



Linear stability of turbulent channel flow with one-point closure

P. V. Kashyap, Y. Duguet , and O. Dauchot 

*Department of Mechanical Engineering, Indian Institute of Science, Bengaluru, India;
LISN-CNRS, Campus Universitaire d'Orsay, Université Paris Saclay, F-91400 Orsay, France;
and Gulliver UMR CNRS 7083, ESPCI Paris, PSL Research University,
10 rue Vauquelin, 75005 Paris, France*



(Received 10 December 2023; accepted 9 May 2024; published 17 June 2024)

For low enough flow rates, turbulent channel flow displays spatial modulations of large wavelengths. This phenomenon has recently been interpreted as a linear instability of the turbulent flow. We question here the ability of linear stability analysis around the turbulent mean flow to predict the onset and wavelengths of such modulations. Both the mean flow and the Reynolds stresses are extracted from direct numerical simulation (DNS) in periodic computational domains of different size. The Orr-Sommerfeld-Squire formalism is used here, with the turbulent viscosity either ignored, evaluated from DNS, or modeled using a simple one-point closure model. Independently of the closure model and the domain size, the mean turbulent flow is found to be linearly stable, in marked contrast with the observed behavior. This suggests that the one-point approach is not sufficient to predict instability, at odds with other turbulent flow cases. For generic wall-bounded shear flows we discuss how the correct models for predicting instability could include fluctuations in a more explicit way.

DOI: [10.1103/PhysRevFluids.9.063906](https://doi.org/10.1103/PhysRevFluids.9.063906)

I. INTRODUCTION

It has been known since the 1960s that many wall-bounded shear flows display a complex organization akin to laminar-turbulent patterning [1] at intermediate Reynolds numbers, Re . This phenomenon was later reproduced, first experimentally in flows of much larger aspect ratio [2–4], then in several numerical studies [5–8]. A large number of recent studies, both experimental and numerical, have described how banded patterns or solitary stripes emerge from the laminar flow after the triggering by localized finite-amplitude disturbances [9–14]. The leading scenario is largely based on the development of a large-scale flow at the interface between laminar and nonlaminar zones [9]. The route from the turbulent flow to the banded patterns, on the contrary, happens spontaneously and does not require any finite-amplitude perturbation. The emergence of the laminar-turbulent patterns starting from the fully turbulent regime is illustrated in Fig. 1 as Re is decreased. The authors of Ref. [3], suspecting an instability of the spatially homogeneous turbulent flow, modeled the emergence of the resulting patterns using a noisy supercritical Ginzburg-Landau formalism. Later numerical evidence [15–17] demonstrated that the precursors of patterning were to be found in the form of low-amplitude modulations of the homogeneous turbulent flow, which is known from simulations at higher Re . Finally, an explicit dispersion relation for the underlying instability was reconstructed from costly numerical data, using a response function procedure, and this spatial modulation could be interpreted as a linear instability of the turbulent flow *as a whole*, i.e., including all its fluctuations [18].

It is natural to wonder whether the onset of modulation can be predicted from simpler concepts, such as linear stability, and at a much lower cost. However, linearization about an unsteady turbulent flow is fraught with mathematical difficulties if all temporal and spatial fluctuations are taken into account. It would thus be desirable to average out the fluctuations and perform the linear stability

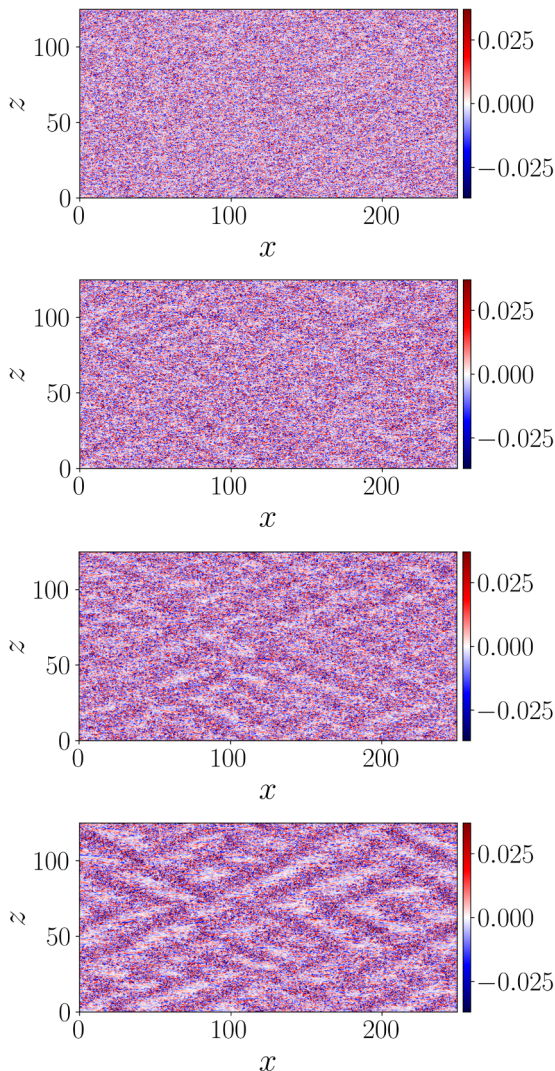


FIG. 1. Onset of laminar-turbulent patterning in turbulent channel flow. Midplane ($y = 0$) wall-normal velocity $v(x, z, t)$. From top to bottom: $\text{Re}_\tau = 100, 90, 85, 80$ in the stationary regime.

analysis of the mean flow. However, because of the nonlinearity of the original equations, that for the mean flow includes correlations of the fluctuations and calls for *turbulent closures*.

Following this line of thought, it was conjectured that turbulent mean flows are marginally stable [19], an idea furthermore suggested as a governing principle used to predict mean flows. Focusing on the canonical case of channel flow between two infinite plates, driven by a pressure gradient, this hypothesis was later dismissed in favor of strict stability [20,21], in the sense that all eigenvalues of the linearized problem are strictly negative. However, the range of values of Re considered in these studies was always too high to be relevant to the pattern formation problem of interest here. The present paper is devoted to a similar stability analysis of turbulent channel flow restricted to the range of Re values where laminar-turbulent patterns are known to occur.

