Analytical solutions for long-time steady state Boussinesq gravity currents flowing along a horizontal boundary of finite length

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This paper presents analytical solutions for a steady turbulent miscible gravity current flowing along a horizontal rigid boundary of finite length into a quiescent uniform environment. These solutions are obtained from the governing equations (mass, momentum, and buoyancy) originally proposed by Ellison and Turner [J. Fluid Mech. **6**, 423 (1959)] for a buoyant layer of fluid in the Boussinesq approximation. For a constant drag coefficient C_d and the specific entrainment law $E \propto \mathrm{Ri}^{-1}$, Ri being the local Richardson number, we first derived a system of coupled ordinary differential equations describing the longitudinal evolution of the velocity u, the height h, the density deficit η, and the Richardson number Ri of the current. For an initially supercritical flow ($\mathrm{Ri}_0 < 1$), explicit relations are found for $u(x)$, $h(x)$, and $\eta(x)$ solely as a function of the Richardson number $\mathrm{Ri}(x)$. The longitudinal evolution of the Richardson number is then theoretically obtained from a universal function F which can be tabulated and, as in the present paper, also plotted. The function F allows us to determine (and only from the knowledge of the boundary conditions at the source) whether the flow remains supercritical over the whole length of the rigid boundary, or might transit towards a subcritical state ($\mathrm{Ri} > 1$). In this latter case, the mathematical resolution is modified by including a discontinuity similar to a hydraulic jump. The location and amplitude of this discontinuity are calculated from an additional universal function G and the injection conditions. The method is finally extended to provide analytical solutions for other classical entrainment laws.

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I. INTRODUCTION

A gravity current is a canonical flow that occurs when a light (heavy) fluid propagates into a heavier (lighter) ambient fluid along a rigid boundary. This flow can involve immiscible or miscible fluids. In the latter case, the current engulfs the surrounding fluid in a process called entrainment, resulting in a longitudinal evolution of the current mass flow rate.

Gravity currents arise in many environmental flows such as katabatic winds (Manins and Sawford [1]), oceanic deep currents (Cenedese and Adduce [2]) or turbidity currents (Meiburg and Kneller

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[3]), to name but a few. They may also appear in hazardous situations such as oil spreading on the sea (Hoult [4]) or fire-induced smoke propagation (Alpert [5]).

Because of both their ubiquity and academic interest, gravity currents have been widely studied. The pioneering works of Von Kármán [6] and later Benjamin [7] were the first to tackle this flow theoretically. One of the main objectives of these works was to determine the dynamics of the current head during the propagation phase. Several authors have subsequently addressed the transient evolution of this flow by proposing models for gravity currents resulting from a fixed-volume release using experimental (Huppert and Simpson [8], Rottman and Simpson [9], Lowe, Rottman, and Linden [10]) numerical (Birman, Battandier, Meiburg, and Linden [11], Bonometti, Ungarish, and Balachandar [12]) and theoretical means (Shin, Dalziel, and Linden [13]), as well as from a fixed-flux release with experiments (Longo, Ungarish, Di Federico, Chiapponi, and Addona [14], Sher and Woods [15], Martin, Negretti, Ungarish, and Zemach [16]), numerical simulations (Shringarpure, Lee, Ungarish, and Balachandar [17], Hogg, Nasr-Azadani, Ungarish, and Meiburg [18]), and theory (Johnson and Hogg [19], Ungarish [20]).

In contrast with the transient phase of gravity currents, which has been addressed in a substantial number of studies, the steady phase (i.e., after a long time subsequent to the reaching of the exit of the rigid boundary by the flow) remains little investigated so far. In their seminal article, Ellison and Turner [21] (hereafter referred to as ET59) developed a theoretical model for a fixed-flux steady gravity current based on the conservation equations for mass, momentum, and buoyancy. Through a system of coupled differential equations, their model allows the evolution of the three variables of the current (velocity, thickness, and density deficit) to be calculated along the longitudinal propagation x axis. Similarly to Morton, Taylor, and Turner [22] for turbulent plumes, they introduced an entrainment coefficient E to quantify the amount of ambient fluid entrained into the current. After some algebraic manipulations, the Richardson number Ri naturally appears in their equations. It is defined as

$$\mathrm{Ri} = \frac{\eta g h}{u^2}, \tag{1}$$

with $\eta = |\rho_a - \rho|/\rho_a$ being the density deficit, ρ_a the density of the ambient, g the gravitational acceleration and h, ρ, and u, the thickness, density, and velocity of the current, respectively. As explained in ET59, the Richardson number, which characterizes the local stability of the current, allows three different flow regimes to be identified: the supercritical regime when $\mathrm{Ri} < 1$, the subcritical regime when $\mathrm{Ri} > 1$, and a so-called critical regime when $\mathrm{Ri} = 1$.

A particular feature of this model (as shown in Sec. III B), which can also be found in some hydraulic problems (Wilkinson and Wood [23]), is the presence of a mathematical singularity which raises an additional problem when $\mathrm{Ri} = 1$.

