Letter

Similarity relations for laminar pipe flows of Bingham fluids in friction coordinates

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We introduce similarity relations, along with their corresponding symmetry groups, for examining the velocity profile and friction factor of laminar Bingham fluid flows using friction coordinates. Specifically, we provide a valuable expression for calculating the friction factor of Bingham plastic fluids when pressure gradient data are accessible. Our findings reveal that there are no clear similarity relations for the mean velocity profile and friction factor formulas in bulk coordinates. Consequently, friction coordinates serve as the most suitable framework for describing this problem.

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Laminar Bingham fluid flows. Viscoplastic liquids can be found in several biological and industrial systems, such as blood, mudflows, mayonnaise, paints, lava flows, and toothpaste. One defining characteristic of these fluids is the presence of a yield stress parameter τ_y. The fluid will only flow when the applied stress exceeds the yield stress, acting as a solid for lower stresses. The Bingham constitutive equation, proposed in Ref. [1], is a simple model used for viscoplastic fluids. Unlike more complex models, such as the Herschel-Bulkley, which displays shear-rate dependency for the dynamic viscosity parameter, the Bingham shear stress satisfies the affine relation $\tau_{rz} = \tau_y - \mu(\frac{dU}{dr})$, for $\tau_{rz} > \tau_y$, where μ is the dynamic viscosity of the fluid.

A special geometric feature of laminar Bingham fluid flows is the presence of a solid pluglike core in the central region of the pipe, where the shear stress satisfies $\tau_{rz} < \tau_y$. The plug's radius R_P is a function of the yield stress τ_y, the wall shear stress τ_w, and the pipe's radius. Indeed, because $\tau_{rz}(r) = -\frac{\partial p}{\partial z}\frac{r}{2}$, at $r = R_P$, the interface with the plug region, the shear stress satisfies $\tau_y = \tau_{rz}(R_P) = \frac{-\partial p}{\partial z}\frac{R_P}{2}$. Because $\tau_w = \tau_{rz}(R) = (-\frac{\partial p}{\partial z})\frac{R}{2}$, where $R = D/2$ is the pipe radius, it follows that $\tau_y/\tau_w = R_P/R$. Figure 1 illustrates a laminar Bingham flow.

In this Letter, we consider laminar Bingham flows in pipes driven by a pressure gradient. For a given rheological configuration (μ, τ_y), fluid's density ρ, and pipe diameter D, the flow is characterized by the imposed conditions either on pressure gradient or on the mass flow rate (this is also true for fully developed turbulent flows). If the condition is imposed on the mass flow rate, we say that the flow is parametrized in bulk coordinates. On the other hand, if the condition is imposed on the pressure gradient, we say that it is parametrized on friction coordinates. The averaged bulk velocity is defined as $\bar{U} = Q/\pi R^2$, where $Q = \int_0^R 2\pi r U\, dr$ is the volumetric flow rate of a pipe flow. In the incompressible context, the mass flow rate is simply $\rho\bar{U}$. Similarly to Newtonian flows, we define the friction velocity $u_\tau = \sqrt{\tau_w/\rho}$. We now define some important dimensionless parameters. Let us start with the bulk Reynolds number Re, and the friction Reynolds number

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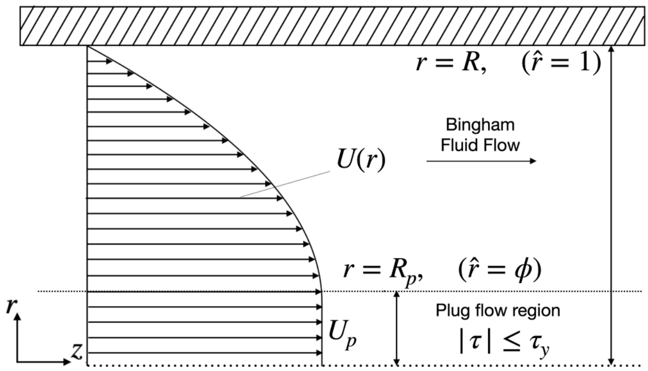

FIG. 1. Mean velocity profile of laminar Bingham fluids.

