

Stationary and nonstationary energy cascades in homogeneous ferrofluid turbulence

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The nonlinear transfer rate of the total energy (transfer rate of kinetic energy + transfer rate due to the work done by the magnetization) for an incompressible turbulent ferrofluid system is studied under the assumption of statistical homogeneity. Using the formalism of the two-point correlators, an exact relation connecting the second-order statistical moments to the average energy injection rate is derived for the scale-to-scale transfer of the total energy. We validate the universality of the exact relation through direct numerical simulations for stationary and nonstationary cascade regimes. For a weak external magnetic field, both kinetic and the total energy cascade with nearly the same cascade rate. A stationary cascade regime is achieved, and hence a good agreement between the exact energy transfer rate and the average energy injection is found. Due to the rapid alignment of the ferrofluid particles in the presence of strong external fields, the turbulence dynamics becomes nonstationary. Interestingly, there too, both kinetic and the total energy exhibit inertial range cascades but with different cascade rates which can be explained using the nonstationary form of our derived exact relation.

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I. INTRODUCTION

To date, turbulence is one of the most challenging problems of physics that needs to be completely understood. In common fluids, turbulence arises from the nonlinear interactions among neighboring fluid layers, which leads to the formation of eddy-like structures of various sizes thus conceiving a wide range of length scales. Energy is fed into the system at large scales and is dissipated at very small scales. Well inside the intermediate inertial length scales, free from the effects of forcing and dissipation, a fully developed turbulence is characterized by a universal cascade of kinetic energy across the scales with a constant transfer rate ε . For homogeneous and isotropic turbulence, ε can be exactly expressed in terms of the third-order moments of two-point velocity fluctuations. For a turbulent magnetohydrodynamic fluid, the total energy (sum of kinetic and magnetic energy) cascades throughout the inertial range. The respective energy transfer rate is expressed in terms of two-point fluctuations of both velocity and magnetic field.

Derivation of the exact relations connecting ε and the two-point fluctuations was pioneered by Kolmogorov [1] for a homogeneous and isotropic incompressible hydrodynamic fluid. Following Kolmogorov, similar exact laws were also derived for different fluid systems including magnetohydrodynamic (MHD) turbulence [2–6]. Without the explicit assumption of isotropy, differential exact relations involving the divergence of the third-order moments have been derived for several other systems including the ones mentioned above [7–11]. This formalism is also extended to compressible systems, where the ε can be written as a divergence of the third-order moments plus some source terms [12–15]. An alternative form of the exact relation involving the second-order statistical

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moments is recently proposed for incompressible HD and MHD turbulence [16–19]. This form is particularly interesting to cases where ε can not be written purely in terms of the divergence of the two-point fluctuations (e.g., energy cascade in incompressible Hall-MHD turbulence, turbulent compressible flows, etc.) [19,20]. One can then be interested to know if a similar type of universal cascade can also be found in more complex fluids such as ferrofluids. Ferrofluids are complex synthetic fluids that contain nanosized ferromagnetic particles suspended in a carrier liquid (water, oil, other organic solvents, etc.). Ferrofluid particles interact via attractive van der Waals forces and dipole-dipole interactions, which may result in aggregations. To prevent immediate aggregation, these ferro-particles are, in general, coated with a surfactant [21–27]. In the absence of an external magnetic field ($\mathbf{H}_0$), the particles are randomly orientated and the fluid have no net magnetism. In the presence of $\mathbf{H}_0$, the magnetic particles respond by forming chainlike structures along the field lines [28,29]. Such chainlike structures reduce the mobility and efficient mixing of different parts of a ferrofluid thereby resisting the development of turbulence. In order to obtain a sustained ferrofluid emulsion, it is therefore necessary to hinder the formation of chainlike structures. This can be achieved by compelling the system to a strong turbulent forcing that introduces very high flow velocities at large Reynolds number. Equivalently, in a fully developed turbulent ferrofluid, as the strength of $\mathbf{H}_0$ increases, the turbulent transfers weaken, which eventually leads to the suppression of turbulence [30–35]. Note that this is different from the emergence of weak turbulence regime due to Alfvén-wave interactions in an MHD fluid [36] or the wave-driven turbulence observed on the surface of a ferrofluid [37–39].

Exploring turbulence within ferrofluid is imperative for investigating the implications of magnetic fields on heat transfer phenomena, encompassing flow management, augmenting heat transfer processes, and mechanisms dedicated to noise reduction through convection [40–43]. Although a number of theoretical and experimental studies have been carried out on ferrofluid flows [24,26,27,44,45], only a few of them are dedicated to study of the turbulent properties of such a flow [30,32–35]. In particular, for homogeneous turbulence, those studies investigated the evolution of the turbulent kinetic energy (both translational and rotational) and the corresponding power spectra [31–33]. In a turbulent ferrofluid, the sum of kinetic energy and work done due to magnetization is an inviscid invariant and is therefore expected to exhibit a universal cascade with a constant flux rate (ε) inside the inertial range. However, the possibility of a universal cascade in terms of the statistical moment of field fluctuations has not been studied until recently where an exact relation for homogeneous ferrofluid turbulence has been derived [46]. Unlike incompressible HD, MHD, and binary fluid dynamics, ε in ferrofluid turbulence cannot be expressed as a divergence of two-point increments, and instead, an alternative form similar to [16,17] of the exact relation has been obtained for the energy cascade in ferrofluid turbulence.

In the current study, we revisit our previous work [46] with a realistic assumption of negligible particle size and rederive a reduced form of the exact relation for the total energy transfer. The derived law is then numerically tested with 3D direct numerical simulations (DNS) ranging from 128^3 to 512^3 grid points to find the signature of scale-independent energy transfer rate. Finally, stationary and nonstationary cascades are studied by varying the strength of $\mathbf{H}_0$ and the effect of an external field on the nature of turbulent cascade in the ferrofluids is discussed.

The paper is organized as follows. In Sec II the governing equations and conservation of total energy for the ferrofluid system are described, whereas Sec. III contains the derivation of the exact relation for negligible ferrofluid particle size. In Sec. IV we present the numerical methods and other simulation details. In Sec. V we present and discuss our findings. Finally, in Sec. VI we summarize and conclude.

II. MODEL AND INVISCID INVARIANT

A. Basic equations

The governing equations for incompressible ferrofluid (the constant density is normalized to unity) consist of the evolution equations for linear momentum, angular momentum, and

magnetization [32,46]:

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mu_0 (\mathbf{M} \cdot \nabla) \mathbf{H} - \zeta \nabla \times (\boldsymbol{\Omega} - 2\boldsymbol{\omega}), \quad (1)$$

$$I(\partial_t + \mathbf{v} \cdot \nabla) \boldsymbol{\omega} = \mu_0 (\mathbf{M} \times \mathbf{H}) + \eta \nabla^2 \boldsymbol{\omega} + 2\zeta (\boldsymbol{\Omega} - 2\boldsymbol{\omega}), \quad (2)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{M} = \boldsymbol{\omega} \times \mathbf{M} - \frac{1}{\tau} (\mathbf{M} - \mathbf{M}_{eq}), \quad (3)$$