There has been a long debate as to whether the linear stability or instability of the mean turbulent flow could be considered as a reliable prediction [21,22]. It fails in some cases when the spectral

distribution of energy of the mean flow is not steep enough [23], but it proved accurate in other situations, such as the two-dimensional cylinder wake [24], in rotating channel flow [25], thermosolutal convection [26], and more recently in turbulent convection [27] and magnetohydrodynamic turbulent flows [28]. Furthermore, finite-time stability [29] of the turbulent mean flow indicates strong transient growth, at spanwise scales consistent with the appearance of both near-wall streaks and large-scale streaks [30,31]. This encouraging result suggests that linearization about a mean flow can contain decisive information about the fluctuation fields and be a predicting tool.

The wall turbulence community has possibly ignored the possibility that such instabilities of the turbulent regime could occur at low rather than at high Re . Nevertheless, linearizing around a well-defined mean flow, as formally done at high Reynolds number, remains *a priori* a relevant approach to capture the emergence of the long wavelength modulation observed in shear flows when *decreasing* the Reynolds number. As a further motivation to consider the instability formalism, a recent low-order model for the mean flow of a companion shear flow (the so-called Waleffe flow) has been recently suggested [32], and the fixed point associated with the mean turbulent flow displays indeed a linear instability precisely at the onset of the turbulent bands.

In this work we revisit the approach using the Orr-Sommerfeld-Squire formalism applied to this problem by focusing on the relevant parameter range. Both the mean flow and the turbulent viscosity are extracted from direct numerical simulation in periodic computational domains of varying size. As an alternative the turbulent viscosity can also be modeled using a simple one-point closure model. The mean turbulent flow is found to be linearly stable, regardless of the choice of one-point model and domain size, in marked contrast with the observed behavior. We conclude that one-point closure models associated with one-dimensional base flows do not capture the essence of the instability mechanism and discuss how the proper fluctuations could be included in a more explicit way.

II. CLOSURE FORMALISM

A. Instability formalism for mean flows

We start from the original momentum equations in vector form,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nabla \cdot (\nu \nabla \mathbf{u}), \quad (1)$$

which together with the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ forms the Navier-Stokes equations. In Eq. (1), $\mathbf{u} = (u_x, u_y, u_z)$ is the (dimensional) three-dimensional velocity field, p the pressure field, ν the kinematic viscosity, and ρ the fluid density. x denotes the streamwise coordinate, y the wall-normal coordinate where the walls are at $y = \pm h$, and z is the spanwise coordinate. A dimensionless Reynolds number $Re_\tau = u_\tau h / \nu$ can be built based on the friction velocity u_τ , related to the imposed mean wall-shear stress τ by the relation $\rho u_\tau^2 = \tau$.

Any averaging $(\overline{\cdot})$ satisfying Reynolds conditions [33] is admissible here, such that *a priori* the average $\mathbf{U} = \overline{\mathbf{u}}$ of a velocity field $\mathbf{u}(x, y, z, t)$ can also depend on x, y, z , and t . By applying the chosen averaging to all terms of the original equation (1) one obtains

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \nabla \cdot (\nu \nabla \mathbf{U}) - \nabla \cdot (\overline{\mathbf{u}' \otimes \mathbf{u}'}), \quad (2)$$

where $\mathbf{u}' = \mathbf{u} - \mathbf{U}$, and the term $\overline{\mathbf{u}' \otimes \mathbf{u}'}$ forms the Reynolds stress tensor. This mean flow equation for \mathbf{U} is not closed. The standard strategy amounts to deriving an evolution equation for the Reynolds stress, which in turn will depend on higher order correlations. To make progress, the resulting hierarchy of equations must then be truncated with a closure model.

When the flow geometry is simple enough—little or no anisotropy, zero or small curvature, etc.—as is the case here, a reasonable assumption, the Boussinesq hypothesis, consists in assuming that the Reynolds stress anisotropy tensor is aligned with the strain tensor of the mean flow. The proportionality coefficient ν_t is called the turbulent eddy viscosity. An explicit expression for ν_t is called a one-point closure model, in contrast to closure models based on the correlations between

several spatial locations. Equation (2) can then formally be rewritten in the more compact form

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \nabla \cdot (v_T \nabla \mathbf{U}), \quad (3)$$

where $v_T = \nu + \nu_t$.

The mean flow equation has a particular steady solution \mathbf{U}^* corresponding to the stationary turbulent flow found at high Re (and different from the laminar Poiseuille solution). The channel flow geometry is furthermore invariant by translation in the streamwise and spanwise directions. In such a case, a homogeneous and steady turbulent flow is observed: all turbulent statistics of this flow, in particular the mean flow and the turbulent viscosity, become independent of t , x , and z . The turbulent viscosity can be evaluated as

$$\nu_t = -\frac{\overline{u'_x u'_y}}{\frac{d}{dy} U_x}. \quad (4)$$

Because Eq. (3) is also a nonlinear PDE, there is no reason why \mathbf{U}^* should be the only stationary solution. More specifically, the symmetries associated with the translation invariance in the streamwise and spanwise directions can be spontaneously broken following a classical instability scenario. In the simplest case, such a scenario would start by the linear destabilization of the solution \mathbf{U}^* , which we now investigate by applying the Orr-Sommerfeld-Squire formalism to the *mean* flow equation (3).

B. Orr-Sommerfeld-Squire system

Assuming a small perturbation $\tilde{\mathbf{U}}(x, y, z, t)$ added to $\mathbf{U}^*(y)$, it obeys the linearized equation

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + (\tilde{\mathbf{U}} \cdot \nabla) \mathbf{U}^* + (\mathbf{U}^* \cdot \nabla) \tilde{\mathbf{U}} = -\nabla \tilde{P} + \nabla \cdot (v_T \nabla \tilde{\mathbf{U}}). \quad (5)$$

The linear stability analysis follows closely the classical Orr-Sommerfeld theory (see, e.g., Ref. [34]), except that (i) the laminar base flow $\mathbf{U}_{\text{lam}}(y)$ is replaced by the mean base flow $\mathbf{U}^*(y)$ and (ii) the viscosity ν_T depends on y [35]. Equation (5) can also be obtained from the triple decomposition suggested in Ref. [21], where the total flow is decomposed into a mean flow, a coherent wave part and less organized turbulent fluctuations.