Re_τ [2,3]:

$$\mathrm{Re} = \frac{\rho \bar{U} D}{\mu}, \quad \mathrm{Re}_\tau = \frac{\rho u_\tau D}{\mu}. \tag{1}$$

The Hedstrom number He and ϕ are respectively defined as

$$\mathrm{He} = \frac{\rho D^2 \tau_y}{\mu^2}, \quad \phi := \frac{R_P}{R} = \frac{\tau_y}{\tau_w} = \frac{\mathrm{He}}{\mathrm{Re}_\tau^2}. \tag{2}$$

The Hedstrom number provides a metric to gauge the impact of yield stress relative to the plastic viscosity in Bingham fluids. Lower Hedstrom values correspond to Newtonian-like behavior, whereas higher values indicate a pronounced, solidlike, non-Newtonian response. Notably, a Hedstrom number of zero, indicative of zero yield stress, corresponds to Newtonian fluid behavior. Although the Hedstrom number depends solely on the geometric setup, rheological properties, and density of the fluid, it is worth emphasizing that the flow conditions can still modulate the roles of yield stress and plastic viscosity in governing the flow dynamics. In the literature, it is common to parametrize the flow with the bulk dimensionless coordinates $(\mathrm{He}, \mathrm{Re})$, instead of the friction dimensionless coordinates $(\mathrm{He}, \mathrm{Re}_\tau)$ (see, e.g., Ref. [4]). This choice is natural if one prescribes conditions on the mass flow rate instead of using the pressure gradient. However, in many applications, this choice is arbitrary, and the pressure gradient may be easier to determine through the use of manometers.

From Buckingham's Π theorem, the pressure gradient satisfies a relation of the form $-\frac{\partial p}{\partial z} = \frac{\rho \bar{U}^2}{D} f$, where f is the so-called friction factor, a function of either $(\mathrm{Re}, \mathrm{He})$ or $(\mathrm{Re}_\tau, \mathrm{He})$. The (Fanning) friction factor is formally defined as [2,3]

$$f = 2\frac{u_\tau^2}{\bar{U}^2} = \frac{2\tau_w}{\rho \bar{U}^2} = \frac{D\left(-\frac{\partial p}{\partial z}\right)}{2\rho \bar{U}^2}. \tag{3}$$

The fact that the friction factor can be written in bulk coordinates, i.e., as $f = \mathcal{H}(\mathrm{Re}, \mathrm{He})$, is related to the invariance of the friction factor by the action of the following similarity group:

$$\mu^* = A_1 \mu, \quad \rho^* = A_2 \rho, \quad D^* = A_3 D,$$
$$\tau_y^* = \frac{A_1^2}{A_2 A_3^2} \tau_y, \quad \bar{U}^* = \frac{A_1}{A_2 A_3} \bar{U}, \quad f^* = f, \tag{4}$$

where A_1, A_2, A_3 are positive real numbers. The group depicted above is derived initially by scaling the dimensionally independent parameters μ, ρ, and D using arbitrary positive constants A_1, A_2, and A_3, collectively termed as *group parameters*. Following this, we compute the appropriate scalings for the dimensionally dependent parameters f, τ_y, and $\bar{U}$ to ensure the constancy of Re, He, and the relationship $\mathcal{H}$. The importance of this similarity group emerges from the transformation (4), which guarantees $\mathrm{Re}^* = \mathrm{Re}$, $\mathrm{He}^* = \mathrm{He}$, and most importantly, $f^* = \mathcal{H}(\mathrm{Re}^*, \mathrm{He}^*)$. Comprehensive details on the derivation of these groups can be referenced in Ref. [5]. On the other hand, the fact that the friction factor can also be written in friction coordinates, as $f = \mathcal{L}(\mathrm{Re}_\tau, \mathrm{He})$, is related to the invariance of the friction by the action of the following similarity group:

$$\mu^* = \hat{A}_1\mu, \quad \rho^* = \hat{A}_2\rho, \quad D^* = \hat{A}_3 D,$$
$$\tau_y^* = \frac{\hat{A}_1^2}{\hat{A}_2\hat{A}_3^2}\tau_y, \quad \left(\frac{\partial p}{\partial z}\right)^* = \frac{\hat{A}_1^2}{\hat{A}_2\hat{A}_3^3}\left(\frac{\partial p}{\partial z}\right), \quad f^* = f, \tag{5}$$

where $\hat{A}_1, \hat{A}_2, \hat{A}_3$ are positive real numbers, and the deriving of such a group follows a similar reasoning as above.

An implicit equation for the friction factor involving the Reynolds number (Re) and the Hedstrom number (He) is applicable for laminar flow of Bingham fluids. This formula, known as the Buckingham-Reiner equation, is briefly derived here for completeness; for a more detailed derivation, please see Ref. [4].