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{H} = \mathbf{0}, \quad (4)$$

where $\mathbf{v}$ is the velocity of the ferrofluid, $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$ is the vorticity, p is the fluid pressure, $\boldsymbol{\omega}$ is the ferrofluid particle spin rate, $\mathbf{M}$ is the magnetization vector, $\mathbf{H}$ is the magnetic field vector, I is the moment of inertia per unit mass for a ferrofluid particle, ν is the kinematic viscosity, ζ is the vortex viscosity, η is the spin viscosity, and τ is the relaxation time. The equilibrium magnetization is given by $\mathbf{M}_{eq} = M_s L(\xi) \mathbf{H} / H$, where $L(\xi) = [\xi \coth(\xi) - 1] / \xi$ is the Langevin function with $\xi = \mu_0 m H / k_B T$ for a ferrofluid at temperature T . The parameters M_s and m are the magnitudes of saturation magnetization and magnetic moment of a single ferrofluid particle, respectively. For small values of ξ , one can write $\mathbf{M}_{eq} = \chi \mathbf{H}$, where χ is magnetic susceptibility. Due to incompressibility, $\mathbf{v}$ is divergenceless, and in the absence of any free current $\mathbf{H}$ is irrotational. Since the magnetic flux ($\mathbf{B}$) is divergence-free, the evolution of the magnetic field can directly be obtained from that of the magnetization as $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$, and hence one can write

$$\nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{H}. \quad (5)$$

Unlike our previous work [46], here we incorporate the fact that the specific moment of inertia I ($\sim 10^{-16} \text{ m}^2$) and the spin viscosity η ($\sim 10^{-15} \text{ kg m s}^{-1}$) are very small and can practically be neglected with respect to the other terms in the evolution equation of angular momentum [32]. Hence, the angular momentum Eq. (2) reduces to

$$\boldsymbol{\omega} = \frac{\boldsymbol{\Omega}}{2} + \frac{\mu_0}{4\zeta} (\mathbf{M} \times \mathbf{H}). \quad (6)$$

Using Eq. (6) in Eqs. (1) and (3), one obtains

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mu_0 (\mathbf{M} \cdot \nabla) \mathbf{H} + \frac{\mu_0}{2} \nabla \times (\mathbf{M} \times \mathbf{H}), \quad (7)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{M} = \frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{M}) + \frac{\mu_0}{4\zeta} (\mathbf{M} \times \mathbf{H}) \times \mathbf{M} - \frac{1}{\tau} (\mathbf{M} - \chi \mathbf{H}). \quad (8)$$

Equations (7) and (8) along with Eqs. (4) and (5) constitute the complete set of equations for ferrofluid turbulence. Note that we have a total of 10 variables, i.e., three components of each of $\mathbf{v}$, $\mathbf{M}$, $\mathbf{H}$, and the scalar pressure p . Since $\mathbf{H}$ is irrotational, it can be written as a gradient of a scalar field $\mathbf{H} = \nabla \phi$. Now, using Eq. (5), one can calculate $\phi = -\nabla^{-2} (\nabla \cdot \mathbf{M})$, and hence $\mathbf{H}$. Finally, p is determined by taking the divergence of Eq. (8) and using the incompressibility condition (4) as $p = -\nabla^{-2} \{ \nabla \cdot [(\mathbf{v} \cdot \nabla) \mathbf{v} - \mu_0 (\mathbf{M} \cdot \nabla) \mathbf{H}] \}$.

B. Conservation of total energy

Similar to ordinary fluids, ferrofluid equations also conserve the total energy in the absence of viscous terms. The total energy consists of the kinetic energy of the fluid and the internal energy resulting from the work performed by the ferrofluid particles in response to the external magnetic field. Using Eq. (7), the kinetic energy evolution equation can be written as

$$\begin{aligned} \partial_t E_{\text{kin}} &= \partial_t \left(\frac{v^2}{2} \right) \\ &= -\nabla \cdot \left[\left(\frac{v^2}{2} + p \right) \mathbf{v} - \frac{\mu_0}{2} (\mathbf{M} \times \mathbf{H}) \times \mathbf{v} \right] + \frac{\mu_0}{2} [(\mathbf{M} \times \mathbf{H}) \cdot \boldsymbol{\Omega} + 2\mathbf{v} \cdot (\mathbf{M} \cdot \nabla) \mathbf{H}] - \nu \Omega^2. \end{aligned} \quad (9)$$

Again using Eq. (8), the differential work performed by the ferrofluid particles due to $\mathbf{H}$ is given by $dE_{\text{mag}} = -\mathbf{H} \cdot d\mathbf{M}$, and hence the corresponding evolution equation is given by

$$\begin{aligned} \partial_t E_{\text{mag}} &= -\mathbf{H} \cdot \partial_t \mathbf{M} \\ &= \mathbf{H} \cdot [(\mathbf{v} \cdot \nabla) \mathbf{M}] - \frac{1}{2} \mathbf{H} \cdot (\boldsymbol{\Omega} \times \mathbf{M}) - \frac{\mu_0}{4\zeta} \mathbf{H} \cdot [(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}] + \frac{1}{\tau} \mathbf{H} \cdot (\mathbf{M} - \chi \mathbf{H}) \\ &= \mathbf{H} \cdot [(\mathbf{v} \cdot \nabla) \mathbf{M}] - \frac{1}{2} \mathbf{H} \cdot (\boldsymbol{\Omega} \times \mathbf{M}) - \frac{\mu_0}{4\zeta} (\mathbf{M} \times \mathbf{H})^2 + \frac{1}{\tau} \mathbf{H} \cdot (\mathbf{M} - \chi \mathbf{H}) \\ &= \mathbf{H} \cdot [(\mathbf{v} \cdot \nabla) \mathbf{M}] - \frac{1}{2} \mathbf{H} \cdot (\boldsymbol{\Omega} \times \mathbf{M}) - \frac{\zeta}{\mu_0} (2\boldsymbol{\omega} - \boldsymbol{\Omega})^2 + \frac{1}{\tau} \mathbf{H} \cdot (\mathbf{M} - \chi \mathbf{H}). \end{aligned} \quad (10)$$

Note that, unlike HD and MHD fluids, here the work done by the magnetic field dE_{mag} cannot be written as an exact differential, and hence, a separate expression for the energy density and the corresponding two-point correlator cannot be obtained. The evolution equation of the total energy is given by

$$\int \partial_t (E_{\text{kin}} + E_{\text{mag}}) d\tau = - \int \nabla \cdot \left[\left(\frac{v^2}{2} + p + \mu_0 \mathbf{M} \cdot \mathbf{H} \right) \mathbf{v} - \frac{\mu_0}{2} (\mathbf{M} \times \mathbf{H}) \times \mathbf{v} \right] d\tau + d, \quad (11)$$

where dissipation term is

$$d = \int \left[-\nu \Omega^2 + \frac{\mu_0}{\tau} \mathbf{H} \cdot (\mathbf{M} - \chi \mathbf{H}) - \zeta (\boldsymbol{\Omega} - 2\boldsymbol{\omega})^2 \right] d\tau. \quad (12)$$

The first term in the r.h.s. of Eq. (11) vanishes by the use of the Gauss divergence theorem, and finally ignoring the dissipation terms, we obtain the conservation of the total energy. In viscous flows, where the dissipative effects are not negligible, the l.h.s. of Eq. (11) does not vanish. In such a situation, one needs to add an additional injection term f which would then lead to the conservation of energy by balancing the dissipation as $d = -f$. In the next section, we shall see if a constant flux of energy can be obtained by ensuring the driving only at large scales and dissipation only at small scales.