In their attempt to solve the ET59 equations, Guo, Li, Ingason, Yan, and Zhu [24] circumvented this problem by freezing the Richardson number at unity once the critical condition was reached. This requires an artificial modification of the velocity which unfortunately fails to conserve fluxes. Recently, similarly to what was done by Dhar, Das, and Das [25] for the flow of a thin water film, Haddad, Vaux, Varrall, and Vauquelin [26] proposed a method to face the singularity problem by introducing a discontinuity, similar to a hydraulic jump, to match the supercritical and subcritical regions. In their paper, they solved the equations numerically with an iterative procedure to determine the location and amplitude of the jump for given values of the domain length and initial release conditions. The authors also compared the results provided by solving the equations of ET59 extended in the non-Boussinesq configuration with the results obtained from large eddy simulations (LES). They found good agreement between the simulations and theory, particularly for fully supercritical flows.

Nevertheless, and this is the purpose of the present paper, it is possible to go further and propose explicit solutions to the ET59 equations via purely analytical means. The methodology used to obtain these analytical solutions is in the same vein as that presented in Michaux and Vauquelin [27] for the Morton *et al.* [22] plume equations.

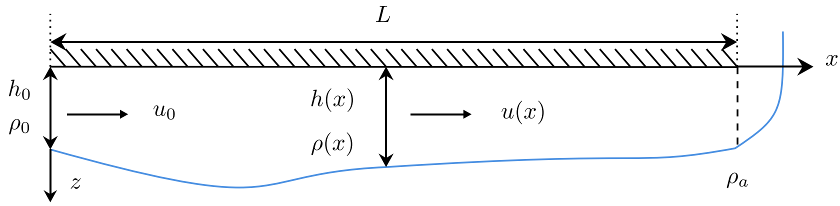

FIG. 1. Schematic of the studied configuration.

The paper is set out as follows: the studied configuration and the governing equations are presented in Sec. II. Analytical solutions to these equations are proposed in Sec. III for a specific entrainment law ($E \propto \mathrm{Ri}^{-1}$) in the case of a supercritical flow, but also for a flow with a regime change (supercritical to subcritical). The method is finally extended to other classical entrainment laws in Sec. IV in order to apply the modeling to a wider range of real-world cases. Conclusions are drawn in Sec. V.

II. CONFIGURATION AND GOVERNING EQUATIONS

As depicted in Fig. 1, we consider a fluid of density ρ_0 (lower than the density ρ_a of the ambient, at rest), injected horizontally from a plane nozzle of height h_0 with a velocity u_0 along a boundary of length L coincident with the horizontal x axis. The flow at the injection is therefore characterised by its Richardson number $\mathrm{Ri}_0 = \eta_0\, g\, h_0/u_0^2$, which is assumed to be less than unity (the flow is then initially supercritical). We consider that the current has reached the end of the domain for a long period and therefore is in a steady state. For the sake of simplicity, the velocity and density profiles along the vertical z axis will be considered uniform (top-hat assumption). Also, shape factors, used by ET59 to take into account the deviation between the real profiles and the top-hat profiles of the velocity and the density, are set to unity, even if they could be adjusted to nonunity values without preventing the algebraic development presented in this article. So, at a given distance x from the injection, $u(x)$, $\eta(x)$, and $h(x)$ stand for the top-hat velocity, the top-hat density deficit and the height of the current, respectively. As in ET59, the conservation equations for mass, momentum, and buoyancy are established over an infinitesimal element of length dx of the current. In the Boussinesq approximation, these equations read

$$\frac{\mathrm{d}\,(u\,h)}{\mathrm{d}x} = E\,u, \tag{2}$$

$$\frac{\mathrm{d}\,(u^2\,h)}{\mathrm{d}x} = -C_d\,u^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\eta\,g\,h^2), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\eta\,g\,u\,h) = 0, \tag{4}$$

with E being the entrainment coefficient and C_d the drag coefficient. The terms on the right-hand side of Eq. (3) represent respectively the turbulent basal drag and the pressure force associated with the change of height and density of the current.

In contrast to the entrainment coefficient for a turbulent plume, which can be considered as a constant, the entrainment coefficient for a gravity current depends on the stability of the flow and therefore on the Richardson number. This is why this coefficient has been the subject of numerous studies. Papers by ET59 and Lofquist [28] were some of the first to address this issue experimentally with salt water experiments. They highlighted the marked dependence of the entrainment coefficient on the Richardson number and proposed laws in which E is a function of Ri only. Other significant contributions include the experimental works of Parker, Fukushima, and Pantin [29], Dallimore, Imberger, and Ishikawa [30], and Wells, Cenedese, and Caulfield [31]. In addition, the reader is referred to the extensive review of Fernando [32] and Chowdhury and Testik [33] and also to the article of Christodoulou [34] in which experimental results from the literature are compiled to provide power laws for several ranges of Richardson number under the form

$$E = \frac{\alpha}{\mathrm{Ri}^n}, \tag{5}$$

where α and n are constants. This modeling can only be used for steady configurations in which the entrainment process concerns a quasihorizontal surface. In the unsteady regime, and as explained and illustrated by Nogueira, Adduce, Alves, and Franca [35] and Sher and Woods [15], the flow morphology is more complex and requires the use of specific entrainment models for the head, the body, and even the tail.