Given that the radial shear stress is $\tau_{rz} = (-\frac{\partial p}{\partial z})\frac{r}{2}$ and the velocity is zero at the pipe wall, we can integrate the Bingham rheological model, $\tau_{rz} = \tau_y - \mu(\frac{dU}{dr})$, with respect to the radial position r. This leads to

$$U(r) = \left(\frac{-\partial p}{\partial z}\right)\frac{R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) - \frac{\tau_y}{\mu}R\left(1 - \frac{r}{R}\right), \tag{6}$$

in the fluidlike region, $R_p \leqslant r \leqslant R$. The corresponding mean velocity profile (MVP) in the plug region can be obtained by substituting $r = R_p$ in the equation above. This yields

$$U_p(r) = \left(\frac{-\partial p}{\partial z}\right)\frac{R^2}{4\mu}\left(1 - \frac{R_p}{R}\right)^2, \tag{7}$$

in the plug region, $0 \leqslant r \leqslant R_p$. After integrating the velocity profiles over the cross-sectional area of the pipe, we arrive at the expression for the mean velocity $\bar{U}$ given by

$$\bar{U} = \frac{Q}{\pi R^2} = \frac{R^2}{8\mu}\left(-\frac{\partial p}{\partial z}\right)\left(1 - \frac{4}{3}\phi + \frac{1}{3}\phi^4\right). \tag{8}$$

Because $\phi = \tau_y/\tau_w = \tau_y/(1/2)f\rho\bar{U}^2$, one can replace this expression into Eq. (8), and divide both sides by $\rho\bar{U}^2$, to obtain the Buckingham-Reiner equation for the laminar friction factor of Bingham fluids in bulk coordinates:

$$f = \frac{16}{\mathrm{Re}}\left[1 + \frac{1}{6}\frac{\mathrm{He}}{\mathrm{Re}} - \frac{1}{3}\frac{\mathrm{He}^4}{f^3\,\mathrm{Re}^7}\right]. \tag{9}$$

Figure 2 shows the laminar friction curves as functions of the bulk Reynolds number for five different values of the Hedstrom parameter. We notice that although the Buckingham-Reiner equation is certainly useful for many engineering applications, it does not reveal any additional symmetries of the flow, besides the one imposed by a dimensional analysis. Moreover, its implicit nature makes it cumbersome and costly to be calculated in large engineering projects regarding the design of piping systems. In the following sections, we argue that the use of friction coordinates is natural in the sense that it yields a simple explicit formula, and that it is closely related to the existence of a renormalization symmetry group of laminar Bingham pipe flows.

Similarity relations and renormalization groups: Definitions. Thus far, our conclusions from a dimensional analysis have been derived exclusively using Buckingham's Π theorem. In line with

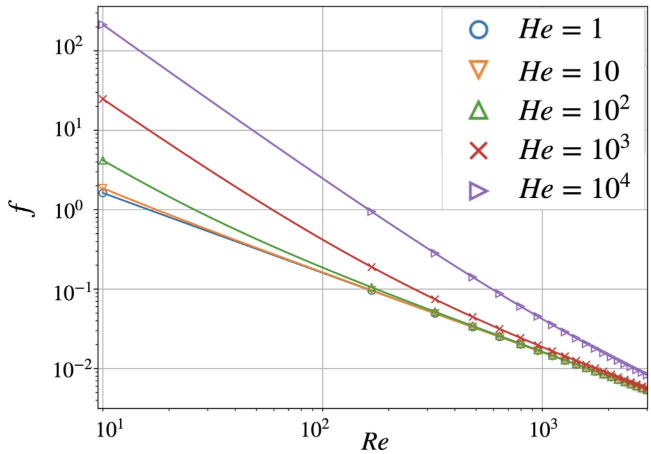

FIG. 2. Friction factor of laminar Bingham fluid flows in bulk coordinates.

the notation presented in Barenblatt's work [5], we know that any physical relation of interest,

$$a = f(a_1, \ldots, a_k, b_1, \ldots, b_m), \tag{10}$$

which depends on k dimensionally independent parameters $a_1, \ldots, a_k$ and m dimensionally dependent parameters $b_1, \ldots, b_m$ can be made nondimensional in the form

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_m), \tag{11}$$

where, for $i = 1, \ldots, m$,

$$\Pi_i = \frac{b_i^{\gamma_i}}{a_1^{\gamma_{1,i}} \cdots a_k^{\gamma_{k,i}}}, \tag{12}$$

and

$$\Pi = \frac{a^{\gamma_0}}{a_1^{\gamma_{1,0}} \cdots a_k^{\gamma_{k,0}}}. \tag{13}$$