III. DERIVATION OF EXACT RELATIONS

In this section, using two-point statistics, we derive an exact relation for the transfer of total energy within the inertial range. For the sake of numerical implementation, it is necessary to cast the constitutive relation (6) and the governing equation (7) and (8) in terms of dimensionless starred variables as

$$\boldsymbol{\omega}^* = \frac{1}{2} \boldsymbol{\Omega}^* + \frac{\text{Re}}{2.2} (\mathbf{M}^* \times \mathbf{H}^*), \quad (13)$$

$$\partial_{t^*} \mathbf{v}^* = \mathbf{v}^* \times \boldsymbol{\Omega}^* - \nabla^* \left(p^* + \frac{v^{*2}}{2} \right) + \frac{1}{\text{Re}} \nabla^{*2} \mathbf{v}^* + (\mathbf{M}^* \cdot \nabla^*) \mathbf{H}^* + \frac{1}{2} \nabla^* \times (\mathbf{M}^* \times \mathbf{H}^*), \quad (14)$$

$$\partial_{t^*} \mathbf{M}^* = -(\mathbf{v}^* \cdot \nabla^*) \mathbf{M}^* + \frac{1}{2} (\boldsymbol{\Omega}^* \times \mathbf{M}^*) + \frac{\text{Re}}{2.2} (\mathbf{M}^* \times \mathbf{H}^*) \times \mathbf{M}^* - \frac{1}{\Gamma} (\mathbf{M}^* - \chi \mathbf{H}^*), \quad (15)$$

where we used $\zeta = 0.55\nu$ [32], $\mathbf{v} = v_{\text{rms}} \mathbf{v}^*$, $\mathbf{M} = \sqrt{\frac{1}{\mu_0}} v_{\text{rms}} \mathbf{M}^*$, $\mathbf{H} = \sqrt{\frac{1}{\mu_0}} v_{\text{rms}} \mathbf{H}^*$, $t = \frac{l_0}{v_{\text{rms}}} t^*$, $p = v_{\text{rms}}^2 p^*$, $\Gamma = \frac{v_{\text{rms}}}{l_0} \tau$, $\text{Re} = \frac{l_0 v_{\text{rms}}}{\nu}$ with l_0 representing the box size and v_{rms} the root-mean-square velocity of the input velocity profile. In order to simplify the notations, we shall omit the stars from the dimensionless variables hereinafter. To achieve a sustained turbulent flow, the system is driven by a large-scale forcing $\mathbf{f}_v$ (delta correlated in time) in the momentum equation. The evolution

equation of the energy correlation function is then given by [using Eqs. (14) and (15)]

$$\begin{aligned}
 \partial_t \mathcal{R} &= \frac{1}{2} \langle (\mathbf{v}' \cdot \partial_t \mathbf{v} + \mathbf{v} \cdot \partial_t \mathbf{v}') - (\mathbf{H} \cdot \partial_t \mathbf{M}' + \mathbf{H}' \cdot \partial_t \mathbf{M}) \rangle \\
 &= \frac{1}{2} \left\langle \mathbf{v}' \cdot (\mathbf{v} \times \boldsymbol{\Omega}) + \mathbf{v} \cdot (\mathbf{v}' \times \boldsymbol{\Omega}') - \mathbf{v} \cdot \nabla' \left(p' + \frac{v'^2}{2} \right) - \mathbf{v}' \cdot \nabla \left(p + \frac{v^2}{2} \right) - \frac{1}{2} \mathbf{H}' \cdot (\boldsymbol{\Omega} \times \mathbf{M}) \right. \\
 &\quad - \frac{1}{2} \mathbf{H} \cdot (\boldsymbol{\Omega}' \times \mathbf{M}') + \mathbf{v} \cdot (\mathbf{M}' \cdot \nabla') \mathbf{H}' + \mathbf{v}' \cdot (\mathbf{M} \cdot \nabla) \mathbf{H} + \mathbf{H}' \cdot (\mathbf{v} \cdot \nabla) \mathbf{M} + \mathbf{H} \cdot (\mathbf{v}' \cdot \nabla') \mathbf{M}' \\
 &\quad + \frac{1}{2} \mathbf{v}' \cdot [\nabla \times (\mathbf{M} \times \mathbf{H})] + \frac{1}{2} \mathbf{v} \cdot [\nabla' \times (\mathbf{M}' \times \mathbf{H}')] \Bigg\rangle + \left\langle \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla'^2 \mathbf{v}' + \frac{1}{\text{Re}} \mathbf{v}' \cdot \nabla^2 \mathbf{v} \right. \\
 &\quad - \frac{\text{Re}}{2.2} \mathbf{H}' \cdot [(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}] - \frac{\text{Re}}{2.2} \mathbf{H} \cdot [(\mathbf{M}' \times \mathbf{H}') \times \mathbf{M}'] + \frac{1}{\Gamma} \mathbf{H}' \cdot (\mathbf{M} - \chi \mathbf{H}) \\
 &\quad \left. + \frac{1}{\Gamma} \mathbf{H} \cdot (\mathbf{M}' - \chi \mathbf{H}') + \mathbf{v}' \cdot \mathbf{f}_v + \mathbf{v} \cdot \mathbf{f}_v' \right\rangle, \tag{16}
 \end{aligned}$$

where unprimed and primed quantities represent the variables at the point $\mathbf{x}$ and $\mathbf{x}' (\equiv \mathbf{x} + \boldsymbol{\ell})$, respectively, $\boldsymbol{\ell}$ is the increment vector, and $\langle \cdot \rangle$ denotes the ensemble average, which is equivalent to the space average due to statistical homogeneity. Again, using incompressibility and homogeneity, one can show $\langle \mathbf{v}' \cdot \nabla (\frac{v^2}{2} + p) \rangle = \langle \mathbf{v} \cdot \nabla' (\frac{v'^2}{2} + p') \rangle = 0$. Equation (16) can further be simplified as

$$\begin{aligned}
 \partial_t \mathcal{R} &= \frac{1}{2} \left\langle -\delta(\mathbf{v} \times \boldsymbol{\Omega}) \cdot \delta \mathbf{v} - \delta \mathbf{v} \cdot \delta[(\mathbf{M} \cdot \nabla) \mathbf{H}] - \delta \mathbf{H} \cdot \delta[(\mathbf{v} \cdot \nabla) \mathbf{M}] - \frac{1}{2} \delta \mathbf{v} \cdot \delta[\nabla \times (\mathbf{M} \times \mathbf{H})] \right. \\
 &\quad + \frac{1}{2} \delta \mathbf{H} \cdot \delta(\boldsymbol{\Omega} \times \mathbf{M}) \Bigg\rangle + \left\langle \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla'^2 \mathbf{v}' + \frac{1}{\text{Re}} \mathbf{v}' \cdot \nabla^2 \mathbf{v} - \frac{\text{Re}}{2.2} \mathbf{H}' \cdot [(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}] \right. \\
 &\quad \left. - \frac{\text{Re}}{2.2} \mathbf{H} \cdot [(\mathbf{M}' \times \mathbf{H}') \times \mathbf{M}'] + \frac{1}{\Gamma} \mathbf{H}' \cdot (\mathbf{M} - \chi \mathbf{H}) + \frac{1}{\Gamma} \mathbf{H} \cdot (\mathbf{M}' - \chi \mathbf{H}') + \mathbf{v}' \cdot \mathbf{f}_v + \mathbf{v} \cdot \mathbf{f}_v' \right\rangle, \tag{17}
 \end{aligned}$$

where we have used the following relations (obtained under the assumption of statistical homogeneity):

$$(i) \quad \langle (\mathbf{v} \times \boldsymbol{\Omega}) \cdot \mathbf{v}' + (\mathbf{v}' \times \boldsymbol{\Omega}') \cdot \mathbf{v} \rangle = -\langle \delta(\mathbf{v} \times \boldsymbol{\Omega}) \cdot \delta \mathbf{v} \rangle, \tag{18}$$

$$\begin{aligned}
 (ii) \quad &\langle \mathbf{v} \cdot (\mathbf{M}' \cdot \nabla') \mathbf{H}' + \mathbf{v}' \cdot (\mathbf{M} \cdot \nabla) \mathbf{H} + \mathbf{H} \cdot (\mathbf{v}' \cdot \nabla') \mathbf{M}' + \mathbf{H}' \cdot (\mathbf{v} \cdot \nabla) \mathbf{M} \rangle \\
 &= \langle -\delta \mathbf{v} \cdot \delta[(\mathbf{M} \cdot \nabla) \mathbf{H}] - \delta \mathbf{H} \cdot \delta[(\mathbf{v} \cdot \nabla) \mathbf{M}] \rangle, \tag{19}
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad &\langle \mathbf{H}' \cdot (\boldsymbol{\Omega} \times \mathbf{M}) + \mathbf{H} \cdot (\boldsymbol{\Omega}' \times \mathbf{M}') - \mathbf{v}' \cdot [\nabla \times (\mathbf{M} \times \mathbf{H})] - \mathbf{v} \cdot [\nabla' \times (\mathbf{M}' \times \mathbf{H}')] \rangle \\
 &= \langle \delta \boldsymbol{\Omega} \cdot \delta[(\mathbf{M} \times \mathbf{H})] - \delta \mathbf{H} \cdot \delta(\boldsymbol{\Omega} \times \mathbf{M}) \rangle. \tag{20}
 \end{aligned}$$