The drag coefficient has also been investigated in the literature, and it is not uncommon to choose it as being constant (Hogg and Woods [36], Baines [37]). We have also made this assumption in the following sections of this paper.

In the next section, when presenting the approach to analytically solving the ET59 equations, we will use, in addition to a constant drag coefficient, the submodel $E = \alpha/\mathrm{Ri}$ proposed by Christodoulou [34] for the range $0.1 < \mathrm{Ri} < 10$. The advantage of selecting this submodel is that it theoretically encompasses a broad range of the super- and sub-critical flows with a unique entrainment model.

III. ANALYTICAL SOLUTIONS

By combining conservation Eqs. (2), (3), (4) and the entrainment submodel (5) with $n = 1$, we obtain the first-order derivatives of the height, velocity, and density deficit as

$$\frac{\mathrm{d}\,h}{\mathrm{d}x} = \frac{\alpha}{2}\,\frac{4 + \mathrm{Ri}(\kappa - 2)}{\mathrm{Ri}\,(1 - \mathrm{Ri})}, \tag{6}$$

$$\frac{\mathrm{d}\,u}{\mathrm{d}x} = -\frac{\alpha}{2}\,\frac{u}{h}\,\frac{2 + \kappa\,\mathrm{Ri}}{\mathrm{Ri}\,(1 - \mathrm{Ri})}, \tag{7}$$

$$\frac{\mathrm{d}\,\eta}{\mathrm{d}x} = -\alpha\,\frac{\eta}{h}\,\frac{1}{\mathrm{Ri}}, \tag{8}$$

where κ is a constant equal to $1 + 2\,C_d/\alpha$.

Additionally, the first-order derivative of the local Richardson number $\mathrm{Ri}(x)$ can be obtained by combining Eqs. (1), (6), (7), and (8). It reads

$$\frac{\mathrm{d}\,\mathrm{Ri}}{\mathrm{d}x} = \frac{3\,\alpha}{2}\,\frac{1}{h}\,\frac{2 + \kappa\,\mathrm{Ri}}{(1 - \mathrm{Ri})}. \tag{9}$$

We notice that Eqs. (6), (7), and (9) present a mathematical singularity when the Richardson number reaches unity. For an initially supercritical flow ($\mathrm{Ri}_0 < 1$), a quick look at the right-hand side of Eq. (9) reveals that Ri has to increase monotonically. With this in mind, two situations have to be considered. In the first one, the current remains supercritical from the injection until the exit of the domain, i.e. the Richardson number never exceeds unity. The set of Eqs. (6), (7), (8), and (9) can be solved without difficulty. In the second one, the Richardson number reaches unity before the end of the domain, meaning that the flow has to transition from a supercritical to a subcritical regime, preventing solutions to be obtained by a straightforward integration. This issue was recently addressed by Haddad *et al.* [26] and discussed by Ungarish [38].

In both cases (the flow remains supercritical or becomes subcritical before the end of the domain), analytical solutions can be obtained. This is the purpose of the following sections.

A. Initially supercritical flow remaining supercritical

At first, by combining the Eqs. (7) and (9), we easily show that the velocity $u(x)$ can be expressed as a function of the Richardson number $\mathrm{Ri}(x)$:

$$-\frac{1}{3}\frac{\mathrm{d\,Ri}}{\mathrm{Ri}} = \frac{\mathrm{d}\,u}{u} \quad \Longrightarrow \quad \frac{u(x)}{u_0} = \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}}, \tag{10}$$

where the subscript 0 refers to values at the injection. We now combine the Eqs. (8) and (9) to write

$$\frac{1-\mathrm{Ri}}{\mathrm{Ri}\,(2+\kappa\,\mathrm{Ri})}\,\mathrm{d\,Ri} = -\frac{3}{2}\,\frac{\mathrm{d}\,\eta}{\eta}, \tag{11}$$

which allows the density deficit $\eta(x)$ to be expressed explicitly as a function of the Richardson number $\mathrm{Ri}(x)$:

$$\frac{\eta(x)}{\eta_0} = \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}}\left[\frac{2+\kappa\,\mathrm{Ri}(x)}{2+\kappa\,\mathrm{Ri}_0}\right]^{\frac{2+\kappa}{3\kappa}}. \tag{12}$$

