Approximate derivation of the power law for the mean streamwise velocity in a turbulent boundary layer under zero-pressure gradient

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Distribution of the mean streamwise velocity in a turbulent boundary layer over a flat plate can be represented by the equation $U \sim \eta^{1/n}$, as was widely used in the past; U and η are the normalized velocity and the wall-normal distance, respectively. However, this $1/n$th-power law is an empirical one. By incorporating either the Reynolds shear stress model of Wei *et al.* [J. Fluid Mech. **969**, A3 (2023)], which is in terms of U and the (normalized) wall-normal velocity (V), or a similar one in the boundary layer equations, it is found that U and V are related as $U^{(H+1)} \sim V^{(H-1)}$ in the outer region of a flat plate boundary layer; H is the flow shape parameter. Along with the distribution of the wall-normal velocity (V_w) of Wei *et al.*, the $1/n$th-power law for U is obtained by equating the derivative (with respect to η) of V with that of V_w. Thus, this empirical power law seems to have a reasonable theoretical basis embedded in it.

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I. INTRODUCTION

In an incompressible and zero-pressure-gradient turbulent boundary layer, the distribution of the mean streamwise velocity u can be represented by the $1/n$th power law (see Schlichting [1]):

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}}, \tag{1}$$

where y is the wall-normal distance, u_∞ is the free stream speed, and δ is the boundary layer thickness. Although this empirical velocity distribution follows from that for flow in a smooth pipe [1], it implies similar velocity profiles. The exponent *1/n* although varies mildly with flow Reynolds number, the 1/7th power law is a popular one.

The power-law Eq. (1) remains an empirical one as no theoretical basis for this equation for the streamwise velocity has been reported in the past. The reason for this seems to be the absence of a simple Reynolds shear stress, $\rho(-\overline{u'v'})$, model in terms of mean velocity components with which one can obtain even an approximate solution of the governing equations easily. For example, the widely used $k-\epsilon$ model requires an equation for the turbulent kinetic energy k, and a model for the turbulence dissipation rate ϵ. With this (or one such model), a coupled momentum equation and the equation for the turbulence kinetic energy require numerical solutions. On the other hand, if one considers the Reynolds shear stress model of Wei *et al.* [2] that is in terms of the streamwise and wall-normal velocities in a flat plate boundary layer, of interest here, then the momentum equation becomes easier to handle. With this Reynolds shear stress model or a similar one considered here, a power-law relationship between the streamwise and wall-normal velocities is

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obtained from the boundary layer equations in the outer region of a flat plate boundary layer. Using this and the distribution of the wall-normal velocity V_w proposed by Wei *et al.* [2], the power-law Eq. (1) for the streamwise velocity U is derived analytically, as reported in this paper.

While the log-law or power-law for the mean streamwise velocity is well established, various aspects of the wall-normal velocity v have been the focus of many recent investigations. For example, Wei and Klewicki [3] have suggested that the mean wall-normal velocity should be scaled by v_∞, the value of v at the boundary layer edge, instead of u_∞. Also, recent studies (Kumar and Dey [4], Kumar and Mahesh [5], Wei *et al.* [2]) suggest that the wall-normal velocity plays an important role in the distribution of the Reynolds shear stress in the boundary layer. By neglecting the viscosity in the outer region, Kumar and Dey [4] have suggested that $(-\overline{u'v'})$ varies linearly with v. Kumar and Mahesh [5] have included the viscosity in proposing an equation for the combined viscous shear and Reynolds shear stresses. Wei *et al.* [2], who have considered the boundary layer equations, proposed the Reynolds shear stresses in the inner and outer regions of a flat plate boundary layer as

$$(-\overline{u'v'})_{\text{inner}} \approx u_\tau^2 - \nu S, \quad (-\overline{u'v'})_{\text{outer}} \approx u_\tau^2\left(1 - \frac{uv}{u_\infty v_\infty}\right), \tag{2}$$

respectively. Here u_τ is the friction velocity, $S = \mathrm{d}u/\mathrm{d}y$, and ν is the kinematic viscosity of fluid. Based on Eq. (2), Wei *et al.* [2] have proposed the Reynolds shear stress model

$$RSS_W \equiv (-\overline{u'v'}) \approx u_\tau^2\left(1 - \frac{uv}{u_\infty v_\infty}\right) - \nu S \tag{3}$$

for zero-pressure-gradient boundary layer flows. This shear stress distribution in the boundary layer is not a linear combination of the inner and outer stresses nor is it based on any matching of the two stresses. In view of the cascading feature of large eddies to smaller eddies, it may be possible to assume that the inner and outer stresses in Eq. (2) are weighted by each other in the distribution of the Reynolds shear stress across the boundary layer. A model based on this is considered here and is found to be comparable with Eq. (3). These Reynolds shear stress models were considered in solving the boundary layer equations in the outer region of a flat plate boundary layer.