To see the effect of the applied external field $\mathbf{H}_0$ on the energy transfer, we apply a uniform external field of strength H_0 along $\hat{z}$ direction. Decomposing the total magnetic field as $\mathbf{H} = \mathbf{H}_0 + \tilde{\mathbf{H}} = H_0 \hat{z} + \tilde{\mathbf{H}}$, we obtain

$$\begin{aligned}
 \partial_t \mathcal{R} &= \frac{1}{2} \left\langle -\delta(\mathbf{v} \times \boldsymbol{\Omega}) \cdot \delta \mathbf{v} - \delta \mathbf{v} \cdot \delta[(\mathbf{M} \cdot \nabla) \tilde{\mathbf{H}}] - \delta \tilde{\mathbf{H}} \cdot \delta[(\mathbf{v} \cdot \nabla) \mathbf{M}] - \frac{1}{2} \delta \boldsymbol{\Omega} \cdot \delta(\mathbf{M} \times \tilde{\mathbf{H}}) \right. \\
 &\quad \left. + \frac{1}{2} \delta \tilde{\mathbf{H}} \cdot \delta(\boldsymbol{\Omega} \times \mathbf{M}) \right\rangle + \left\langle \frac{1}{2} \mathbf{H}_0 \cdot (\delta \boldsymbol{\Omega} \times \delta \mathbf{M}) \right\rangle + D + F, \tag{21}
 \end{aligned}$$

where D consists of the two-point dissipative terms

$$D = \frac{1}{2} \left\langle \frac{1}{\text{Re}} \mathbf{v} \cdot \nabla^2 \mathbf{v}' + \frac{1}{\text{Re}} \mathbf{v}' \cdot \nabla^2 \mathbf{v} - \frac{\text{Re}}{2.2} \tilde{\mathbf{H}}' \cdot [(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}] - \frac{\text{Re}}{2.2} \tilde{\mathbf{H}} \cdot [(\mathbf{M}' \times \mathbf{H}') \times \mathbf{M}'] \right. \\ \left. + \frac{1}{\Gamma} \tilde{\mathbf{H}}' \cdot (\mathbf{M} - \chi \mathbf{H}) + \frac{1}{\Gamma} \tilde{\mathbf{H}} \cdot (\mathbf{M}' - \chi \mathbf{H}') \right\rangle, \quad (22)$$

and F consists of purely the large-scale contributions, where

$$F = \frac{1}{2} \langle \mathbf{v}' \cdot \mathbf{f}_v + \mathbf{v} \cdot \mathbf{f}_v' \rangle - \left\langle \frac{\text{Re}}{2.2} \mathbf{H}_0 \cdot [(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}] - \frac{1}{\Gamma} \mathbf{H}_0 \cdot (\mathbf{M} - \chi \mathbf{H}) \right\rangle. \quad (23)$$

Assuming a statistically stationary state where $\partial_t \mathcal{R} = 0$ and ignoring the dissipative effects inside the inertial range, finally, we obtain the exact relation as

$$A(\ell) = A_1(\ell) + A_2(\ell) = 2\varepsilon, \quad (24)$$

where

$$A_1(\ell) = \left\langle \delta \mathbf{v} \cdot \{ \delta (\mathbf{v} \times \boldsymbol{\Omega}) + \delta [(\mathbf{M} \cdot \nabla) \tilde{\mathbf{H}}] \} + \delta \tilde{\mathbf{H}} \cdot \left\{ \delta [(\mathbf{v} \cdot \nabla) \mathbf{M}] - \frac{\delta (\boldsymbol{\Omega} \times \mathbf{M})}{2} \right\} + \delta \boldsymbol{\Omega} \cdot \frac{\delta (\mathbf{M} \times \tilde{\mathbf{H}})}{2} \right\rangle, \quad (25)$$

$$A_2(\ell) = - \left\langle \frac{1}{2} \mathbf{H}_0 \cdot (\delta \boldsymbol{\Omega} \times \delta \mathbf{M}) \right\rangle, \quad (26)$$

and $\varepsilon = F \approx \langle \mathbf{v} \cdot \mathbf{f}_v - \mathbf{H}_0 \cdot \{ \frac{\text{Re}}{2.2} [(\mathbf{M} \times \mathbf{H}) \times \mathbf{M}] - \frac{1}{\Gamma} (\mathbf{M} - \chi \mathbf{H}) \} \rangle = \varepsilon_{\text{inj}} - \varepsilon_{H_0}$ is the mean energy injection rate.

Equation (24) is the main analytical result of this paper. It gives the energy cascade rate for ferrofluid turbulence, where the ferrofluid particles are of negligible size. Whereas $A_1(\ell)$ denotes the part of ε coming exclusively from the fluctuations, $A_2(\ell)$ gives the contribution of external magnetic field H_0 to ε . Note that $A_2(\ell)$ is not entirely coming from the mean-field contribution of the momentum evolution Eq. (14) but also comes from the nonlinear terms in Eq. (15), which are proportional to H_0 and are obtained while calculating $\mathbf{H} \cdot \partial_t \mathbf{M}'$ or $\mathbf{H}' \cdot \partial_t \mathbf{M}$. As shown above, the external magnetic field ($\mathbf{H}_0$) not only actively contributes to the inertial range energy transfer but also modifies the input energy injection (ε) coming from large scales to the inertial range. In the next section, we numerically investigate if ferrofluid turbulence shows an inertial range energy cascade and eventually calculate the cascade rate.

IV. NUMERICAL METHOD AND SIMULATION DETAILS

We perform three-dimensional direct numerical simulations (DNS) of Eqs. (14) and (15) using a pseudospectral method with periodic boundary conditions. The box length is taken to be 2π with N grid points in each direction. We run three simulations with 128^3 , 256^3 , and 512^3 grid points with input Reynolds numbers $\text{Re} = 250$, 500 , and 1428 , respectively. The aliasing error is removed by a standard 2/3-dealiasing method, thus limiting the maximum available wave number to $N/3$. The code is parallelized using an MPI-based slab decomposition scheme [47]. The velocity field is initialized from a fully developed pure Navier-Stokes (NS) flow. A random initial condition is used for the magnetization field, and a uniform time-independent external magnetic field of strength $H_0 = 0.1$ is applied along the z direction. The energy is injected by forcing the momentum evolution equation with a large-scale Taylor-Green forcing $\mathbf{f}_v \equiv f_0 [\sin(k_o x) \cos(k_o y) \cos(k_o z), -\cos(k_o x) \sin(k_o y) \cos(k_o z), 0]$, where $k_o = 2$ is the energy-injection scale and $f_0 = 0.5$ is the forcing amplitude. The system is time evolved using a fourth-order Runge-Kutta (RK4) method until a statistical stationary state is achieved. In the following, using the

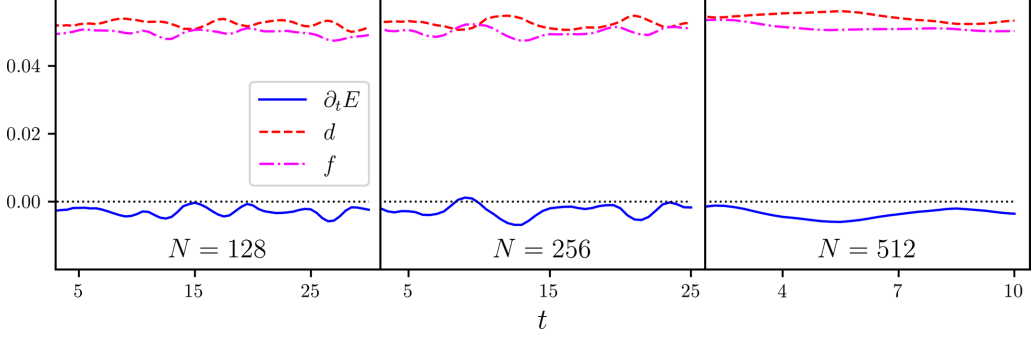


FIG. 1. Time variation of $\partial_t E$, energy injection rate $f = \langle \mathbf{f}_v \cdot \mathbf{v} \rangle$ and dissipation rate d for the external field strength $H_0 = 0.1$. See run 1a (left), run 2a (center), and run 3a (right) from Table I for simulation parameters.

exact law derived above, we shall numerically investigate if a universal energy cascade of energy is obtained in fully developed ferrofluid turbulence and the contribution of different nonlinear terms.