Because of the spatial and temporal translation symmetries of Eq. (3), we use the classical velocity ansatz $\tilde{\mathbf{U}} \sim \hat{\mathbf{U}}(y) e^{i(\alpha x + \beta z - \omega t)}$, where α and β are respectively the axial and spanwise wave number and $\omega = \omega_r + i\omega_i$ a complex number, with ω_r the angular frequency and ω_i the growth rate of the perturbations. Expanding all scalar fields on a basis of N Chebyshev polynomials ($N = 65$, throughout the study), the original system takes, for each choice of α and β , the form of a generalized matrix eigenvalue problem [34,36] $\mathbf{A} \mathbf{q} = -i\omega \mathbf{q}$, where

$$\mathbf{A} = \begin{pmatrix} \Delta^{-1} \mathcal{L}_{OS} & 0 \\ -i\beta U' & \mathcal{L}_{SQ} \end{pmatrix}, \quad (6)$$

with

$$\mathcal{L}_{OS} = -i\alpha(U\Delta - U'') + \nu_T \Delta^2 + 2\nu_T' \Delta \mathcal{D} + \nu_T''(\mathcal{D}^2 + k^2) + \nu_T \Delta + \nu_T' \mathcal{D} \quad (7)$$

and

$$\mathcal{L}_{SQ} = -i\alpha U + \nu_T \Delta + \nu_T' \mathcal{D}. \quad (8)$$

The vector \mathbf{q} contains the wall-normal velocity \hat{U}_y and the wall-normal vorticity $\hat{\omega}_y$, respectively, expressed at all (discrete) collocation points in y . The differential operators are $\mathcal{D} = d/dy$ (just like the prime symbol) and $\Delta := \mathcal{D}^2 - k^2$, with $k^2 = \alpha^2 + \beta^2$. The formulation is identical to that in Ref. [35].

C. Teachings from the Squire theorem

Since the formalism is mathematically similar to the Orr-Sommerfeld-Squire, a legitimate question is the applicability and the consequences of the Squire transformation. As a consequence of this transformation, if there exists a solution of the Orr-Sommerfeld eigenvalue problem its growth rate satisfies the scaling $\omega_i(\alpha, \beta, \text{Re}) = \alpha c_i(k^2, \alpha \text{Re})$, where c_i is the imaginary part of a complex phase velocity. Accordingly for any solution parametrized by $(\alpha_1, \beta_1, \omega_{i1}, R_1)$, there exists an identical solution with $k_2^2 = k_1^2$, $\omega_{i2} = \alpha_2 \omega_{i1} / \alpha_1$, and $R_2 = \alpha_1 R_1 / \alpha_2$. This is usually used to prove that if there is a three-dimensional linear instability of a laminar plane shear flow ($\omega_{i1} = 0$, $\alpha_1 > 0$, $\beta_1 > 0$) for a given critical Reynolds number Re_c , then there is a two-dimensional one ($\omega_{i2} = 0$, $\alpha_2 > \alpha_1$, $\beta_2 = 0$) emerging at a lower critical Reynolds number [37]. But it also says that there are other marginally stable modes at higher Reynolds number ($\omega_{i2} = 0$, $\alpha_2 < \alpha_1$, $\beta_2 > \beta_1$). In the usual case, the instability sets in when increasing the Reynolds number and this second part of the theorem is useless, since the base flow is already unstable. Here, in contrast, we are looking for an instability taking place when decreasing Re . Using the theorem, one would conclude that there are unstable modes at all Reynolds numbers ($\omega_{i2} = 0$, $\alpha_2 \rightarrow 0$, $\beta_2 > \beta_1$, $R_2 \rightarrow \infty$). This is certainly not backed by experimental or numerical evidence.

Note that the scaling leading to the Squire transformation does not hold strictly, because the base flow and the turbulent viscosity depend on the Reynolds number, in contrast with the laminar flow. Nevertheless, close enough to the three-dimensional critical value of Re , one does not expect any rapid change of the base flow and a weaker formulation of the theorem should hold. In such a scenario, given that all observations point towards a well defined three-dimensional pattern, this suggests that the Orr-Sommerfeld-Squire formulation, or equivalently any linear stability analysis of the mean flow alone, is likely to fail at predicting the observed instability where both critical α and β are nonzero. In the present work, we will actually show explicitly that this is indeed the case for the plane channel flow.

D. Models for the turbulent viscosity

The Boussinesq eddy viscosity model itself does not prescribe the form of the turbulent viscosity $\nu_t(y)$, which depends on the flow geometry and the Reynolds number. We consider here three different simple explicit models for ν_t .

The most drastic one trivially assumes vanishing Reynolds stresses: $\nu_t = 0$. It amounts to the wishful hypothesis that linearization around the mean flow could be carried out as if it were a solution to the Navier-Stokes equations. Counterintuitively perhaps, this procedure leads in some flow cases to physically correct results [23,24] and it should hence not be immediately discarded.

Second, we use an analytical closure model, popular for high-Reynolds-number channel flow, proposed by Cess [31,38]. The Cess formula reads

$$\frac{\nu_T}{\nu} = \frac{1}{2} \left(1 + \frac{\kappa^2 \text{Re}_\tau^2 (1 - \eta^2)^2 (1 + 2\eta^2)^2 \{1 - \exp[(|\eta| - 1) \text{Re}_\tau / A]^2 \}}{9} \right)^{\frac{1}{2}} + \frac{1}{2}, \quad (9)$$

where $\eta = y/h$ and with $\kappa = 0.426$ and $A = 25.4$ as in [31]. The Cess formula has been used also in similar studies of turbulent mean flow stability in, e.g., Refs. [21,27,30,35]. Note that the previous model with vanishing Reynolds stresses corresponds to a special case of the Cess closure with $\kappa = 0$.

Other popular modeling possibilities for turbulent channel using one-point closures are the $k - \varepsilon$ and $k - \Omega$ closures, where ν_t is imposed as a scalar field built on the mean kinetic energy and dissipation. Linear stability analysis of the $k - \Omega$ model with prescribed constants has been shown to predict strict stability of the mean plane Couette flow [39]. Because of the many constants involved in this model, usually determined from benchmarks performed at high Re rather than in the transitional regime, we did not pursue this approach here.