Finally, given the buoyancy conservation (4) and Eqs. (10) and (12), we immediately obtain for the height $h(x)$ of the current:

$$\frac{h(x)}{h_0} = \left[\frac{\mathrm{Ri}(x)}{\mathrm{Ri}_0}\right]^{\frac{2}{3}}\left[\frac{2+\kappa\,\mathrm{Ri}_0}{2+\kappa\,\mathrm{Ri}(x)}\right]^{\frac{2+\kappa}{3\kappa}}. \tag{13}$$

The three relations (10), (12), and (13) allow the calculation of the longitudinal evolution of the characteristic variables of the current (velocity, density deficit, and height) exclusively from the knowledge of the longitudinal evolution of the Richardson number. This latter can be determined by solving the Eq. (14) below, obtained from Eqs. (9) and (13):

$$\frac{\mathrm{d\,Ri}}{\mathrm{d}\,x} = \frac{1}{\Lambda_0}\,\frac{(2+\kappa\,\mathrm{Ri})^{\frac{2+4\kappa}{3\kappa}}}{(1-\mathrm{Ri})\mathrm{Ri}^{\frac{2}{3}}}, \tag{14}$$

where the constant Λ_0, which has the dimension of a length, reads

$$\Lambda_0 = \frac{2\,h_0(2+\kappa\,\mathrm{Ri}_0)^{\frac{2+\kappa}{3\kappa}}}{3\,\alpha\,\mathrm{Ri}_0^{\frac{2}{3}}}. \tag{15}$$

By integrating Eq. (14) from the injection to an abscissa x, it becomes

$$\int_{\mathrm{Ri}_0}^{\mathrm{Ri}(x)} \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\kappa}{3\kappa}}}\,\mathrm{d}\,\zeta = \frac{1}{\Lambda_0}\int_0^x \mathrm{d}\,\xi. \tag{16}$$

We then introduce a universal function $F(X)$ defined by

$$F(X) = \int_0^X \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\kappa}{3\kappa}}}\,\mathrm{d}\,\zeta, \tag{17}$$

which allows the relation (16) to be rewritten as:

$$F[\mathrm{Ri}(x)] = \frac{x}{\Lambda_0} + F(\mathrm{Ri}_0). \tag{18}$$

In practice, once the values of α and C_d are set, the different steps of the calculation are as follows:

(1) From the injection conditions u_0, η_0, and h_0, Ri_0 and Λ_0 are first calculated.

(2) The value of $F(\mathrm{Ri}_0)$ is then determined using Eq. (17).

(3) For a given abscissa x, the quantity x/Λ_0 is added to this value, in order to obtain $F[\mathrm{Ri}(x)]$, according to Eq. (18).

(4) $\mathrm{Ri}(x)$ is then obtained from $F[\mathrm{Ri}(x)]$ via the inverse of the universal function F.

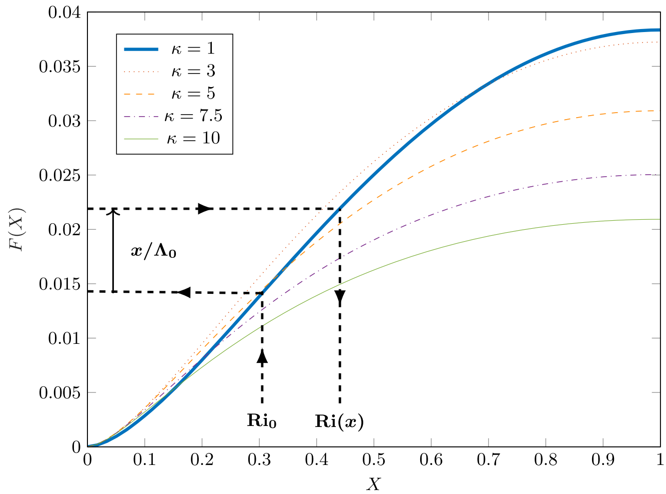

FIG. 2. Illustration of the graphical determination of the longitudinal evolution of the Richardson number via the universal function F for several values of κ. The dashed lines are provided to illustrate the graphical procedure for $\kappa = 1$.

(5) Finally, Eqs. (10), (12), (13) allow the characteristic variables of the current, $u(x)$, $\eta(x)$, and $h(x)$, to be calculated.

Although the function F can be calculated numerically without difficulty, we propose in Fig. 2 a graphical illustration of the method for several values of the constant κ.

Note that, from relation (18), it is possible to determine the critical length L_c at which the flow theoretically reaches the critical condition (i.e., Ri $= 1$). We therefore obtain

$$L_c = \Lambda_0\,[F(1) - F(\mathrm{Ri}_0)]. \tag{19}$$

This relation is particularly interesting since it allows, from the knowledge of the injection conditions and the length of the domain, to know immediately whether or not the gravity current is likely to transition from a supercritical to a subcritical regime. This issue was discussed by Kostic and Parker [39] in the particular case of a turbidity current developing along a boundary of finite length.

B. Initially supercritical flow becoming subcritical

We now consider the case where the length L of the domain is greater than the critical length L_c, i.e., the case when the flow transitions from a supercritical to a subcritical regime.

