chen [11] based on numerical simulations. In the present work, the relationship is derived from the theoretical concept of St^*.

Equation (10) can be rewritten as

$$\frac{1}{St} = A' + B'\left(\frac{1}{Re}\right), \tag{11}$$

which shows a linear relationship between St^{-1} and Re^{-1}.

Now, the following *ansatz* is made: A' and B' are related to (1) the critical values of Reynolds, Re_0, and Strouhal, St_0, numbers that identify the onset of vortex shedding, and (b) the limiting Strouhal number, St_∞, for very large Re. This relationship is formulated as follows:

$$A' = \frac{1}{2St_\infty},\ B' = Re_0\left(\frac{1}{St_0} - \frac{1}{St_\infty}\right). \tag{12}$$

Experimental results for a circular cylinder [12] show that $Re_0 = 47$, $St_0 = 0.118$, and $St_\infty = 0.215$. This leads to $A' = 4.23$, and $B' = 182.2$, that provides good agreement with Williamson and Brown's experimental data [13]; see Fig. 3. Maximum and mean deviations between measured and predicted results were 1.3 and 0.49%, respectively.

Since, typically, $St_\infty \approx 2St_0$ the $St-Re$ relationship (11)–(12) could be further simplified. Substitution of $St_\infty \approx 2St_0$ into Eq. (11) leads to

$$\frac{1}{St} \approx \frac{1}{2St_0} + \left(\frac{Re_0}{2St_0}\right)\frac{1}{Re}. \tag{13}$$

Relation (13), which predicts the $St-Re$ relationship as a function of critical quantities, (Re_0, St_0) only, could be regarded as having a universal nature. Concerning its validation, Fig 4.

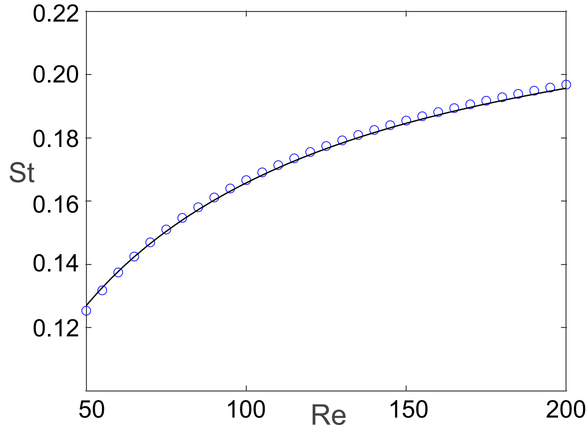

FIG. 3. $St-Re$ relationship for a circular cylinder. Blue circles: experimental data from Williamson and Brown [13]. Solid line: Eq. (10) with *A'* and *B'* taken from relation (12) with $Re_0 = 47$, $St_0 = 0.118$, and $St_\infty = 0.215$.

presents plots of $1/St - 1/(2St_0)$ versus $Re_0/(2St_0Re)$ for different cases. The results (either numerical or experimental) were taken from Refs. [13–15], and involve circular, square, and triangular cross-section shapes. It could be observed that data collapses reasonably around the straight line of slope 1. This confirms the validity of relation (13) for the representative cross-section shapes just mentioned. Obviously, a systematic study involving a larger number of shapes is still needed.

Vortex shedding with unsteady incoming flow at high Re. This subsection considers the case in which the incoming flow velocity varies with time; i.e.: $U = U(t)$. First, it is assumed, as previously, that the following linear relationship holds: $U_{SL}(t) = AU(t)$, where A is a constant. Then,

$$\int_0^{t_v} U_{SL}(t)dt = A\int_0^{t_v} U(t)dt. \tag{14}$$

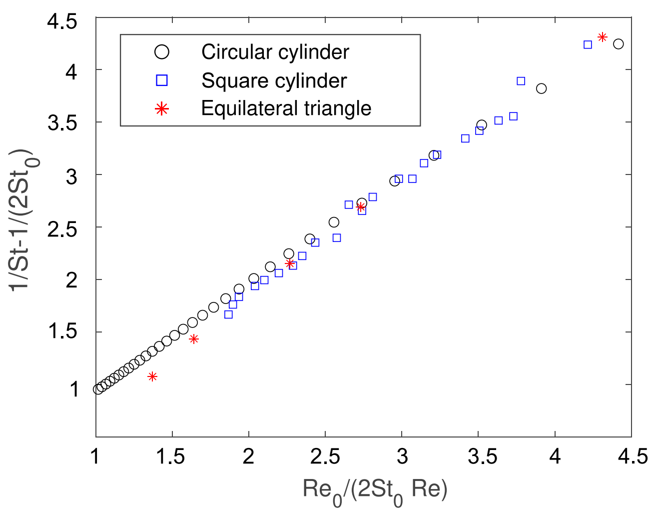

FIG. 4. Plot of the $St-Re$ relationship (13). Results (either numerical or experimental) taken from the literature. Black circles: circular cylinder ($Re_0 = 47$ and $St_0 = 0.118$) [12]. Blue squares: square cylinder ($Re_0 = 64$ and $St_0 = 0.118$) [14]. Red asterisks: equilateral triangle ($Re_0 = 50$ and $St_0 = 0.116$) [15].