In the framework of unsteady RANS modeling (URANS), closures obtained by the compressible form of the Spalart-Allmaras turbulence model have also been considered. In particular, in the study of Ref. [40], the turbulent viscosity is considered as an unknown fluctuating variable in its own right. Fluctuations of ν_t are hence also taken into consideration through the use of yet another closure model. However, the models used in such studies, for instance, the Spalart-Allmaras model in Ref. [40], are phenomenological rather than based on any governing principle. This approach has hence not been pursued here.

Instead, we propose a more direct approach, in avoiding the formulation of any model and directly estimating $\nu_t(y)$ from direct numerical simulation (DNS), numerically evaluating the Reynolds stresses and the mean flow $U_x^*(y)$, and using Eq. (4). The apparent singularity at $y = 0$, where the denominator vanishes, is avoided by using L'Hôpital's rule. Stability around a numerically determined turbulent mean flow is now routinely evaluated, e.g., in resolvent analysis studies [41]. However, considering the Reynolds stresses under a data-driven approach (as in, e.g., Ref. [42]) has, to our knowledge, not been attempted in linear stability studies despite the marginal additional cost.

III. NUMERICAL DATA

All simulations were performed with the MPI-parallel spectral solver Channelflow2.0 [17,43] in a domain of size $L_x \times L_z = 250 \times 125$ and averaged over times up to $t = 4 \times 10^3$ time units. They ran at the French computation center IDRIS using up to 640 cores. The pseudospectral resolution is based on 1024 modes in both x and z (including dealiasing with the 2/3 rule) and 65 in y , which is comparable to [16].

The averages considered in this study, i.e., the steady mean flow and the Reynolds stresses,

$$\mathbf{U}(y) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{u}(x, y, z, t) dt, \quad (10)$$

$$\overline{u'_x u'_y}(y) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [(u_x - U_x)u_y](x, y, z, t) dt, \quad (11)$$

have been evaluated in practice as time averages and computed over 4000 time units after excluding transients.

Different computational domains have been tested for this study, respectively $L_x = 2L_z = 4\pi, 25, 50$, and 100. Note that the computational domain size is not directly related to the wave numbers α and β emerging from stability analysis. For $L_x = 4\pi$, the Reynolds stresses are not symmetric in y over moderate observation times and statistics are slow to converge. This asymmetry in y of the flow fields in computational domains that are too small has long been documented [44] and we discard the corresponding data. For the larger system sizes $L_x = 25, 50$, and 100, the results are quantitatively similar as demonstrated, for instance, by the mean flow profiles $U_x^*(y)$, computed by DNS for different values of Re_τ from 110 down to 96 and for different domain sizes that are compared in Fig. 2. All results displayed in the following were obtained with a system size $L_x = 100$. We also checked that the fluctuations extracted from the present DNS simulations match those in the literature; see Fig. 3.

Finally Fig. 4 compares the turbulent viscosity profiles, evaluated directly from the DNS simulations using Eq. (4) or from the Cess formula, Eq. (9). Not so surprisingly, the Cess formula, designed for high Reynolds flows, underestimates the turbulent viscosity.

IV. STABILITY RESULTS

The eigenvalues of the linear operator defined by Eqs. (6), (7), and (8) have been computed using Python for values of α and β on a discrete grid. To illustrate the results, we have picked two specific

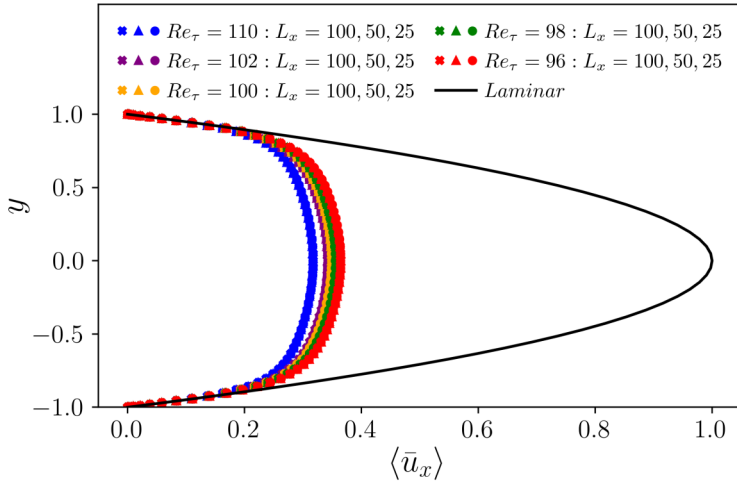


FIG. 2. Mean flow $U_x^*(y)$ from DNS for Re_τ from 110 to 96, used for the linear stability analysis. Different domain sizes have been used without any strong influence on the statistics.

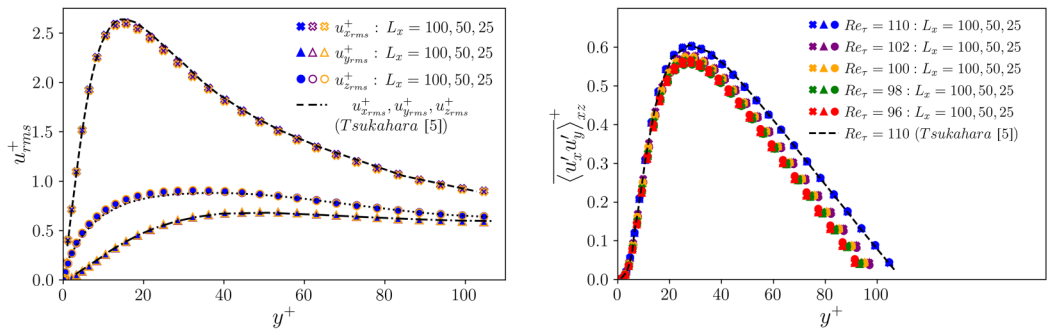


FIG. 3. rms streamwise velocity fluctuations for $Re_\tau = 110$ (left) and Reynolds stress $\overline{u_x' u_y'}$ for Re_τ from 110 to 96 (right) versus y^+ . Includes a comparison to [5].