To take into account this transition in the equations of ET59, Haddad *et al.* [26] introduced a mathematical discontinuity similar to a hydraulic jump at a location L_1, which leads the Richardson number and the height of the current to suddenly increase to match the subcritical regime.

Assuming that the density deficit in the current does not change on either side of the discontinuity, the governing equations (mass and momentum) of the jump are

$$u_1\,h_1 = u_2\,h_2, \tag{20}$$

$$u_1^2\,h_1 + \frac{\eta_1\,g\,h_1^2}{2} = u_2^2\,h_2 + \frac{\eta_2\,g\,h_2^2}{2}, \tag{21}$$

in which the subscripts 1 and 2 are used for the quantities just upstream and just downstream of the jump, respectively. By combining these two equations, we obtain the jump relation, also known as

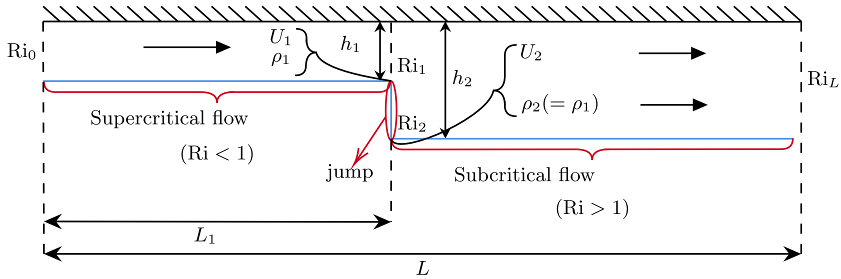

FIG. 3. Representation of a gravity current including an idealized transition between the supercritical and the subcritical flows.

Bélanger equation (or Rankine-Hugoniot condition):

$$\frac{h_2}{h_1} = \left(\frac{\mathrm{Ri}_2}{\mathrm{Ri}_1}\right)^{\frac{1}{3}} = \sigma(\mathrm{Ri}_1), \tag{22}$$

with

$$\sigma(X) = \frac{\sqrt{1+\frac{8}{X}}-1}{2}. \tag{23}$$

The theoretical problem of ET59 can be addressed by considering the connection between a supercritical flow (from Ri_0 to Ri_1 over a length L_1) and a subcritical flow (over a length $L - L_1$, from Ri_2 to the Richardson number at the exit of the domain, denoted Ri_L), as illustrated in Fig. 3.

Thus, based on the mathematical developments presented above, the location of the jump L_1 as well as the corresponding Richardson number Ri_1 just upstream can be found by solving the following system of equations:

$$F(\mathrm{Ri}_1) = \frac{L_1}{\Lambda_0} + F(\mathrm{Ri}_0), \tag{24}$$

$$F(\mathrm{Ri}_L) = \frac{L-L_1}{\Lambda_2} + F(\mathrm{Ri}_2), \tag{25}$$

where Λ_2 is given by

$$\Lambda_2 = \frac{2\, h_2\, (2+\kappa\,\mathrm{Ri}_2)^{\frac{2+\kappa}{3\kappa}}}{3\,\alpha\,\mathrm{Ri}_2^{\frac{2}{3}}}. \tag{26}$$

At this stage, the system composed of Eqs. (24) and (25) contains five unknowns, namely, Ri_1, L_1, Ri_2, Ri_L, and h_2. First, concerning the Richardson number Ri_L at the exit, as explained by Henderson [40], it should be close to unity for a subcritical current in an open channel. We will therefore consider hereafter that $\mathrm{Ri}_L = 1$. In addition, the use of the Bélanger Eq. (22) allows, on the one hand, to express Ri_2 as a function of Ri_1, and on the other hand, by using it in the relation (13), to obtain explicitly the value of the current height h_2 after the jump:

$$h_2 = h_0 \left(\frac{\mathrm{Ri}_1}{\mathrm{Ri}_0}\right)^{\frac{2}{3}} \left(\frac{2+\kappa\,\mathrm{Ri}_0}{2+\kappa\,\mathrm{Ri}_1}\right)^{\frac{2+\kappa}{3\kappa}} \sigma(\mathrm{Ri}_1). \tag{27}$$

After some algebra, we can show that Ri_1 is given by the following relation:

$$G(\mathrm{Ri}_1) = \frac{L}{\Lambda_0} + F(\mathrm{Ri}_0), \tag{28}$$

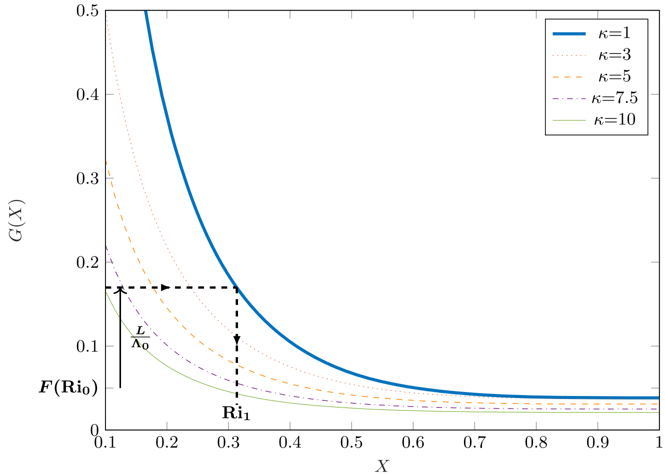

FIG. 4. Illustration of the graphical determination of the Richardson number Ri_1 just upstream of the jump via the universal function G for several values of κ. The dashed lines are provided to illustrate the graphical procedure for $\kappa = 1$.