II. ANALYSIS

An incompressible and turbulent boundary layer flow over a semi-infinite plate is considered here. The velocity components in the streamwise x and wall-normal y directions are u and v, respectively; the corresponding fluctuating velocity components are denoted by u' and v', respectively. The constant free stream speed is denoted by u_∞ and the value of v at the boundary layer edge is denoted by v_∞.

The governing continuity and momentum equations, under the boundary approximation, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial(-\overline{u'v'})}{\partial y}, \tag{5}$$

respectively. Here, the terms involving Reynolds direct stresses are neglected. The Reynolds shear stress model Eq. (3) is in terms of the streamwise and wall-normal velocities, as mentioned earlier. However, this model is not a linear-combination type nor based on the matching of the inner and outer layer stresses. Since large eddies break into smaller eddies, it is assumed here that the $(-\overline{u'v'})_{\text{inner}}$ and $(-\overline{u'v'})_{\text{outer}}$ stresses in Eq. (2) may weigh on each other like

$$RSS_P \equiv (-\overline{u'v'}) \approx \left(u_\tau^2 - \nu S\right)\left(1 - \frac{uv}{u_\infty v_\infty}\right). \tag{6}$$

A comparison of this model with Eq. (3) is shown in Fig. 1, for various Reynolds numbers Re; $\text{Re} = u_\infty\theta/\nu$ is the Reynolds number based on the momentum thickness θ. Here (and hereafter),

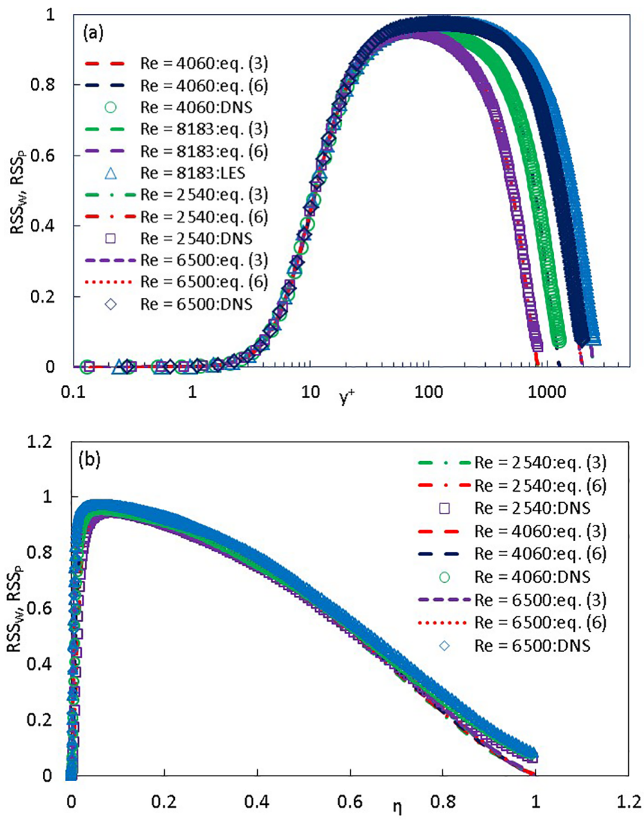


FIG. 1. Comparison of the Reynolds shear stress models Eq. (3) and Eq. (6) with actual values (a) in y^+ scale, and (b) in $\eta = y/\delta$. Simulated data from references [6–8] are used in all plots reported in this work.

DNS data of Schlatter and Örlü [6] for Re = 2540, 4060 and that of Simens *et al.* [7] for Re = 4000 and 6500, and LES data of Eitel-Amor *et al.* [8] for Re = 8183 are used. Figure 1(a) is in $y^+(=yu_\tau/\nu)$ scale, and Fig. 1(b) is in y/δ scale. It can be seen that the difference between Eq. (3) and Eq. (6) is indistinguishable; towards the boundary layer edge, both of these models show the same extent of difference with the simulated data.