The exponents $\gamma_i, \gamma_{1,i}, \ldots, \gamma_{k,i}$ must be chosen in order to make Π_i and Π dimensionless parameters. This normalization introduces a class of similarity groups, which we call *Buckingham's similarity groups*. These are well exemplified in Eqs. (4) and (5), and a more thorough discussion can be found in Refs. [5–7]. Although the Π theorem is proven to be effective in both theory and experimental design, there are extended notions of similarity relations that are very useful in scientific literature. The first one is called *complete similarity*, in which the function Φ above converges sufficiently fast to a a nonzero limit when some of the parameters $\Pi_{l+1}, \ldots, \Pi_m$ go to zero or infinity. In such a case, we can study an idealized version of the problem while also having a clear instance of dimensionality reduction:

$$\Pi = \Phi^{(0)}(\Pi_1, \ldots, \Pi_l). \tag{14}$$

Regrettably, the situation described is not the most common or general instance of similarity, since the dependence on parameters $\Pi_{l+1}, \ldots, \Pi_m$ typically exists regardless of their magnitude, whether it is small or large. Nevertheless, there exists an additional category of similarity known as *incomplete similarity*, which also occurs when the aforementioned parameters exhibit extremely small or large values. For this kind of similarity, relation (11) becomes

$$\Pi = \Pi_{l+1}^{\alpha_{l+1}} \cdots \Pi_m^{\alpha_m} \Phi^{(1)}\left(\frac{\Pi_1}{\Pi_{l+1}^{\beta_1} \cdots \Pi_m^{\delta_1}}, \ldots, \frac{\Pi_l}{\Pi_{l+1}^{\beta_l} \cdots \Pi_m^{\delta_l}} \right). \tag{15}$$

This formulation can also be regarded as an instance of dimensionality reduction. In the context of Buckingham's Π theorem, this type of similarity originates from the property of generalized homogeneity associated with the function Φ. As a result, it is logical that it also gives rise to an incomplete similarity group, referred to as the *renormalization group*. This group contains $m - l$ parameters and we will briefly discuss the process of obtaining it.

First, the dimensionally independent parameters will be fixed, i.e., $a_1^* = a_1, \dots, a_k^* = a_k$, and we will have the freedom of scaling the last $m - l$ dimensionally dependent parameters by arbitrary positive constants, that is, $b_{l+1}^* = B_{l+1} b_{l+1}, \dots, b_m^* = B_m b_m$. Now, in order to find the proper scalings for $b_1, \dots, b_l$ and a, we will demand that the relationship (15) remains unaltered. This implies that, for $j = 1, \dots, l$, we must have

$$\frac{\Pi_j^*}{{\Pi_{l+1}^*}^{\beta_j} \cdots {\Pi_m^*}^{\delta_j}} = \frac{\Pi_j}{\Pi_{l+1}^{\beta_j} \cdots {\Pi_m}^{\delta_j}}, \tag{16}$$

and also

$$\frac{\Pi^*}{{\Pi_{l+1}^*}^{\alpha_{l+1}} \cdots {\Pi_m^*}^{\alpha_m}} = \frac{\Pi}{\Pi_{l+1}^{\alpha_{l+1}} \cdots \Pi_m^{\alpha_m}}. \tag{17}$$

By solving the system of equations defined by (16) and (17), we arrive at the aforementioned renormalization group:

$$\begin{aligned} a_1^* &= a_1, \quad a_2^* = a_2, \quad \dots, \quad a_k^* = a_k, \\ b_1^* &= B_{l+1}^{\beta_1 \gamma_{l+1}/\gamma_1} \cdots B_m^{\delta_1 \gamma_m/\gamma_1} b_1, \quad \dots, \\ b_l^* &= B_{l+1}^{\beta_l/\gamma_l} \cdots B_m^{\delta_l/\gamma_l} b_l, \\ b_{l+1}^* &= B_{l+1} b_{l+1}, \quad \dots, \quad b_m^* = B_m b_m, \\ a^* &= B_{l+1}^{\alpha_{l+1}\gamma_{l+1}/\gamma_0} \cdots B_m^{\alpha_m \gamma_m/\gamma_0} a. \end{aligned} \tag{18}$$

The significance of this group lies in the fact that, by construction, the relationship expressed in (15) remains unaltered under the transformation (18) for all positive scaling constants $B_{l+1}, \dots, B_m$, which we can also call group parameters. Consequently, if incomplete similarity is identified in any physical phenomena, it can greatly simplify the processes of modeling and experimental design for both physicists and engineers.