with G being a universal function defined by

$$G(X) = \left(\frac{2 + \kappa\, X\, \sigma^3(X)}{2 + \kappa\, X}\right)^{\frac{2+\kappa}{3\kappa}} \frac{F(1) - F[X\, \sigma^3(X)]}{\sigma(X)} + F(X). \tag{29}$$

Practically, for a fixed value of κ, we first calculate Ri_0, Λ_0, and $F(\mathrm{Ri}_0)$. Then, we add to $F(\mathrm{Ri}_0)$ the quantity L/Λ_0 which allows Ri_1 to be calculated with the Eq. (28) and the inverse of the function G. A graphical illustration of the method is proposed in Fig. 4.

Once Ri_1 is known, Ri_2 is calculated with the Bélanger Eq. (22), L_1 by Eq. (24), h_2 by Eq. (27), and Λ_2 by Eq. (26). Finally, according to Eq. (16), the longitudinal evolution of the Richardson number in the subcritical region can be obtained graphically (see Fig. 5), or numerically from the following equation:

$$F[\mathrm{Ri}(x)] = \frac{x - L_1}{\Lambda_2} + F(\mathrm{Ri}_2). \tag{30}$$

Once $\mathrm{Ri}(x)$ is known over the whole domain, the characteristic variables $u(x)$, $h(x)$, and $\eta(x)$ of the current are calculated immediately from (10), (12), and (13) in the supercritical region, and in the subcritical region with the following relations:

$$\frac{u(x)}{u_2} = \left[\frac{\mathrm{Ri}_2}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}}, \quad \frac{\eta(x)}{\eta_2} = \left[\frac{\mathrm{Ri}_2}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}} \left[\frac{2 + \kappa\, \mathrm{Ri}(x)}{2 + \kappa\, \mathrm{Ri}_2}\right]^{\frac{2+\kappa}{3\kappa}},$$
$$\text{and} \quad \frac{h(x)}{h_2} = \left[\frac{\mathrm{Ri}(x)}{\mathrm{Ri}_2}\right]^{\frac{2}{3}} \left[\frac{2 + \kappa \mathrm{Ri}_2}{2 + \kappa \mathrm{Ri}(x)}\right]^{\frac{2+\kappa}{3\kappa}}. \tag{31}$$

IV. ANALYTICAL SOLUTIONS FOR OTHER ENTRAINMENT LAWS

The methodology for obtaining analytical solutions presented in the previous section considered the specific entrainment model $E = \alpha/\mathrm{Ri}$. In this section we propose to extend the approach for other entrainment laws from the literature. These laws are often associated with particular flow regimes, as will be specified.

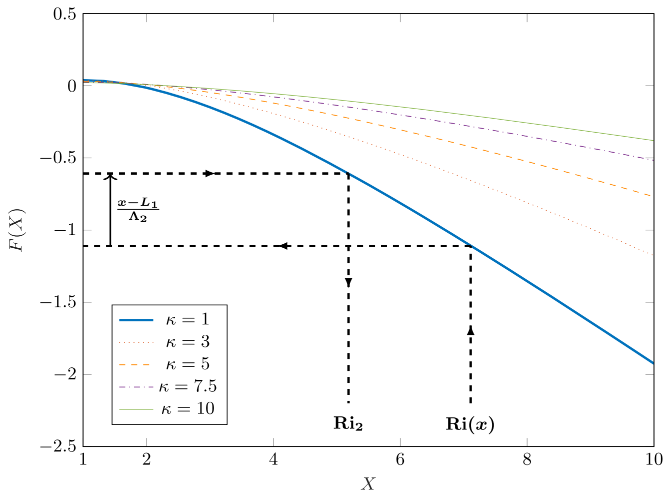

FIG. 5. Illustration of the graphical determination of the longitudinal evolution of the Richardson number downstream of the jump (subcritical regime) via the universal function F for several values of κ. The dashed lines are provided to illustrate the graphical procedure for $\kappa = 1$.

A. No entrainment

A null entrainment coefficient can be associated with a strongly buoyant gravity current for which buoyancy forces inhibit the mixing such as long-distance submarine currents with strong stratification. The no-entrainment configuration can also correspond to hydraulic or immiscible flows.