In Fig. 1 we plot the rate of total energy [$\partial_t E = \int \partial_t (E_{\text{kin}} + E_{\text{mag}}) d\tau$], which is determined by the volume integration over the rate of translational kinetic energy and the rate of work due to magnetization in the presence of an external field of the fluid [see Eq. (11)]. As is evident from the figure, the average energy dissipation rate (d) is balanced by the average energy injection rate (f) leading to a statistically stationary state for the total energy, i.e., $\partial_t E \approx 0$. All the statistics are done when the statistical steady state is achieved. The presence of structures at all scales in the velocity, magnetization, and magnetic field intelligibly signifies that the turbulence is fully developed (see Fig. 2).

Simulation parameters are summarized in Table I. All the runs are well resolved as the ratio of the maximum wave number $k_{\text{max}} = N/3$ to the Kolmogorov wave number $k_\eta = 2\pi/\eta$ where $\eta = (\nu^3/\varepsilon)^{1/4}$ is greater than 1. One could expect to better resolve the Kolmogorov scale by increasing the grid size, i.e., the ratio k_{max}/k_η should increase with the increase in grid size. However, at the same time, we are decreasing the fluid viscosity which corresponds to a higher value of k_η . Thus, we could achieve $k_{\text{max}}/k_\eta \sim 1.5$ for 512^3 grid points.

In addition to the aforesaid simulations, we have also performed a series of three simulations for a comparatively strong field strength $H_0 = 1.0$. Rather than searching for a stationary state, we have used the same forcing ($f_0 = 0.5$) as used for the weaker H_0 and collected the data for a state where the average energy is not constant but changing at a constant rate in time (see Fig. 3). In this case, we investigate whether the kinetic and the total energy still cascade with constant rates. Note that the chosen values of H_0 are comparable to realistic situations. For example, in [32] the three values of H_0

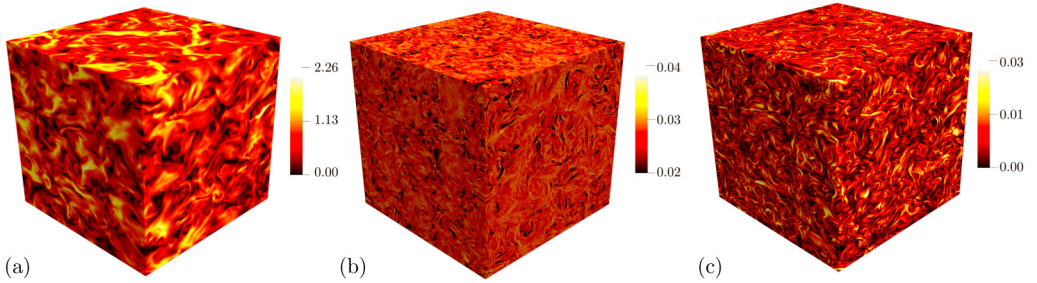


FIG. 2. Snapshot of the modulus of (a) velocity ($\mathbf{v}$) (b) magnetization ($\mathbf{M}$) and (c) magnetic field ($\tilde{\mathbf{H}}$) at simulation time $t = 10$ for run 3a (Table I).

TABLE I. Simulation parameters: “NS” represents Navier-Stokes flow, the subscript “a” represents runs with $H_0 = 0.1$, and “b” represents runs with $H_0 = 1.0$. U_{rms} is the stationary root mean square (rms) velocity, M_0 is the mean magnetization, $\lambda = [5 \int E(k) dk / \int k^2 E(k) dk]^{1/2}$ is the Taylor length scale, and $L = (3\pi/4) \int E(k) k^{-1} dk / \int E(k) dk$ is the integral length scale, where $E(k)$ is the kinetic energy spectrum [48]. $\text{Re}_L = U_{\text{rms}} L / \nu$ is the integral-scale Reynolds number, and $\text{Re}_\lambda = U_\lambda \lambda / \nu$ is the Taylor-scale Reynolds number, where U_λ is the velocity corresponding to the scales λ and is calculated using the phenomenological relation $U_\lambda / U_{\text{rms}} \sim (\lambda / L)^{1/3}$ [49]. For all runs, dimensionless relaxation time $\Gamma = 0.1$ is used.

Run	N	U_{rms}	M_0	L	λ	η	Re_L	Re_λ	k_{max}/k_η	L/λ	L/η
NS	512	1.219	—	0.929	0.194	0.044	1619	201	1.193	4.78	21.11
1a	128	0.739	0.01	0.613	0.343	0.21	113	52	1.432	1.79	2.92
2a	256	0.772	0.01	0.541	0.247	0.125	209	73	1.698	2.19	4.33
3a	512	0.797	0.01	0.506	0.15	0.057	576	114	1.553	3.37	8.88
1b	128	0.731	0.11	0.651	0.377	0.222	118	57	1.508	1.73	2.93
2b	256	0.776	0.11	0.57	0.284	0.133	221	87	1.812	2.00	4.29
3b	512	0.803	0.11	0.514	0.17	0.061	589	135	1.662	3.02	8.43

are 158 Oe, 316 Oe, and 1264 Oe which map to 1.1, 2.2, and 8.8 nondimensional H_0 , respectively. Among these, the value 1.1 is very close to $H_0 = 1.0$ of our case. $H_0 = 8.8$ or further higher values are not taken since the current magnetization equation is applicable only to moderately high values of H_0 as reported by [32]. The value $H_0 = 0.1$ is taken to check the possibility of a nearly stationary cascade.

For a fully developed turbulent flow, inside the inertial range, a flat region is expected for $A(\ell)$ when plotted as a function of increment vector ℓ . The flatness indicates the scale-independent nature of the energy cascade rate and this constant flux rate is equal to the average injection rate i.e., $A(\ell) = \text{const} = 2\varepsilon$. The average $\langle(\cdot)\rangle$ is calculated over the possible pairs of $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \ell$, i.e., $A(\ell) = \langle A(\mathbf{x}, \ell) \rangle_{\mathbf{x}}$. This average is sufficient for the calculation of all flux terms. However, in practice, exact scaling laws are written assuming statistical isotropy. In order to achieve that, we vary the increment vector over 73 directions spanned by the base vectors $\in \{(1,0,0), (1,1,0), (1,1,1), (2,1,0), (2,1,1), (2,2,1), (3,1,0), (3,1,1)\}$ (in units of the grid resolution). Finally, a one-dimensional interpolation is done before the final averaging over all the 73 directions, which gives the isotropic $A(\ell) = \sum A(\ell)/73$ [18,50,51]. Interestingly, in our derived exact relation, $A(\ell)$ is found to be more or less identical in all 73 directions and hence $A(\ell) \simeq A(\ell)$ (see Fig. 4). This is because our derived $A(\ell)$ is according to the alternative form ([16,17,19]) and is free of any global divergence.