$t = 0$ refers to the instant when the last vortex was convected downstream, and the vortex begins its formation. In time-averaged terms

$$\overline{U_{SL}} = \frac{1}{t_v}\int_0^{t_v} U_{SL}(t)dt = A\frac{1}{t_v}\int_0^{t_v} U(t)dt = A\bar{U}, \tag{15}$$

where $\bar{U}$ is the time-averaged $U(t)$. Substitution into Eq. (7) leads to

$$f_v = \frac{1}{2\pi D_v} A\bar{U}. \tag{16}$$

For high enough Re, D_v depends on D only, and from what was said above, the following relation holds: $D_v = BD$, where B is a constant. Then, Eq. (16) could be written in dimensionless form as

$$St = \frac{f_v D}{\bar{U}} = \frac{A}{2\pi B}. \tag{17}$$

This suggests that under unsteady incoming flow conditions, the classical Strouhal law holds, but using $\bar{U}$ instead of $U(t)$. This is in agreement with the experimental results of Gharib *et al.* [16], who experimentally found that $\bar{U}$ was a suitable velocity scale to predict vortex rings' formation time.

To test the validity of Eq. (17), 2D lattice Boltzmann simulations (algorithm details provided in Refs. [7,8]) were carried out on the problem of an accelerating flow past a cylinder at a standstill. Cylinder circunference had $D = 30$ lattice points. Computational domain size was 20D × 10D, with a total of $600 \times 300 = 180000$ lattice points. During an initial time period $t = 0$ to $t = t_s$ inflow velocity was constant (U_0) at the letft boundary of the domain. This was needed to achieve a well-developed Kármán vortex street before acceleration started. From t_s onwards, the velocity prolfile was $U(t) = U_0 + a(t - t_s)$ until a final velocity of $2U_0$ was reached. Computational parameters were lattice time $\Delta t = 1$, lattice distance $\Delta x = 1$, $U_0 = 0.05$, and $a = 1 \cdot 10^{-6}$. A no-slip boundary condition was applied at the body surface using the bounce-back rule. Kinematic viscosity was chosen so that Re varied form 200 to 400. It is known that 3D effects appear in the flow past a circular cylinder for $Re > 180$. However, experimental results show that these 3D effects have a limited influence on the Strouhal number. In fact, this number remains nearly constant (around 0.21) up to $Re \sim 10^5$ when the drag crisis occurs. Then, the present 2D analysis represents a reasonable approximation in the parametric range being considered ($200 < Re < 400$). In any case, future work in the field may require the detailed numerical simulation of 3D flow fields to quantify these effects more accurately.

Fluid forces on the body were computed via the Method of Momentum Exchange Algorithm [17]. The numerical results obtained are presented in Fig. 5. It presents the computed frequency of vortex shedding ($f_v D$) versus a series of discrete $\bar{U}$ values between consecutive lift peaks. These discrete numerical results can be fitted to the following linear relationship: $f_v D/\bar{U} = 0.218$. On the other hand, taking $A = 1.5$ and $B = 1$ in Eq. (17) (as in earlier in this Letter) leads to $f_v D/\bar{U} = 0.238$. It is observed that the discrepancy between the computed fitting slope (0.218) and the slope predicted by the model (0.238) is 9%. This lends theoretical support to previous hypothesis postulated after experimental observations.

The numerical results obtained (black circles in Fig. 5) show small deviations around the linear fit (dashed line). The reasons for the deviations might be twofold: (1) the lattice Boltzmann method itself, which is a second-order formulation for the weakly compressible Navier-Stokes equations, and (2) the level of spatial discretization chosen for the lattice. For the simulations that have been performed, the numerical uncertainty for the variables is estimated to be ±5%.

Conclusions. This work presented in this Letter revisits the problem of the existence of a universal Strouhal number in the separated flow past a stationary cylinder. This study consists of the presentation of an approach based, solely, on flow kinematics. The model that has been developed rests on two main hypotheses: (1) Vorticity generated in the boundary layer–shear layer of the cylinder is convected downstream and concentrated into a single circular vortex; and (2) the average flow speed