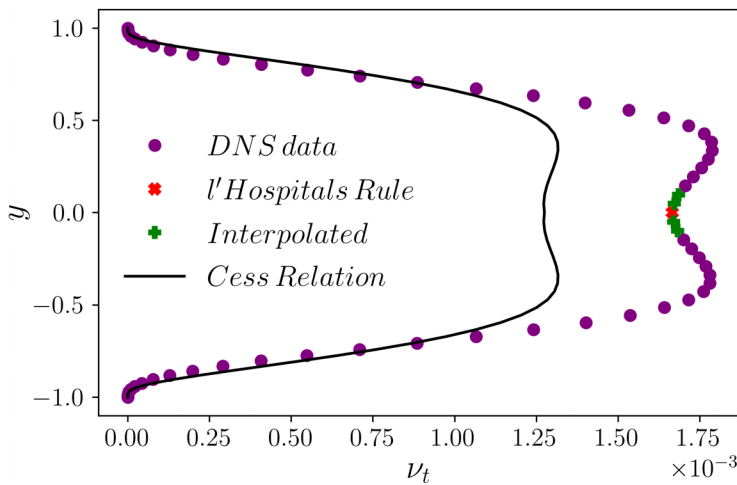


FIG. 4. Turbulent viscosity ν_t for $Re_\tau = 102$, evaluated directly from DNS using Eq. (4) (dots), interpolated near the midplane $y = 0$ using l'Hospital's rule and using the Cess formula of Eq. (9) (line).

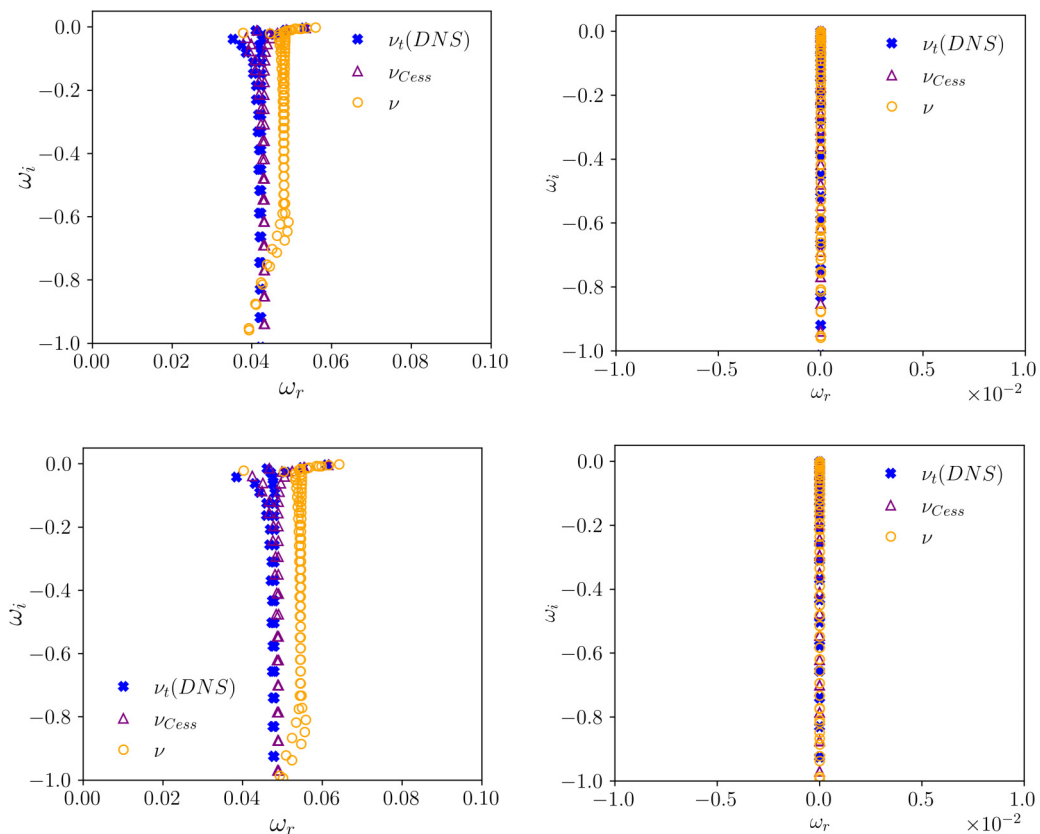


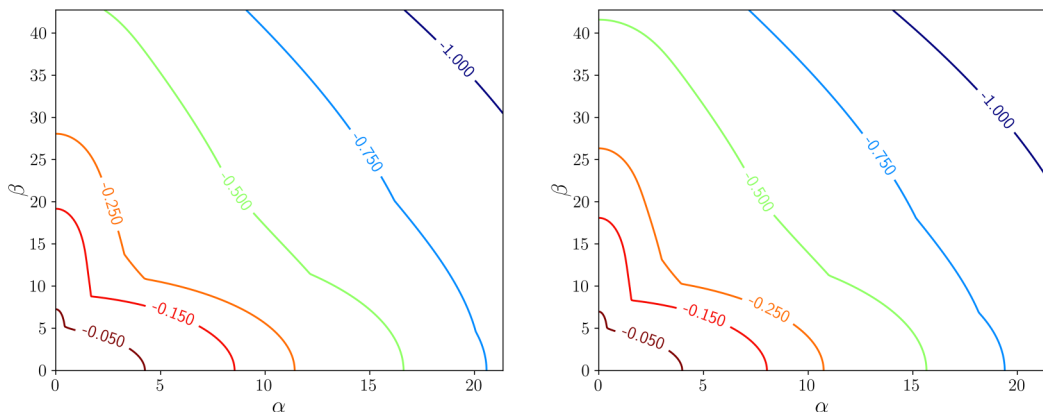
FIG. 5. Complex eigenvalues in the ω plane for $\text{Re}_\tau = 110$ (top row) and $\text{Re}_\tau = 96$ (bottom row). Left column: $(\alpha, \beta) = (0.18, 0.42)$ (the critical wave vector in Ref. [18]). Right column: $(\alpha, \beta) = (0, 0)$.

values of (α, β) , respectively $(0.18, 0.42)$, the marginal wave vector expected from the response analysis conducted in [18], expressed in units of $2\pi/h$, and $(0, 0)$. Figure 5 displays the growth rate of the perturbation for two values of Re_τ larger than the expected instability threshold Re_τ^c , estimated in [18]. In all cases the growth rates are strictly negative except for two neutral modes corresponding to the neutral directions x and z : perturbations to the mean relax towards the linearly stable mean flow U^* . For $(\alpha, \beta) = (0, 0)$ all modes are stationary and decaying. For $(\alpha, \beta) = (0.18, 0.42)$ the modes are propagating.