Incorporating the Reynolds shear stress from Eq. (6) in the momentum equation Eq. (5), we have

$$u\frac{\partial u}{\partial x}\left(H-1+\frac{\nu S}{u_\tau^2}\right)+v\frac{\partial u}{\partial y}\left(H+1-\frac{\nu S}{u_\tau^2}\right)=\left(\frac{uv}{u_\tau^2}\right)\nu\frac{\partial^2 u}{\partial y^2}. \tag{7}$$

Here, $u_\infty^+ v_\infty^+ \, (=u_\infty v_\infty/u_\tau^2) = H$ (Wei and Klewicki [3]) is used; $H(=\delta_1/\theta)$ is the shape factor, and δ_1 is the displacement thickness. The shape parameter being based on the integral length scale represents the viscous effect via the profile shape of the streamwise velocity, while the terms with ν are the local viscous quantities from the inner region. This equation shows that the inner layer stress is convected, and the viscous diffusion term is multiplied by the Reynolds shear stress in the outer region. In the outer region, of interest here, the molecular viscosity is usually neglected leading to the approximate momentum equation,

$$u\frac{\partial u}{\partial x}(H-1)+v\frac{\partial u}{\partial y}(H+1)=0. \tag{8}$$

Using the continuity equation (4), Eq. (8) becomes

$$\frac{(H-1)}{v}\frac{\partial v}{\partial y} \approx \frac{(H+1)}{u}\frac{\partial u}{\partial y}. \tag{9}$$

It is worth mentioning that the Reynolds shear stress model of Wei *et al.* [2], Eq. (3), leads to the same Eq. (9); the algebra being simple is skipped here.

We consider the following variables:

$$\eta = \frac{y}{\delta(x)}, \quad U = \frac{u}{u_\infty}, \quad V = \frac{v}{v_\infty}. \tag{10}$$

In terms of these variables, Eq. (9) becomes

$$(H-1)\frac{d(lnV)}{d\eta} \approx (H+1)\frac{d(lnU)}{d\eta}. \tag{11}$$

Integrating Eq. (11), we have

$$V^{(H-1)} \approx CU^{(H+1)} \Rightarrow U(\eta) \approx [(V(\eta))^{(H-1)/(H+1)}]D, \tag{12}$$

where C and D are constants. In order to compare this with the 1/7-th-power, $H = 1.3$ is assumed in Eq. (12). That is,

$$U(\eta) \approx [V(\eta)]^{0.13}D \approx DV^{\frac{1}{7.7}}. \tag{13}$$

The constant D was evaluated at the boundary layer edge and its value is unity, i.e., $D = 1$. Thus, the simple power-law relationship between the streamwise and wall-normal velocities is

$$U \approx V^{\frac{1}{7.7}}. \quad u^+ \approx u^+_\infty\left(\frac{v}{v_\infty}\right)^{\frac{1}{7.7}}. \tag{14}$$

The second expression is in terms of the inner velocity scale u_τ: $u^+ = u/u_\tau$. Using the simulated data of Schlatter and Örlü [6], Eitel-Amor *et al.* [8], and Simens *et al.* [7], mentioned above, the estimated u^+ is compared with the actual u^+ in Figs. 2 and 3 in y^+ scale. The log-law for the streamwise velocity

$$u^+ = 5.6 \log{(y^+)} + 5.2$$

is also shown here. For clarity, different Reynolds number cases are shown separately in Figs. 2 and 3. It may be noted that the value of H was chosen for a better match with the data for the following reason: The numerical value of H_1 (say) $= u^+_\infty v^+_\infty$ was found to differ from H_2 (say) $= \delta_1/\theta$ slightly. For example, for $Re = 6500$, H_1 and H_2 are 1.28 and 1.36, respectively; $H = 1.39$ was used, which is close to the value of H_2. Similarly, for Re $= 2540$, $H = H_1 = 1.306$ was used, while H_2 is 1.4. In Fig. 4, the estimated U is compared with the actual data in η scale; for clarity, only a few flows are shown here. It can be seen in Figs. 2 and 3 that the present proposal covers even the log-layer, which is in the outer region. A power-law relationship between the streamwise velocity and the wall-normal velocity has been presented here.

For a constant pressure boundary layer, Wei *et al.* [2] have proposed the distribution of the wall-normal velocity in terms of the mean streamwise velocity,

$$V_w \approx \frac{\delta}{\delta_1}\int_0^\eta (1-U)d\eta - \frac{\delta}{\delta_1}\eta(1-U). \tag{15}$$