Incomplete similarity relation for the friction factor. For a specific Hedstrom number and sufficiently low Reynolds number, the Reynolds stress term can be dismissed. Consequently, the laminar velocity profiles represented by Eqs. (6) and (7) serve as reasonable approximations for Bingham fluid flow within their respective valid domains, as detailed in Ref. [4].

Upon integrating these velocity expressions across the flow domain, Eq. (8) emerges, providing a representation for the bulk velocity. Utilizing the definition of Re_τ, this equation can be reformulated as follows:

$$\bar{U} = u_\tau \,\mathrm{Re}_\tau \left[\frac{1}{8} - \frac{\phi}{6} + \frac{\phi^4}{24}\right]. \tag{19}$$

Inserting (19) into Eq. (3), one obtains

$$f = 2\frac{u_\tau^2}{\bar{U}^2} = \frac{2}{\mathrm{Re}_\tau^2 \left[\frac{1}{8} - \frac{\phi}{6} + \frac{\phi^4}{24}\right]^2}. \tag{20}$$

Therefore, the friction factor satisfies an incomplete similarity relation in the sense that

$$f = \mathcal{L}(\mathrm{Re}_\tau, \mathrm{He}) = \mathrm{Re}_\tau^{-2} F\left(\frac{\mathrm{He}}{\mathrm{Re}_\tau^2}\right), \tag{21}$$

where $F(\phi) = \frac{2}{[\frac{1}{8} - \frac{\phi}{6} + \frac{\phi^4}{24}]^2}$.

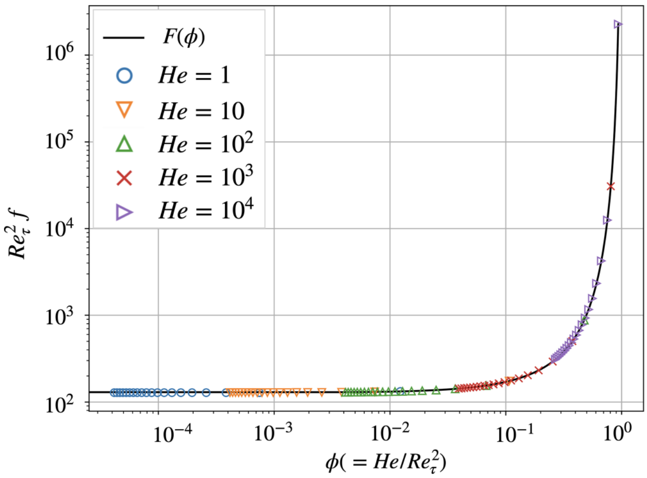

FIG. 3. Friction factor of laminar Bingham fluid flows in bulk coordinates—incomplete similarity collapse of data.

This means that if one plots the quantity $f\,\mathrm{Re}_\tau^2$ against $\phi = \mathrm{He}/\mathrm{Re}_\tau^2$, one obtains a single curve for every possible laminar flow configuration. This is illustrated in Fig. 3, which shows the collapse of all the points shown in Fig. 2 over the graph of the function F.

By following the reasoning in the previous section, one can use our incomplete similarity relation to arrive at the following renormalization group,

$$\mu^* = \mu, \quad \rho^* = \rho, \quad D^* = D,$$
$$\tau_y^* = B_1 \tau_y, \quad \left(\frac{\partial p}{\partial z}\right)^* = B_1\left(\frac{\partial p}{\partial z}\right), \quad f^* = B_1^{-1} f, \tag{22}$$

where B_1 is a positive real number. We remark that this symmetry group cannot be obtained through pure dimensional reasoning, and that there is no similar invariance relation in purely bulk coordinates.