We have explored a large range of values for the wave numbers (α, β) . Figure 6 displays the isolevels of the growth rate in the (α, β) plane for the same two values of Re_τ . In both cases $(\alpha, \beta) = (0, 0)$ is the least stable mode. This points out the lack of selection of an oblique structure under linear processes.

Finally, we explore the dependence of the growth rate on Re_τ . As shown in Fig. 7 for $\beta = 0$ and in Table I, the decay rate of the least stable mode is weakly affected by the value Re_τ in the range of values investigated. This result suggests that the stable eigenvalues in question are robust and that minute modifications of other parameters (such as spatial resolution, averaging time, etc.) will not have a strong influence on the conclusions.

To conclude this analysis, we compare eigenspectra obtained from different closures (Fig. 8). A least stable branch of eigenvalues emerges in the case $\nu_t = 0$, without any counterpart once Reynolds stresses are included, whether based on the Cess formula or on the DNS data: Reynolds stresses stabilize the homogeneous mean turbulent flow U^* .


 FIG. 6. Isocontours of growth rate $\omega_i(\alpha, \beta)$. $\text{Re}_\tau = 110$ (left) and $\text{Re}_\tau = 96$ (right).

V. DISCUSSION AND OUTLOOKS

The dynamical origin of laminar-turbulent patterns and localized turbulent bands in wall-bounded shear flows has puzzled researchers for decades. The formation of such structures can be either investigated as a finite-amplitude instability of the laminar flow, as done in several recent studies [10,12,13], or directly as a linear instability of the homogeneous turbulent flow [3]. In the case of pressure-driven channel flow, recent work has concluded that laminar-turbulent patterns emerge from homogeneous turbulent flow below a Reynolds number Re_τ of about 95 via a modulational instability [18]. In the present numerical study we have focused on the channel case and investigated the ability of linear stability analysis conducted around the mean flow to predict the onset of modulation. Several simple one-point closures have been employed to link the Reynolds stresses to the mean velocity field. In all cases the homogeneous mean flow, which depends only on the wall-normal variable y , appears as linearly stable for all values of the Reynolds number investigated. Moreover, the least stable mode has in all cases $k_x = k_z = 0$, at odds with the nonzero wave vector of the modulated state. This result appears robust with respect to domain size and numerical resolution. It is moreover consistent with the predictions from the Squire theorem although this theorem does not strictly apply here because the base flow depends on Re .

There are several lessons to be learned from the failure of the analysis in predicting the observed instability. First, because the linear instability of the mean flow is not better predicted when the turbulent viscosity profile is directly measured from the DNS data, we can conclude that it is not the specific choice of a turbulent viscosity model that matters most. It is therefore tempting to conclude that any one-point closure or eddy viscosity model is doomed to fail at capturing the mechanism of the instability. In fact, this is precisely what our discussion about the applicability of the Squire transformation in Sec. II C suggests.

 TABLE I. Growth rate associated with the least stable eigenvalue as a function of Re_τ for the three closure models investigated.

Re_τ	DNS	Cess	$A = 0$
110	-0.00115673	-0.00122346	-0.00040783
102	-0.00124783	-0.00133929	-0.00047432
100	-0.00127355	-0.00137158	-0.00049348
98	-0.00129956	-0.00140540	-0.00051383
96	-0.00132725	-0.00144086	-0.00053546

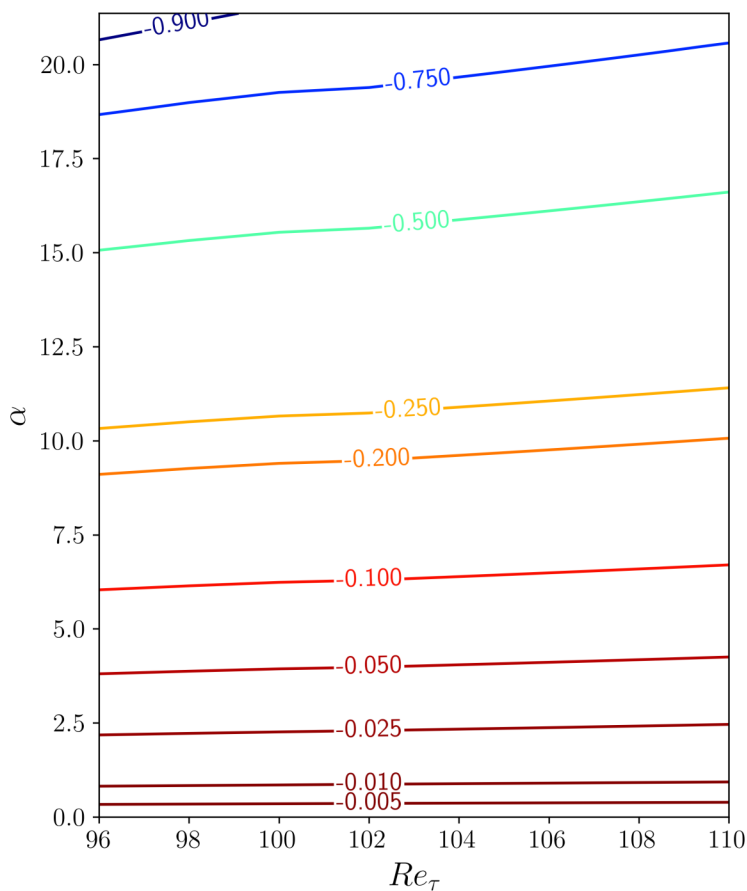


FIG. 7. Stability in DNS. Isocontours of growth rate in the (α, Re_τ) plane for $\beta = 0$.