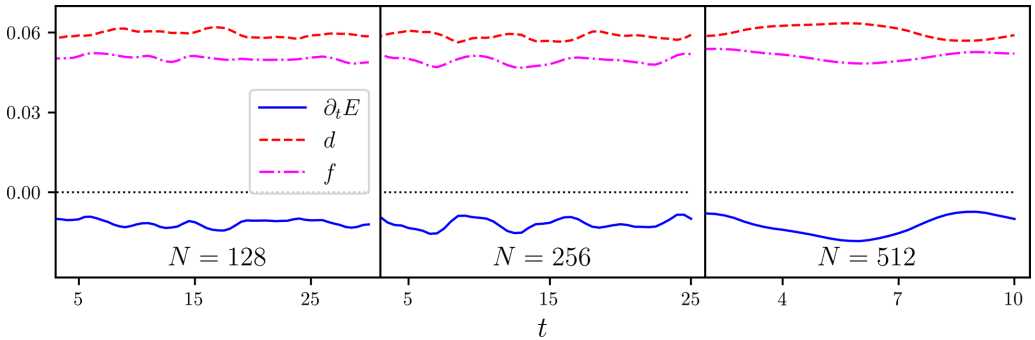


FIG. 3. Time variation of $\partial_t E$, f , and d for the external field strength $H_0 = 1.0$. See run 1b (left), run 2b (center), and run 3b (right) from Table I for simulation parameters.

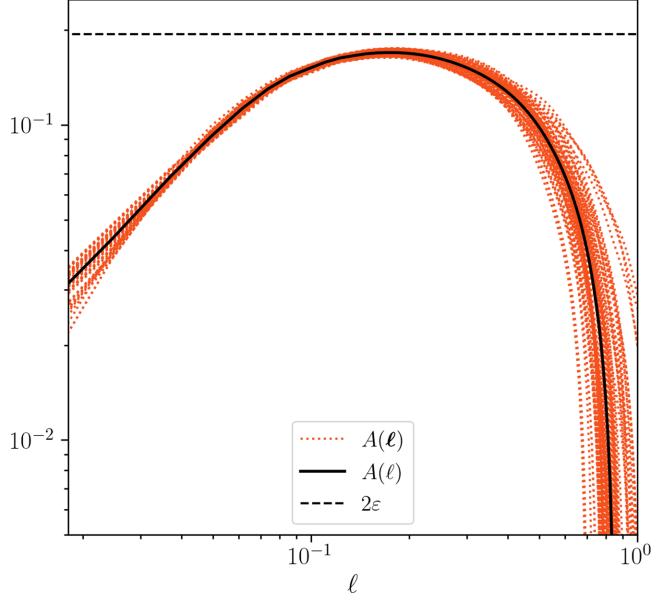


FIG. 4. Energy cascade rate along all the 73 directions (dotted curves) and the average over all the directions (solid black curve) for a typical simulation (run 3a).

V. RESULTS

In Fig. 5 we plot $A(\ell)$, $A_1(\ell)$, $A_2(\ell)$ for external field strengths $H_0 = 0.1$ and 1.0 . While $A_2(\ell)$ consists of the flux contributions due to the initial external field $\mathbf{H}_0$, $A_1(\ell)$ accounts for the contributions due to $\tilde{\mathbf{H}}$. A flat region in $A(\ell)$ (black solid curve) is clearly found within the

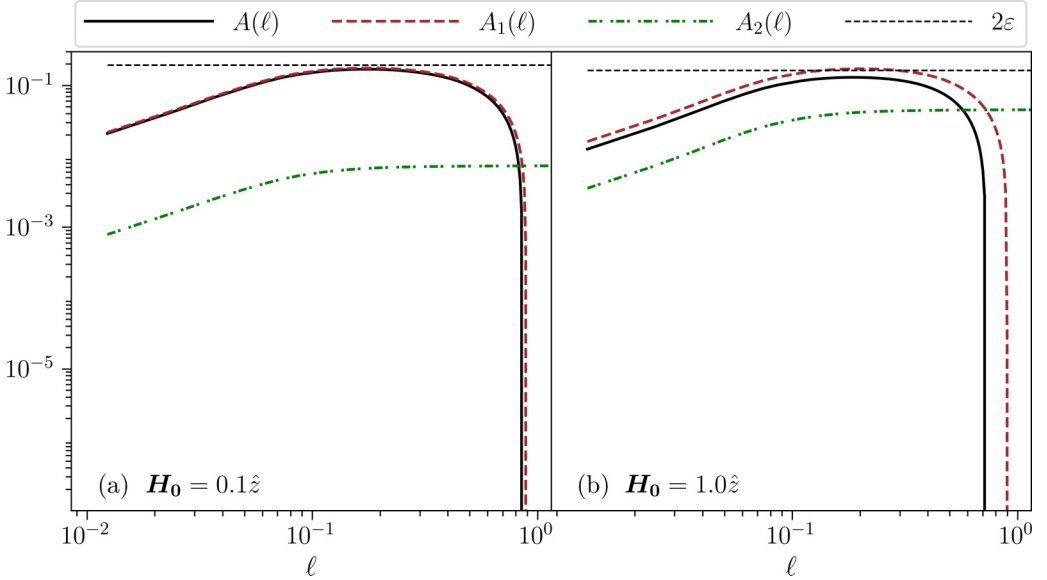


FIG. 5. Energy cascade rates as a function of ℓ for a box with 512^3 grid points for $\mathbf{H}_0 = 0.1\hat{z}$ (left) and $\mathbf{H}_0 = 1.0\hat{z}$ (right).

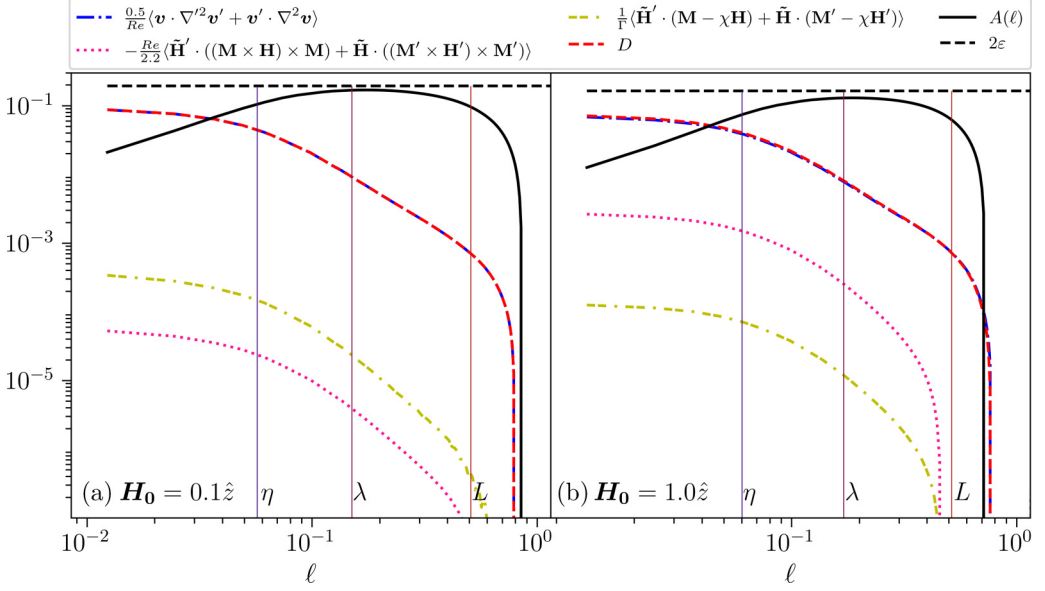


FIG. 6. Various components of two point energy dissipation rate and $A(\ell)$ as a function of ℓ for a box with 512^3 grid points for $\mathbf{H}_0 = 0.1\hat{z}$ (left) and $\mathbf{H}_0 = 1.0\hat{z}$ (right).