Incomplete similarity relation for the laminar velocity profile. We now extend the similarity relations to the mean velocity profile. Let $\hat{r} = r/R$. By Buckingham's Π theorem, the velocity profile can be rewritten either in bulk coordinates, as $U = \mathcal{U}(r; D, \rho, \mu, \tau_y, \bar{U}) = \bar{U}\Psi(\hat{r}, \mathrm{He}, \mathrm{Re})$, which is related to the following Buckingham's similarity group,

$$\mu^* = A_1\mu, \quad \rho^* = A_2\rho, \quad D^* = A_3 D, \quad r^* = A_3 r,$$
$$\tau_y^* = \frac{A_1^2}{A_2 A_3^2}\tau_y, \quad \bar{U}^* = \frac{A_1}{A_2 A_3}U, \quad U^* = \frac{A_1}{A_2 A_3}U, \tag{23}$$

or in friction coordinates, as $U = \mathcal{U}(r; D, \rho, \mu, \tau_y, \frac{\partial p}{\partial z}) = u_\tau \Phi(\hat{r}, \mathrm{He}, \mathrm{Re}_\tau)$, which is related to the symmetry

$$\mu^* = A_1\mu, \quad \rho^* = A_2\rho, \quad D^* = A_3 D, \quad r^* = A_3 r,$$
$$\tau_y^* = \frac{A_1^2}{A_2 A_3^2}\tau_y, \quad \left(\frac{\partial p}{\partial z}\right)^* = \frac{A_1^2}{A_2 A_3^3}\left(\frac{\partial p}{\partial z}\right),$$
$$U^* = \frac{A_1}{A_2 A_3}U. \tag{24}$$

In friction coordinates, an explicit formula can be obtained by simple manipulation of Eq. (6),

$$\frac{U}{u_\tau} = \mathrm{Re}_\tau \frac{1}{4}[(1-\phi)^2 - (\hat{r}-\phi)^2], \tag{25}$$

for $\phi \leqslant \hat{r} \leqslant 1$, and

$$\frac{U}{u_\tau} = \mathrm{Re}_\tau \frac{1}{4}(1-\phi)^2, \tag{26}$$

for $0 \leqslant \hat{r} \leqslant \phi$. This is a statement of incomplete similarity in friction coordinates in the sense that

$$\frac{U}{u_\tau} = \Phi(\hat{r}, \mathrm{He}, \mathrm{Re}_\tau) = \mathrm{Re}_\tau\, \Phi^{(1)}\left(\hat{r}, \frac{\mathrm{He}}{\mathrm{Re}_\tau^2}\right), \tag{27}$$

for $\phi \leqslant \hat{r} \leqslant 1$, where

$$\Phi^{(1)}(x_1, x_2) = \frac{1}{4}[(1-x_2)^2 - (x_1 - x_2)^2], \tag{28}$$

and

$$\frac{U}{u_\tau} = \Phi_p(\mathrm{He}, \mathrm{Re}_\tau) = \mathrm{Re}_\tau\, \Phi_p^{(1)}\left(\frac{\mathrm{He}}{\mathrm{Re}_\tau^2}\right), \tag{29}$$

for $0 \leqslant \hat{r} \leqslant \phi$, where $\Phi_p^{(1)}(x) = \frac{1}{4}(1-x)^2$.

This is related to the scaling of the mean velocity profile by the action of the following renormalization group,

$$\begin{aligned} &\mu^* = \mu, \quad \rho^* = \rho, \quad D^* = D, \ r^* = r, \\ &\tau_y^* = B_1\tau_y, \quad \left(\frac{\partial p}{\partial z}\right)^* = B_1\left(\frac{\partial p}{\partial z}\right), \quad U^* = B_1 U, \end{aligned} \tag{30}$$

where B_1 is a positive real number.

Let us illustrate this scaling phenomenon with the following example, following Ref. [8]: Consider a set of four different configurations of Bingham plastic fluids with $\mu = 0.035$ Pa s and density $\rho = 1200$ kg/m^3. Let us also assume a pipe effective diameter $D = 0.1$ m. Because $\mathrm{Re}_\tau = (\rho D/\mu)u_\tau$, and $u_\tau = \sqrt{(-\partial p/\partial z)D/4\rho}$, one can choose four different values of the parameter $\partial p/\partial z$ so that $\mathrm{Re}_\tau \in \{10, 100, 200, 400\}$, as well as four different values of the parameter τ_y so that $\phi = \mathrm{He}/\mathrm{Re}_\tau^2 = 0.2$. In Fig. 4, we show three level curves of the parameter ϕ in the $\mathrm{Re}_\tau \times \mathrm{He}$ plane. We also show the points associated with the parameters $\mathrm{Re}_\tau \in \{10, 100, 200, 400\}$ over the level curve $\phi = 0.2$. In Fig. 5, we show the velocity profiles associated with this set of parameters. In Fig. 6, we illustrate the incomplete similarity phenomenon with the collapse of all curves on the graph of $\Phi^{(1)}(\hat{r}, \frac{\mathrm{He}}{\mathrm{Re}_\tau^2} = 0.2)$.