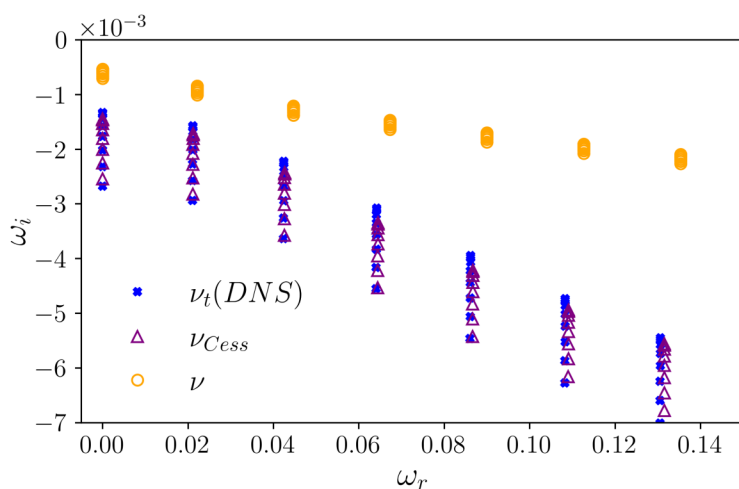


FIG. 8. Eigenvalues in the complex ω plane for $Re_\tau = 96$. The values of the wave numbers are varied over the range $\alpha \leq 0.4$, $\beta \leq 0.9$.

There are however alternative directions to investigate. One direction is to look beyond the Boussinesq theory for the constitutive relation; another is to consider a more complex base flow. When performing the linear stability analysis of Eq. (3), the turbulent viscosity is assumed to remain independent of x and z . In other words, the fluctuations were averaged over the three homogeneous coordinates, respectively the streamwise direction x , the spanwise direction z , and the time t . It is likely that such an extreme spatial coarsening of the fluctuations carried out here is responsible for the fact that the linear instability does not develop. In particular, at the moderate Reynolds numbers considered here, the so-called homogeneous turbulent flow is mainly structured around elongated coherent structures such as streamwise *streaks* (see the $k_x - k_z$ energy spectra in Ref. [18], which have a peaked distribution about $100\nu/u_\tau$). These structures are not properly captured by the mean flow which is based on z averaging. In addition, the infinitely long time scale associated with the mean flow is probably not needed given that modulations can appear in DNS over a time scale of $O(100)$ time units [17].

This suggests a more careful coarsening of the fluctuations, taking into account such structures as streaks into the base flow of interest for the stability study. The instability of optimally growing streaks has already been invoked in several works [45,46] as a plausible explanation for the larger-scale streaks observed in high-Re turbulent shear flows. Moreover, as is visually clear from Fig. 1, streaks are typically small scale compared to the large scale modulations. It is thus only as a collective instability that the present large-scale modulations found below $Re \approx 100$ can arise. Investigating spatially subharmonic instabilities of streaks at the relevant parameter values, for instance using recent methods based on the Bloch wave formalism [47,48], is hence a promising pathway in the quest for predicting the onset of laminar-turbulent patterns.

ACKNOWLEDGMENTS

The authors thank O. Semeraro, C. Cossu, and P. Morra for sharing their numerical code and useful discussions. J.-C. Robinet is also acknowledged for interesting discussions on the literature. L. Tuckerman and D. Barkley, as well as S. Gomé, are also acknowledged for discussing their unpublished work on plane Couette flow. Computations have been run at LISN, on the supercalculator Jean-Zay at IDRIS, and on the cluster mesoPSL at the Observatoire de Paris, financed by the region Ile-de-France.

-
- [1] D. Coles, Transition in circular couette flow, *J. Fluid Mech.* **21**, 385 (1965).
 - [2] A. Prigent, La spirale turbulente: Motif de grande longueur d'onde dans les écoulements cisailés turbulents, Ph.D. thesis, Université Paris Sud-Paris XI, 2001.
 - [3] A. Prigent, G. Grégoire, H. Chaté, O. Dauchot, and W. van Saarloos, Large-scale finite-wavelength modulation within turbulent shear flows, *Phys. Rev. Lett.* **89**, 014501 (2002).
 - [4] A. Prigent, G. Grégoire, H. Chaté, and O. Dauchot, Long-wavelength modulation of turbulent shear flows, *Physica D* **174**, 100 (2003).
 - [5] T. Tsukahara, Y. Seki, H. Kawamura, and D. Tochio, Dns of turbulent channel flow at very low reynolds numbers, *TSFP Digital Library Online* (Begel House Inc., Danbury, CT, 2005).
 - [6] D. Barkley and L. S. Tuckerman, Computational study of turbulent laminar patterns in couette flow, *Phys. Rev. Lett.* **94**, 014502 (2005).
 - [7] A. Meseguer, F. Mellibovsky, M. Avila, and F. Marques, Instability mechanisms and transition scenarios of spiral turbulence in taylor-couette flow, *Phys. Rev. E* **80**, 046315 (2009).
 - [8] Y. Duguet, P. Schlatter, and D. S. Henningson, Formation of turbulent patterns near the onset of transition in plane couette flow, *J. Fluid Mech.* **650**, 119 (2010).
 - [9] Y. Duguet and P. Schlatter, Oblique laminar-turbulent interfaces in plane shear flows, *Phys. Rev. Lett.* **110**, 034502 (2013).