intermediate range of scales. It indicates the existence of a scale-independent energy cascade within the internal range and thus numerically validates the Kolmogorov type of universality for our derived exact relation [Eq. (24)] for ferrofluids. For statistical stationary state, in addition, the constant scale-independent cascade rate should be equal to twice the average energy injection rate (ε) at large scales. This equality expressed in Eq. (24) holds reasonably well for $H_0 (=0.1)$ where we observe $A(\ell) \approx 2\varepsilon$ within the inertial range [see Fig. 5(a)]. Note that energy flux rate $A(\ell)$ is balanced with the modified injection rate ε , which is not merely the external energy input rate $\langle \mathbf{f}_v \cdot \mathbf{v} \rangle$. This is well expected as we achieved a statistical stationary state for one-point average energy with $H_0 = 0.1$. Also, $A_2(\ell)$ is small compared to $A_1(\ell)$ (~ 20 times smaller), and consequently the total energy transfer is mainly governed by $A_1(\ell)$ as $A_1(\ell) \approx A(\ell) \approx 2\varepsilon$. Similar to Fig. 5(a), both $A(\ell)$ and $A_1(\ell)$ in Fig. 5(b) become constant for a range of scales thus exhibiting a clear signature of both kinetic and the total energy cascades.

However, for stronger external field ($H_0 = 1.0$), $A_2(\ell)$ increases and becomes only ~ 4 times smaller than $A_1(\ell)$, which remains unaffected by any change in H_0 . Due to negative contribution of $A_2(\ell)$ in $A(\ell)$, the latter decreases with respect to $A_1(\ell)$. Note that, 2ε also decreases as H_0 increases and is slightly dominated by $A_1(\ell)$ in the inertial range as is evident from Fig. 5(b). Globally, one can easily see that the exact relation $A(\ell) = 2\varepsilon$ is not closely verified for $H_0 = 1.0$ as there appears a gap between $A(\ell)$ and 2ε . This could be due to the fact that scale-dependent dissipative terms are no longer negligible in the inertial length scales. In order to probe this, we have also plotted different contributions from the total dissipation in Fig. 6. For both values of H_0 , all the dissipation effects are found to remain small in the inertial range and become important at small scales. It is therefore natural to understand that the gap between $A(\ell)$ and 2ε is mainly because a proper stationary state was not obtained for $H_0 = 1.0$ and the exact law is calculated in the presence of a nonzero but constant $\partial_t R$. This is particularly interesting as it gives an evidence of energy cascade even in a nonstationary regime where the more general exact relation $A(\ell) = \partial_t R + 2\varepsilon$ is satisfied and affirms a constant $A(\ell)$ when $\partial_t R$ is a nonzero constant. In Fig. 7 we plot $A(\ell)$ along with its components by varying the number of grid points from 128^3 to 512^3 . As is expected, for both $H_0 = 0.1$ and $H_0 = 1.0$, the extent of the flat region increases with the number of grid points and a better convergence towards $A(\ell) = 2\varepsilon$ is obtained [see Figs. 7(e) and 7(f)].

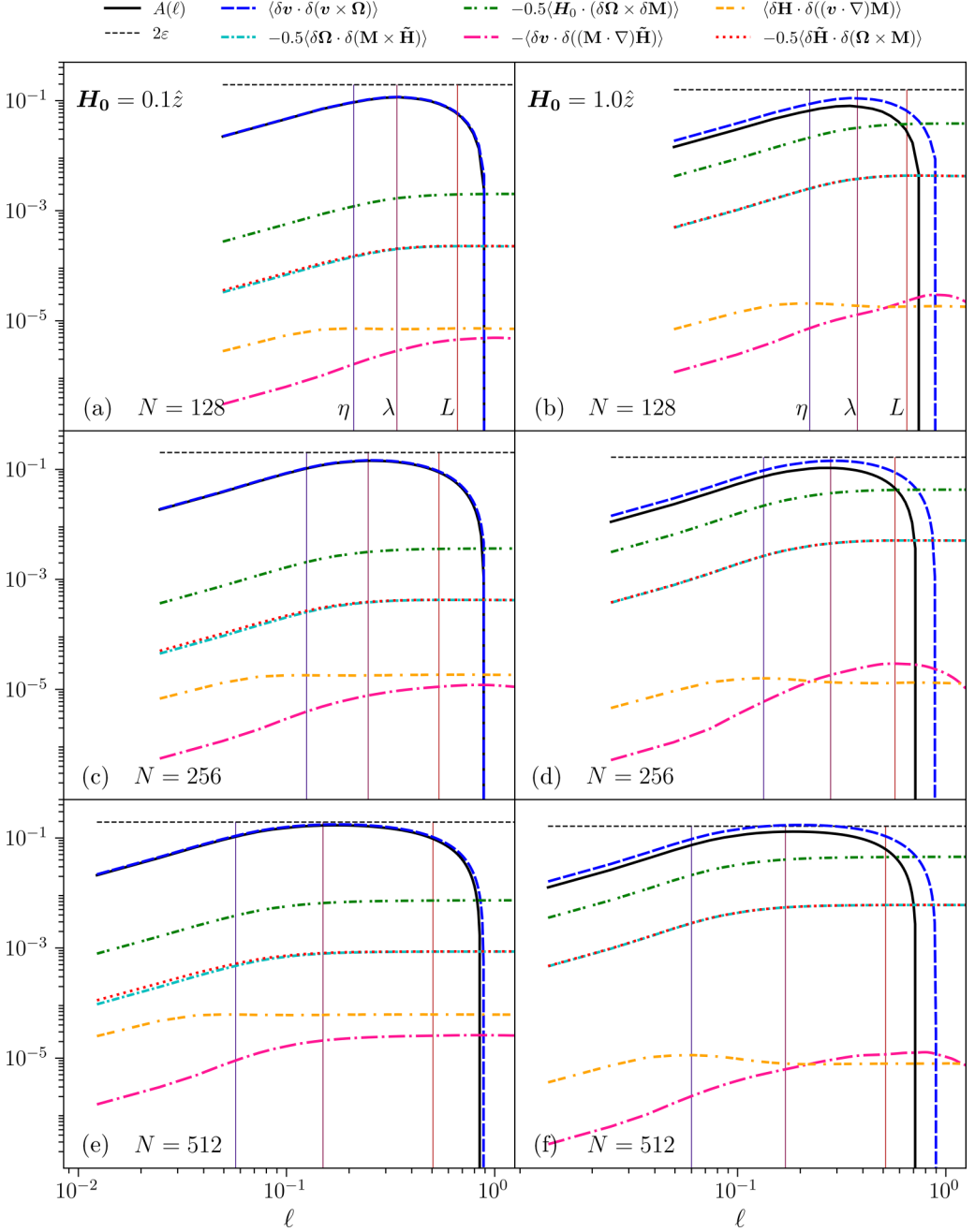


FIG. 7. Comparison of the energy cascade rates $A(\ell)$ and its components for grid points 128^3 to 512^3 (top to bottom) with external magnetic field strength $H_0 = 0.1$ and $H_0 = 1.0$ (left and right).

The extent of the inertial range is described in terms of the largest energy-containing integral scale (L) and the Kolmogorov scale (η) corresponding to the smallest eddies. As one goes on increasing the resolution (or grid points), the large-scale Reynolds number Re increases and so does the ratio (L/η) , thus corresponding to a wider inertial range. From the plots, the increase in

TABLE II. Order of magnitude of different terms of $A(\ell)$ for grid points 512³.