In this case, the density deficit turns out to be a constant of the problem and the two primary variables $u(x)$ and $h(x)$ are immediately expressed as a function of the local Richardson number:

$$\frac{u(x)}{u_0} = \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}} \quad \text{and} \quad \frac{h(x)}{h_0} = \left[\frac{\mathrm{Ri}(x)}{\mathrm{Ri}_0}\right]^{\frac{1}{3}}. \tag{32}$$

Substituting $h(x)$ into Eq. (9) by its expression as a function of the Richardson number, the integration is straightforward and we obtain the longitudinal evolution of the Richardson number from the following implicit equation:

$$[4 - \mathrm{Ri}(x)]\mathrm{Ri}(x)^{1/3} = \frac{4}{3\,\Lambda_0}\,x + [4 - \mathrm{Ri}_0]\mathrm{Ri}_0^{1/3}, \tag{33}$$

with

$$\Lambda_0 = \frac{h_0}{3\,C_d\,\mathrm{Ri}_0^{1/3}}. \tag{34}$$

B. Constant entrainment coefficient

A constant entrainment coefficient (as in plumes and jets) can be associated with a strongly inertial gravity current for which the stable density gradient of the flow weakly affects the mixing, as in the case of a ceiling jet from an impinging plume. However, this model is no longer appropriate when the Richardson number increases and the effects of buoyancy appear (when Ri reaches 0.01 according to Christodoulou [34]).

From a mathematical point of view, returning to Eqs. (2), (3), and (4) with $E = \alpha$, a pure constant, the first derivatives of the primary variables and the local Richardson number read

$$\frac{\mathrm{d}\,h}{\mathrm{d}x} = \frac{\alpha}{2}\,\frac{3+\kappa-\mathrm{Ri}}{(1-\mathrm{Ri})}, \tag{35}$$

$$\frac{\mathrm{d}\,u}{\mathrm{d}x} = -\frac{\alpha}{2}\,\frac{u}{h}\,\frac{1+\kappa+\mathrm{Ri}}{(1-\mathrm{Ri})}, \tag{36}$$

$$\frac{\mathrm{d}\,\eta}{\mathrm{d}x} = -\alpha\,\frac{\eta}{h}, \tag{37}$$

$$\frac{\mathrm{d}\,\mathrm{Ri}}{\mathrm{d}x} = \frac{3\,\alpha}{2}\,\frac{1}{h}\,\frac{\mathrm{Ri}(1+\kappa+\mathrm{Ri})}{(1-\mathrm{Ri})}, \tag{38}$$

with $\kappa = 1 + 2\,C_d/\alpha$.

In the same way as in Sec. III, we first express the primary variables as a function of the Richardson number:

$$\begin{aligned}
\frac{u(x)}{u_0} &= \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}},\\
\frac{\eta(x)}{\eta_0} &= \left[\frac{\mathrm{Ri}(x)+1+\kappa}{\mathrm{Ri}_0+1+\kappa}\right]^{\frac{2(2+\kappa)}{3(1+\kappa)}}\left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{2}{3(1+\kappa)}},\\
\frac{h(x)}{h_0} &= \left[\frac{\mathrm{Ri}_0+1+\kappa}{\mathrm{Ri}(x)+1+\kappa}\right]^{\frac{2(2+\kappa)}{3(1+\kappa)}}\left[\frac{\mathrm{Ri}(x)}{\mathrm{Ri}_0}\right]^{\frac{3+\kappa}{3(1+\kappa)}},
\end{aligned} \tag{39}$$

and by substituting $h(x)$ in Eq. (38), we introduce the universal function F:

$$F(X) = \int_0^X \frac{(1-\zeta)\zeta^{\frac{-2\kappa}{3(1+\kappa)}}}{(\zeta+1+\kappa)^{\frac{7+5\kappa}{3(1+\kappa)}}}\,\mathrm{d}\,\zeta, \tag{40}$$

which allows the longitudinal evolution of the Richardson number to be determined from the following relation:

$$F[\mathrm{Ri}(x)] = \frac{x}{\Lambda_0} + F(\mathrm{Ri}_0), \tag{41}$$

where Λ_0 is a constant equal to

$$\Lambda_0 = \frac{2\,h_0\,(\mathrm{Ri}_0+1+\kappa)^{\frac{2(2+\kappa)}{3(1+\kappa)}}}{3\,\alpha\,\mathrm{Ri}_0^{\frac{(3+\kappa)}{3(1+\kappa)}}}. \tag{42}$$

C. Entrainment law in the Parker style

In their paper on turbidity currents, Parker *et al.* [29] proposed to express the entrainment coefficient as

$$E = \frac{\alpha}{\beta + \mathrm{Ri}}, \tag{43}$$

with α and β being two constants. This model is justified by the fact that it allows the entrainment coefficient to tend towards a universal asymptotic value for low Richardson numbers.