-
- [10] J. J. Tao, B. Eckhardt, and X. M. Xiong, Extended localized structures and the onset of turbulence in channel flow, *Phys. Rev. Fluids* **3**, 011902(R) (2018).
- [11] C. Paranjape, Onset of turbulence in plane Poiseuille flow, Ph.D. thesis, IST Austria, 2019.
- [12] X. Xiao and B. Song, The growth mechanism of turbulent bands in channel flow at low reynolds numbers, *J. Fluid Mech.* **883**, R1 (2020).
- [13] E. Parente, J.-C. Robinet, P. De Palma, and S. Cherubini, Linear and nonlinear optimal growth mechanisms for generating turbulent bands, *J. Fluid Mech.* **938**, A25 (2022).
- [14] L. Zhang and J. Tao, Large-scale mean flow and inclination of isolated turbulent band in channel flow, [arXiv:2310.03238](https://arxiv.org/abs/2310.03238).
- [15] L. S. Tuckerman, M. Chantry, and D. Barkley, Patterns in wall-bounded shear flows, *Annu. Rev. Fluid Mech.* **52**, 343 (2020).
- [16] M. Shimizu and P. Manneville, Bifurcations to turbulence in transitional channel flow, *Phys. Rev. Fluids* **4**, 113903 (2019).
- [17] P. V. Kashyap, Y. Duguet, and O. Dauchot, Flow statistics in the transitional regime of plane channel flow, *Entropy* **22**, 1001 (2020).
- [18] P. V. Kashyap, Y. Duguet, and O. Dauchot, Linear instability of turbulent channel flow, *Phys. Rev. Lett.* **129**, 244501 (2022).
- [19] W. Malkus, Outline of a theory of turbulent shear flow, *J. Fluid Mech.* **1**, 521 (1956).
- [20] W. Reynolds and W. Tiederman, Stability of turbulent channel flow, with application to Malkus's theory, *J. Fluid Mech.* **27**, 253 (1967).
- [21] W. Reynolds and A. Hussain, The mechanics of an organized wave in turbulent shear flow. Part 3. theoretical models and comparisons with experiments, *J. Fluid Mech.* **54**, 263 (1972).
- [22] S. Beneddine, D. Sipp, A. Arnault, J. Dandois, and L. Lesshafft, Conditions for validity of mean flow stability analysis, *J. Fluid Mech.* **798**, 485 (2016).
- [23] D. Sipp and A. Lebedev, Global stability of base and mean flows: A general approach and its applications to cylinder and open cavity flows, *J. Fluid Mech.* **593**, 333 (2007).
- [24] D. Barkley, Linear analysis of the cylinder wake mean flow, *Europhys. Lett.* **75**, 750 (2006).
- [25] G. Brethouwer, P. Schlatter, Y. Duguet, D. S. Henningson, and A. V. Johansson, Recurrent bursts via linear processes in turbulent environments, *Phys. Rev. Lett.* **112**, 144502 (2014).
- [26] S. E. Turton, L. S. Tuckerman, and D. Barkley, Prediction of frequencies in thermosolutal convection from mean flows, *Phys. Rev. E* **91**, 043009 (2015).
- [27] C. Cossu, Onset of large-scale convection in wall-bounded turbulent shear flows, *J. Fluid Mech.* **945**, A33 (2022).
- [28] Y. Ponty, P. D. Mininni, J.-F. Pinton, H. Politano, and A. Pouquet, Dynamo action at low magnetic prandtl numbers: Mean flow versus fully turbulent motions, *New J. Phys.* **9**, 296 (2007).
- [29] P. J. Schmid, Nonmodal stability theory, *Annu. Rev. Fluid Mech.* **39**, 129 (2007).
- [30] J. C. Del Alamo and J. Jimenez, Linear energy amplification in turbulent channels, *J. Fluid Mech.* **559**, 205 (2006).
- [31] G. Pujals, M. García-Villalba, C. Cossu, and S. Depardon, A note on optimal transient growth in turbulent channel flows, *Phys. Fluids* **21**, 015109 (2009).
- [32] S. J. Benavides and D. Barkley, Model for transitional turbulence in a planar shear flow, [arXiv:2309.12879](https://arxiv.org/abs/2309.12879).
- [33] A. Monin and A. Yaglom, *Statistical Fluid Dynamics (MIT Press, Cambridge, MA, 1971)*, Vols. I and II.
- [34] P. J. Schmid, D. S. Henningson, and D. Jankowski, Stability and transition in shear flows. applied mathematical sciences, Vol. 142, *Appl. Mech. Rev.* **55**, B57 (2002).
- [35] P. Morra, O. Semeraro, D. S. Henningson, and C. Cossu, On the relevance of reynolds stresses in resolvent analyses of turbulent wall-bounded flows, *J. Fluid Mech.* **867**, 969 (2019).
- [36] P. J. Schmid and L. Brandt, Analysis of fluid systems: Stability, receptivity, sensitivity lecture notes from the flow-nordita summer school on advanced instability methods for complex flows, Stockholm, Sweden, 2013, *Appl. Mech. Rev.* **66**, 024803 (2014).
- [37] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, UK, 2004).

- [38] R. Cess, A survey of the literature on heat transfer in turbulent tube flow, *Res. Rep.* **1958**, 8 (1958).
- [39] L. S. Tuckerman, D. Barkley, and O. Dauchot, Instability of uniform turbulent plane couette flow: Spectra, probability distribution functions and k-closure model, *Seventh IUTAM Symposium on Laminar-Turbulent Transition: Proceedings of the Seventh IUTAM Symposium on Laminar-Turbulent Transition, Stockholm, Sweden, 2009* (Springer, Berlin, 2010), pp. 59–66.
- [40] J. Crouch, A. Garbaruk, and D. Magidov, Predicting the onset of flow unsteadiness based on global instability, *J. Comput. Phys.* **224**, 924 (2007).
- [41] Y. Hwang and C. Cossu, Amplification of coherent streaks in the turbulent couette flow: An input–output analysis at low reynolds number, *J. Fluid Mech.* **643**, 333 (2010).
- [42] M. Farano, S. Cherubini, J.-C. Robinet, and P. De Palma, Optimal bursts in turbulent channel flow, *J. Fluid Mech.* **817**, 35 (2017).
- [43] J. Gibson, F. Reetz, S. Azimi, A. Ferraro, T. Kreilos, H. Schrobendorff, M. Farano, A. Yesil, S. Schütz, M. Culpo *et al.*, *Channelflow 2.0* (unpublished).
- [44] J. Jiménez and P. Moin, The minimal flow unit in near-wall turbulence, *J. Fluid Mech.* **225**, 213 (1991).
- [45] J. Park, Y. Hwang, and C. Cossu, On the stability of large-scale streaks in turbulent couette and poiseuille flows, *C. R. Mec.* **339**, 1 (2011).
- [46] F. Alizard, Linear stability of optimal streaks in the log-layer of turbulent channel flows, *Phys. Fluids* **27**, 105103 (2015).
- [47] P. J. Schmid, M. F. de Pando, and N. Peake, Stability analysis for n-periodic arrays of fluid systems, *Phys. Rev. Fluids* **2**, 113902 (2017).
- [48] A. Jouin, N. Ciola, S. Cherubini, and J.-C. Robinet, Detuned secondary instabilities in three-dimensional boundary-layer flow, *Phys. Rev. Fluids* **9**, 043901 (2024).