Conclusions. We have demonstrated that the friction factor and velocity profile of Bingham fluids adhere to an incomplete similarity relation when expressed in friction coordinates. These similarity relations correspond to the invariance of flow under the influence of their respective symmetry groups.

It is crucial to highlight that the presence of yield stress breaks the symmetry of the laminar velocity profile for simple Newtonian flows, which is otherwise characterized by complete similarity in bulk coordinates. Indeed, when the yield stress (τ_y) is equal to zero, the laminar profile fulfills the similarity condition in bulk coordinates,

$$\frac{U}{\bar{U}} = 2(1-\hat{r}^2), \tag{31}$$

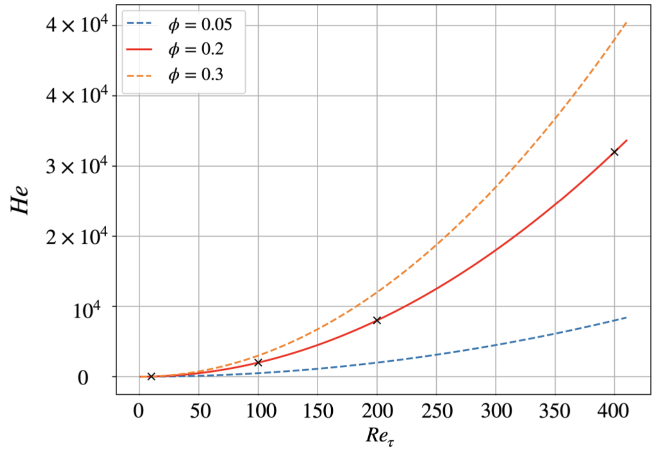

FIG. 4. Three different level curves for the parameter ϕ.

which is associated with the similarity group

$$\begin{aligned} \mu^* = \mu, \quad \rho^* = \rho, \quad D^* = D, \\ \bar{U}^* = B_1\bar{U}, \quad U^* = B_1 U, \end{aligned} \tag{32}$$

where B_1 is a positive real number. The corresponding relation for Bingham fluids is

$$\frac{U}{\bar{U}} = \frac{1}{4}\frac{[(1-\phi)^2 - (\hat{r}-\phi)^2]}{\left[\frac{1}{8} + \frac{\phi}{6} + \frac{\phi^4}{24}\right]}, \tag{33}$$

for $\phi \leqslant \hat{r} \leqslant 1$, and

$$\frac{U}{\bar{U}} = \frac{1}{4}\frac{(1-\phi)^2}{\left[\frac{1}{8} + \frac{\phi}{6} + \frac{\phi^4}{24}\right]}, \tag{34}$$

for $0 \leqslant \hat{r} \leqslant \phi$. These relations mix both bulk and friction coordinates, so that the complete similarity relation (31) cannot be restored for positive values of ϕ.

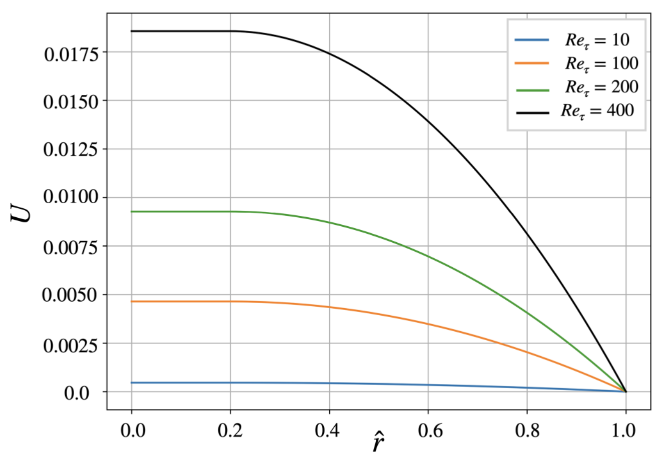

FIG. 5. Velocity profiles for $\phi = 0.2$ and four different values of Re_τ.