Mathematically, with this entrainment law, after a little algebra, the conservation equations allow the primary variables $u(x)$, $\eta(x)$ and $h(x)$ to be written as a function of the local Richardson number

Ri(x) as

$$\frac{u(x)}{u_0} = \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}},$$
$$\frac{\eta(x)}{\eta_0} = \left[\frac{\sigma + \kappa\,\mathrm{Ri}(x)}{\sigma + \kappa\,\mathrm{Ri}_0}\right]^{\frac{2(\kappa+\sigma)}{3\kappa\sigma}} \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{2}{3\sigma}},$$
$$\frac{h(x)}{h_0} = \left[\frac{\sigma + \kappa\,\mathrm{Ri}_0}{\sigma + \kappa\,\mathrm{Ri}(x)}\right]^{\frac{2(\sigma+\kappa)}{3\kappa\sigma}} \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{2+\sigma}{3\sigma}}, \tag{44}$$

with $\kappa = 1 + 2\,C_d/\alpha$ and $\sigma = 2\,(1 - \beta + \beta\kappa)$.

We then introduce the universal function F:

$$F(X) = \int_0^X \frac{(1-\zeta)\zeta^{\frac{2-2\sigma}{3\sigma}}(\beta+\zeta)}{(\sigma+\kappa\,\zeta)^{\frac{\kappa(2+3\sigma)+2\sigma}{3\sigma\kappa}}}\,\mathrm{d}\,\zeta, \tag{45}$$

which allows us to calculate the evolution of the Richardson number with the relation

$$F[\mathrm{Ri}(x)] = \frac{x}{\Lambda_0} + F(\mathrm{Ri}_0), \tag{46}$$

where Λ_0 is a constant defined as

$$\Lambda_0 = \frac{2\,h_0\,(\sigma + \kappa\,\mathrm{Ri}_0)^{\frac{2(\kappa+\sigma)}{3\sigma\kappa}}}{3\,\alpha\,\mathrm{Ri}_0^{\frac{2+\sigma}{3\sigma}}}. \tag{47}$$

For the sake of brevity, we do not discuss here the implementation of the results introduced in the Sec. IV when a jump appears. However, it is entirely possible to combine these results with the methodology presented in Sec. III B to adapt the entrainment law before or after the jump. A new function G is then obtained in order to determine the magnitude and the location of the jump.

V. CONCLUSIONS AND DISCUSSIONS

This paper reports an analytical method to solve the equations of Ellison and Turner [21] describing the longitudinal evolution of a steady Boussinesq miscible gravity current. The buoyant fluid that forms the current is continuously injected from a plane nozzle along a horizontal rigid boundary of finite length.

First, expressions of the primary variables of the current (velocity u, height h, and density deficit η) as explicit functions of the Richardson number Ri(x) are established. Then, the longitudinal evolution of the Richardson number is obtained from a universal function F (which depends only on the entrainment and drag coefficients). This universal function is given by an indefinite integral which can be easily calculated or, alternatively, tabulated. Here, we have illustrated the method by means of a graphical representation which, for prescribed injection conditions, allows the Richardson number to be determined readily at any abscissa x along the horizontal boundary.

The ET59 equations present a mathematical singularity when $\mathrm{Ri} = 1$, so the developed method applies therefore easily as long as the flow remains supercritical ($\mathrm{Ri} < 1$) on the whole domain. Note that, from the knowledge of the injection conditions and the length of the domain, it is possible to determine theoretically thanks to the universal function F whether the flow remains supercritical or should transition to a subcritical state ($\mathrm{Ri} > 1$).

If there is a transition (i.e. if the Richardson number reaches unity before the exit of the domain), the two regimes (supercritical and subcritical) can be matched by introducing in the ET59 equations a mathematical discontinuity similar to a hydraulic jump. For the sake of simplicity, we have modeled this jump using the Bélanger relation, as that was previously done by Haddad *et al.* [26]. In practice, the location of the jump is obtained from the injection conditions and a second universal function G. The evolution of the primary variables is then obtained for each zone (upstream and downstream of the jump) from the first function F.

The approach presented in this article has led to analytical solutions to the ET59 equations for different entrainment laws: $E = 0$, $E \propto \mathrm{Ri}^0$, $E \propto \mathrm{Ri}^{-1}$, and $E \propto (\beta + \mathrm{Ri})^{-1}$. It can also be applied to other laws proposed in the literature, such as those of ET59, Hebbert, Patterson, Loh, and Imberger [41], Dallimore *et al.* [30], Wells *et al.* [31], Johnson and Hogg [19], and van Reeuwijk *et al.* [42], for example.

Finally, in this article, we studied theoretically a gravity current in a steady state, which is the consequence of a transient development. In some configurations, we had to introduce a mathematical discontinuity in the theoretical model (similar to a hydraulic jump). This raises the question of the existence (or not) and development of this discontinuity in the transient phase. Ungarish [38] has recently addressed this issue using the shallow water equations. He has provided interesting results, but no definitive answers, making this a still-open question.

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