This self-similar velocity distribution, though approximate, covers almost the entire boundary layer (see Fig. 2 of Ref. [2]). The present proposal for the wall-normal velocity in the outer region, from (12) (with $D = 1$), is

$$V(\eta) \approx [U(\eta)]^{\frac{H+1}{H-1}}. \tag{16}$$

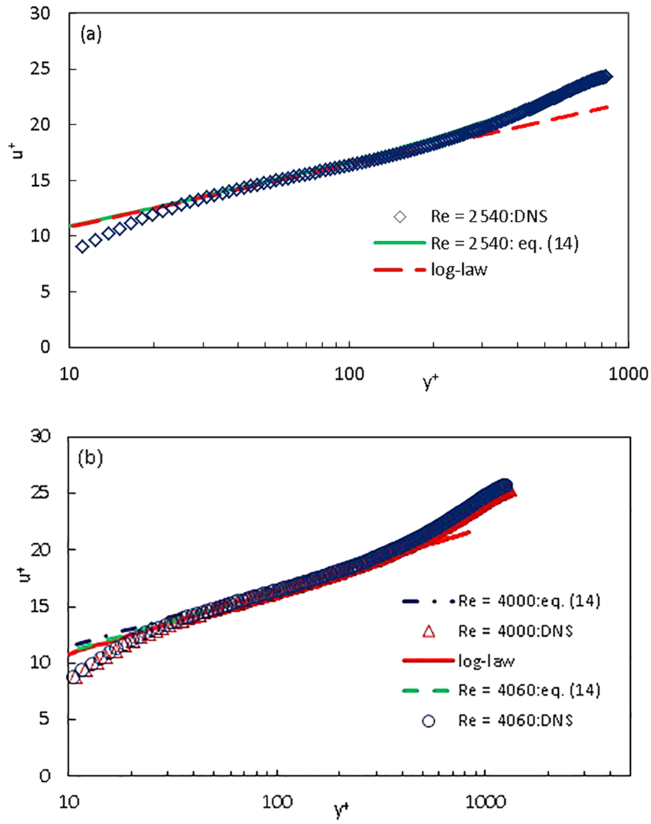


FIG. 2. Comparison of u^+ estimated using Eq. (14) with actual data.

While both Eq. (15) and Eq. (16) are approximate, Eq. (15) involves integration, compared to the power-law result Eq. (16). However, Eq. (15) is useful in arriving at an interesting result, as in the following.

Differentiating Eq. (15) and Eq. (16) with respect to η, and equating the two derivatives $(dV_w/d\eta)$ and $(dV/d\eta)$, we have

$$\left(\frac{H+1}{H-1}\right)U^{2/(H-1)}\frac{dU}{d\eta} \approx \frac{\delta}{\delta_1}\eta\frac{dU}{d\eta}. \tag{17}$$

This step is like the matching of V_w, which covers a large part of the boundary layer, with V for the outer region. It can be seen that Eq. (17) immediately leads to the $1/n$th-power law for the streamwise velocity,

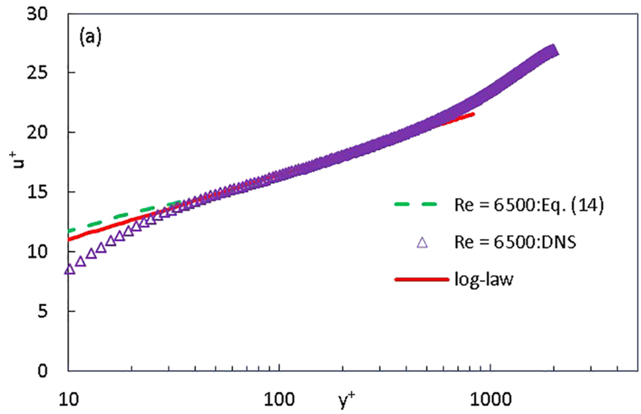
$$U \approx \left[\frac{(H-1)\delta}{(H+1)\delta_1}\right]^{\frac{H-1}{2}} \eta^{\frac{H-1}{2}}. \tag{18}$$

The exponent 1/n of η in Eq. (1) is related to the shape factor [1] as

$$\frac{1}{n} = \frac{H-1}{2}. \tag{19}$$

It is interesting to note that the same exponent of η appears in Eq. (18). Assuming $\delta/\delta_1 = 8$, for the 1/7th power law [1] and $H = 1.3$, the equation for the streamwise velocity from Eq. (18) is

$$U \approx 1.006\eta^{0.15}. \Rightarrow U \approx 1.006\eta^{1/6.7}. \tag{20}$$



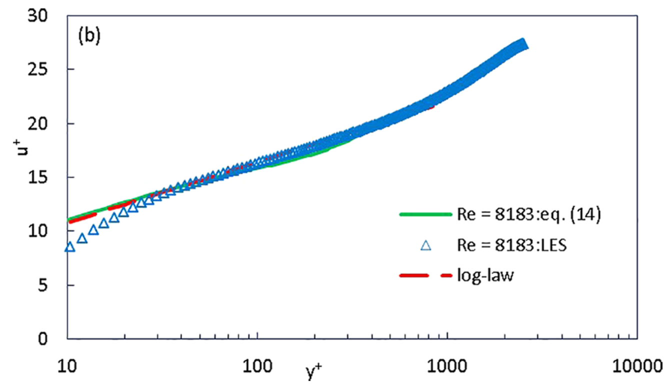


FIG. 3. Comparison of u^+ estimated using Eq. (14) with simulated data.