H_0	$\langle \delta \mathbf{v} \cdot \delta(\mathbf{v} \times \boldsymbol{\Omega}) \rangle$	$\langle \delta \mathbf{v} \cdot \delta[(\mathbf{M} \cdot \nabla) \tilde{\mathbf{H}}] \rangle$	$\langle \delta \tilde{\mathbf{H}} \cdot \delta[(\mathbf{v} \cdot \nabla) \mathbf{M}] \rangle$	$\langle \delta \tilde{\mathbf{H}} \cdot \delta(\boldsymbol{\Omega} \times \mathbf{M}) \rangle$	$\langle \delta \boldsymbol{\Omega} \cdot \delta(\mathbf{M} \times \tilde{\mathbf{H}}) \rangle$	$\langle \mathbf{H}_0 \cdot (\delta \boldsymbol{\Omega} \times \delta \mathbf{M}) \rangle$
0.1	0.1	10^{-5}	10^{-5}	10^{-3}	10^{-3}	10^{-2}
1.0	0.1	10^{-5}	10^{-4}	10^{-2}	10^{-2}	10^{-1}

the inertial range with grid size is evident from the considerable shift of η towards smaller scales, whereas L shifts very slightly towards the left. Similar to η , the Taylor microscale (λ), which denotes the length scale where dissipation begins to impact turbulent eddies, also appears to migrate towards smaller scales leading to a wider inertial range. All these three length scales are denoted by vertical lines for all the runs plotted in Fig. 7. Further, for a particular grid resolution, the ratios L/λ and L/η can be used to quantify the coupling between large and small scales. Smaller values of L/λ and L/η indicate more coupling and vice versa. For, e.g., the ratio L/λ decreases from 4.78 to 3.37 as we move from a hydrodynamic case (run NS) to a ferrofluid system with $H_0 = 0.1$ (run 3a). The same ratio decreases to 3.02 as H_0 increases to 1.0 (run 3b) indicating a stronger coupling in the second case. The stronger coupling results from the large-scale effect coming from the external mean field [52]. This is also true for L/η . Thus we conclude that the coupling between the large and the small scales is the smallest in the pure HD case ($H_0 = 0$) and increases with the increase of H_0 , and hence the inertial range becomes smaller for stronger magnetic field strengths. Interestingly, when we increase $H_0 = 0.1$ to 1.0, the value of U_{rms} does not change much, which is consistent with the observation in [32]. However, the mean magnetization M_0 is increased from 0.01 to 0.11 indicating an increased alignment of the ferrofluid particles with the external field.

For all runs, the pure kinetic term $\langle \delta \mathbf{v} \cdot \delta(\mathbf{v} \times \boldsymbol{\Omega}) \rangle$ is found to be the most dominating term in $A(\ell)$ (see Table II for the relative order of magnitude of various terms). Among other terms, the energy transfer due to $\langle \delta \boldsymbol{\Omega} \cdot \delta(\mathbf{M} \times \tilde{\mathbf{H}}) \rangle$ and $\langle \delta \tilde{\mathbf{H}} \cdot \delta(\boldsymbol{\Omega} \times \mathbf{M}) \rangle$ increases in magnitude with H_0 . However, these two contributions cancel each other within the inertial range and thus do not affect the total energy transfer. The remaining terms $\langle \delta \tilde{\mathbf{H}} \cdot \delta[(\mathbf{v} \cdot \nabla) \mathbf{M}] \rangle$ and $\langle \delta \mathbf{v} \cdot \delta[(\mathbf{M} \cdot \nabla) \tilde{\mathbf{H}}] \rangle$ are several orders of magnitude less than the kinetic term and are almost unaffected by the strength of external field. Finally, as mentioned above, the effect of the external field enters through $\langle \mathbf{H}_0 \cdot (\delta \boldsymbol{\Omega} \times \delta \mathbf{M}) \rangle$ which increases with H_0 .

VI. DISCUSSION AND CONCLUSIONS

Using two-point statistics, we have derived the exact relation corresponding to the transfer of total energy for three-dimensional incompressible homogeneous ferrofluid turbulence. The exact relation is equally applicable to isotropic and anisotropic flows. For anisotropic flows, unlike the divergence form of the exact relation, the energy cascade rate can be calculated in a straightforward manner without worrying about the geometry of the system. Using direct numerical simulations, we have numerically calculated the cascade rate owing to our derived exact relation. For weak external magnetic fields, we have studied the turbulent dynamics of a statistical stationary state where the kinetic energy almost equals the total energy and exhibits an inertial range cascade. For a stronger external magnetic field, we have studied the dynamics of a nonstationary regime with a constant energy dissipation rate. Note that, for a strong H_0 , achieving a statistical stationary state is found to be practically difficult as ferromagnetic particles tend to align themselves quickly along a strong external field thereby leading to a constant leakage of energy from the fluid to the external field. This distinguishes ferrofluid turbulence from ordinary fluid turbulence, where the stationary energy cascade is observed. For ferrofluids, one can, however, obtain a nonstationary cascade with a constant cascade rate if the energy is dissipated at a uniform rate. This is similar to the current study where both the kinetic and the total energy cascade with a uniform transfer rate both for stationary and nonstationary regimes. However, the kinetic energy cascade rate is found to be greater than the

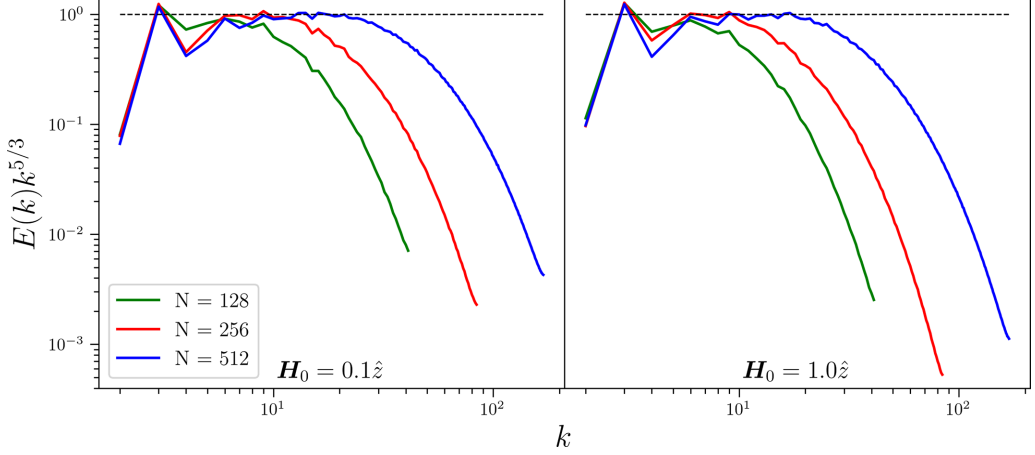


FIG. 8. Compensated kinetic energy spectrum for all runs in Table I.

total energy cascade rate, which includes the loss of energy due to magnetic field. With the increase in resolution from 128^3 to 512^3 , we also observed that the flat region between Kolmogorov scale and integral scale increases and a better convergence towards the exact law is achieved.

In spectral space, the kinetic energy power spectrum gives a $k^{-5/3}$ spectrum for both values of H_0 thereby justifying the existence of a Kolmogorov-type cascade for kinetic energy in ferrofluid turbulence (see Fig. 8). Similar to the physical space, wider range for $k^{-5/3}$ spectrum is also found for simulations with increasing grid points. Finding power spectra for total energy is, however, not evident for ferrofluid turbulence where the energy density function cannot be written due to the inexact differential form of the magnetic energy part.

With the help of the derived exact relation, one can also study the turbulent relaxation in ferrofluids using the recently proposed principle of vanishing nonlinear transfer [53]. This principle has recently been used to successfully predict the relaxed states in binary fluid turbulence [54] and is currently being implemented to find the relaxation in ferrofluids in a separate study. Complementary to the current study, one can also investigate if the energy conservation is also satisfied in triads and search for the corresponding mode to mode transfers [55]. Using the said concept, a natural continuation would be to study contributions from local and nonlocal triads in the energy cascade in ferrofluids in the presence of an external magnetic field of various strengths. The current study can also be extended to the compressible ferrofluid system including the temperature evolution equation into account [56].

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