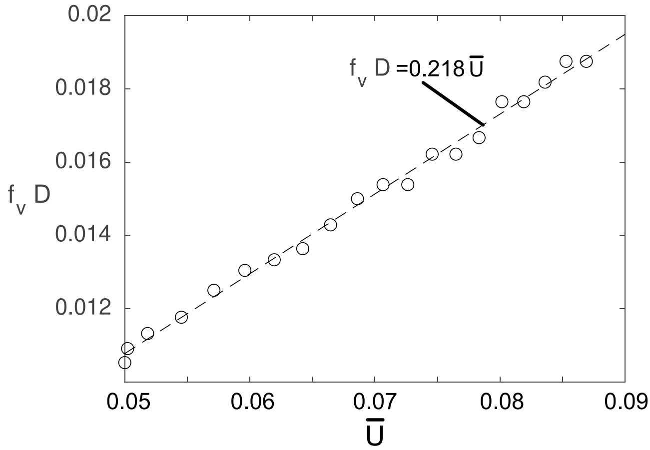

FIG. 5. ($f_v D$) vs incoming mean flow speed, $\bar{U}$, between consecutive lift peaks. Open circles: lattice Boltzmann numerical results. Dashed line: linear fit of the numerical results.

at the vortex enclosing streakline coincides with the mean shear velocity, $\overline{U_{SL}}$, at the body surface point where the flow separates. Combination of both hypotheses yields a universal Strouhal number $St^* = 1/2\pi$. This theoretically predicted St^* agrees well with experimental results obtained and discussed by other authors.

The formulation can be expanded to predict a $St-Re$ relationship in the supercritical regime that has a universal nature since it depends on the critical values of Strouhal and Reynolds numbers only. The relation has been validated with previous experimental and numerical results described in the specialized literature. They involve representative cross-section shapes such as the circumference, the square, and the triangle.

Finally, the vortex-shedding model may be applied to the problem of an accelerating incoming flow. The formulation thus developed lends theoretical support to the idea, suggested experimentally by previous researchers, that time-averaged flow speed during a vortex formation cycle governs vortex dynamics in this unsteady case.

Acknowledgments. This work has been funded by the Spanish Ministry of Science and Innovation under Research Contracts No. PID2019-109619GB-C21 and No. TED2021-132378B-I00.

---

[1] A. Roshko, On the wake and drag of bluff bodies, J. Aero. Sci. **22**, 124 (1955).

[2] P. W. Bearman, On vortex street wakes, J. Fluid Mech. **28**, 625 (1967).

[3] J. E. L. Simmons, Similarities between two-dimensional and axisymmetric vortex wakes, Aero. Quaterly **28**, 15 (1977).

[4] O. M. Griffin, Universal similarity in the wakes of stationary and vibrating structures, J. Fluids Eng.-Trans. ASME **103**, 52 (1981).

[5] Y. Nakamura, Vortex shedding from bluff bodies and universal Strouhal number, J. Fluids Struct. **10**, 159 (1996).

[6] E. Levi, A universal Strouhal law, J. Eng. Mech **109**, 718 (1983).

[7] D. Vicente-Ludlam, A. Barrero-Gil, and A. Velazquez, Flow-induced vibration of a rotating circular cylinder using position and velocity feedback, J. Fluids Struct. **72**, 127 (2017).

[8] A. Barrero-Gil, D. Vicente-Ludlam, D. Gutierrez, and F. Sastre, Enhance of energy harvesting from transverse galloping by actively rotating the galloping body, Energies **13**, 91 (2020).

[9] P. Roushan and X. L. Wu, Structure-based interpretation of the Strouhal-Reynolds number relationship, Phys. Rev. Lett. **94**, 054504 (2005).

[10] I. Kim and X. L. Wu, Unified Strouhal-Reynolds number relationship for laminar vortex streets generated by different-shaped obstacles, Phys. Rev. E **92**, 043011 (2015).

[11] H. Jiang and L. Cheng, Strouhal-Reynolds number relationship for flow past a circular cylinder, J. Fluid Mech. **832**, 170 (2017).

[12] F. Ponta and H. Aref, Strouhal-Reynolds number relationship for vortex streets, Phys. Rev. Lett. **93**, 084501(R) (2004).

[13] C. H. K. Williamson and L. G., Brown A series in $1/\mathrm{Re}$ to represent the Strouhal-Reynolds number relationship of the cylinder wake, J. Fluids Struct. **12**, 1073 (1998).

[14] S. C. Luo, X. H. Tong, and B. C. Khoo, Transition phenomena in the wake of a square cylinder, J. Fluids Struct. **23**, 227 (2007).

[15] K. Arnab and D. Amaresh, Numerical simulation of unconfined flow past a trianglar cylinder, Int. J. Numer. Methods Fluids **52**, 801 (2006).

[16] M. Gharib, E. Rambod, and K. Shariff, A universal time scale for vortex ring formation, J. Fluid Mech. **360**, 121 (1998).

[17] K. Timm, H. Kusumaatmaja, A. Kuzmin, O. Shardt, G. Silva, and E. Viggen, *The Lattice Boltzmann Method: Principles and Practice* (Springer International Publishing AG, Cham, Switzerland, 2016).