That is same as the widely used 1/7 th-power law, the nonsignificant difference in the exponent of η is due to the choice of $H = 1.3$ made here. One conclusion here is that the power-law Eq. (1) seems more appropriate in the outer region of a flat plate boundary layer. The extent of empiricism in the present analysis is in the Reynolds shear stress models Eq. (3) and Eq. (6), but this is acceptable in view of the result that the distribution $U \sim \eta^{(H-1)/2}$ is based on the solution of the boundary layer

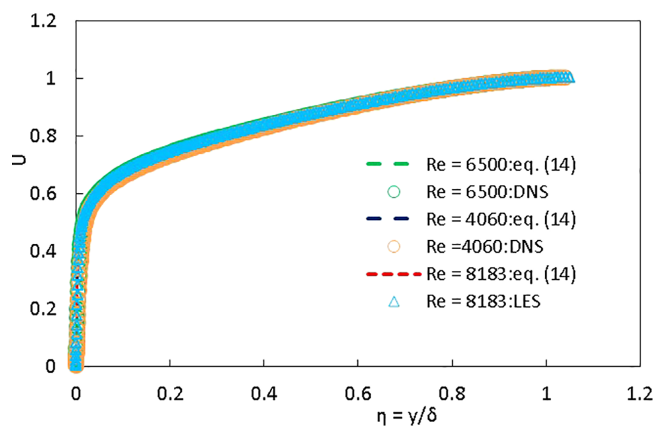


FIG. 4. Comparison of U estimated using Eq. (14) with actual data in y/δ scale.

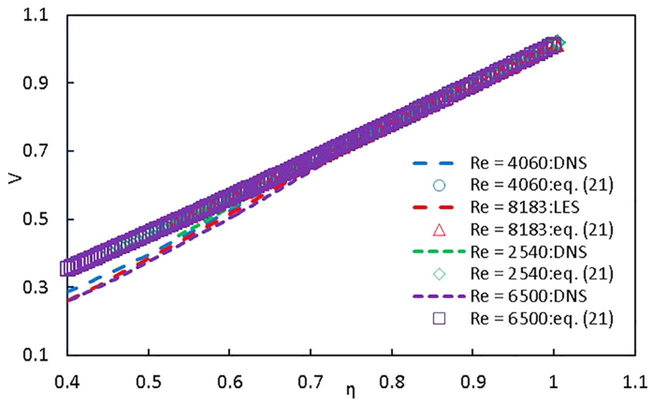


FIG. 5. Comparison of wall-normal velocity estimated using Eq. (21) with simulated values in y/δ scale.

equations here and that by Wei *et al.* [2]. One can thus conclude that Eq. (1) for the streamwise velocity cannot be labeled as purely empirical.

Similar to the power-law equation (20) for the streamwise velocity, the power-law equation for the wall-normal velocity that follows from Eq. (16) and Eq. (20) is

$$V \approx 1.047\eta^{1.144}. \tag{21}$$

This distribution of $V\,(=v/v_\infty)$ is compared with the actual data in Fig. 5. We may note that, instead of the factor 1.047, a factor of 1.012 was used for a good match with the data. It can be seen that agreement with the simulated data is good for $\eta\,(=\mathrm{y}/\delta) \geqslant 0.70$, i.e., mostly in the outer region, as expected. It may be noted that with $U = \eta^{1/7}$ in the approximate distribution of the wall-normal velocity Eq. (15), V_w is

$$V_w \approx \eta^{1.143}. \tag{22}$$

which is the same as Eq. (21).

III. CONCLUSION

Similar to the Reynolds shear stress model of Eq. (3), proposed by Wei *et al.* [2], the Reynolds shear stress model Eq. (6) is considered here. Incorporating either of these two Reynolds shear stress models in the boundary layer equations, a simple power-law relationship between the streamwise and wall-normal velocities $V \sim U^{(H+1)/(H-1)}$ [Eq. (16)] is obtained as a solution of the governing equations in the outer region of a flat plate boundary layer. By considering $dV/d\eta = dV_w/d\eta$ (in the sense of matching the two velocity profiles), the $1/n$th power law Eq. (1) is obtained analytically [Eq. (18)]. Thus, it appears that the power-law Eq. (1), rather than a purely empirical one, has some reasonable basis embedded in it.

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