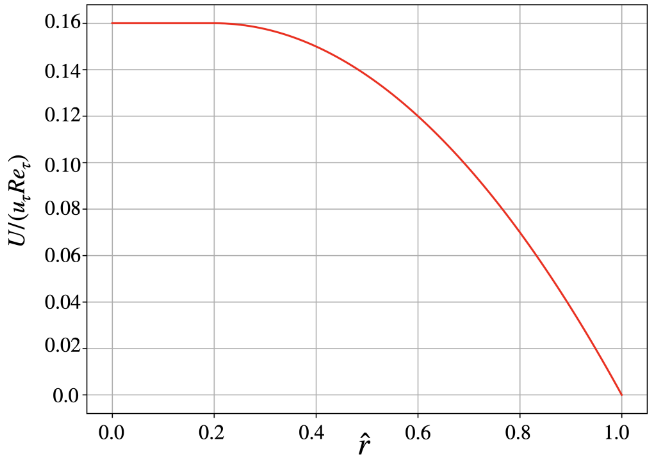

FIG. 6. Collapse of the velocity profiles in Fig. 5—incomplete similarity.

Determining the appropriate definition of the Reynolds number for non-Newtonian fluids has long been a challenging endeavor [9,10]. We believe that our results will persuade readers that, when the pressure gradient is accessible, friction coordinates (and thus Re_τ) should be preferred for reasons of both similarity and simplicity. Our future work will focus on exploring similarity and coordinate choices in other non-Newtonian flow categories, such as turbulent Bingham, as well as laminar and turbulent Herschel-Bulkley flows.

From a practical standpoint, this work's derivation of the incomplete similarity result carries significant implications, enabling straightforward conversion of experimental data from pressure drop terms into flow rate terms. As outlined in the Introduction, flow rate, and pressure drop cannot be jointly prescribed for a given fluid with fixed parameters and constant geometrical setup (for pipe flows, this refers to the radius).

In the scenario of laminar Newtonian flows, a straightforward relationship exists. Multiplying the pressure drop by a factor B_1 results in an equivalent multiplication of the flow rate. This relationship, however, does not hold true for laminar Bingham fluids.

As defined by the friction factor, we have $\bar{U}^2 = (D/2\rho)(\partial p/\partial z)f^{-1}$. This equation indicates that altering the flow rate of a laminar Bingham fluid is not a simple matter of multiplying by a factor B_1, since f is also dependent on τ_y.

The renormalization group identities, as specified in Eq. (30), illustrate an incomplete similarity relation. This implies that if one wants to adjust the flow rate of a laminar Bingham fluid by a factor of B_1, one must proportionally scale both the pressure drop and the yield stress. Therefore, in order to achieve this, in addition to multiplying the pressure drop by B_1, the yield stress must also be multiplied by the same factor.

In conclusion, the behavior of Bingham fluid flow is more intricate than that of Newtonian flows, requiring adjustments to both the pressure drop and yield stress in order to modify the flow rate.

This relationship cannot be deduced from straightforward dimensional analysis, nor from the Buckingham-Reiner equation, which is formulated in purely bulk coordinates. The findings underscore the distinctive rheological characteristics of Bingham fluids in laminar flow, challenging conventional understanding and introducing different parameters to consider in fluid dynamics analysis.

It is important to note that the conclusions drawn from this study may not be directly applicable to turbulent flow conditions. This is primarily due to the limitations of Eqs. (6) and (7), which are based on the assumption of a stationary flow and were formulated through an examination of the plug region.

In the realm of turbulent flows, this stable plug region does not persist, as referenced in Ref. [4]. When the Reynolds number increases for a given Hedstrom number, the flow transitions into a turbulent state. In this turbulent regime, the influence of Reynolds stresses becomes substantial and cannot be ignored in the equations of motion, resulting in the disintegration of the plug region.

For turbulence scenarios, several semiempirical formulas have been proposed, as mentioned in Refs. [4,11,12]. These formulas attempt to capture the complex dynamics of turbulent flows, illustrating the nuanced nature of this flow regime compared to the laminar flow of Bingham fluids.

To conclude, in the second section, we defined the incomplete similarity based on the classical concept introduced by Barenblatt [5]. This idea involves a relationship that emerges when one or more nondimensional quantities are notably small or large. For Bingham fluid flows with a specific He number, laminarity is achieved when Re_τ values are sufficiently low. Semiempirical methodologies have been proposed to represent this transition, as illustrated in Refs. [8,13]. This finding suggests the presence of an *incomplete similarity region* in the $\mathrm{Re}_\tau \times \mathrm{He}$ plane, where the laminar approximation remains valid. As a result, a more comprehensive definition of incomplete similarity is required, potentially benefiting various applications. We plan to create and introduce such an extension in future research.

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[1] E. Bingham, An investigation of the laws of plastic flow, Bull. Bur. Stand. (US) **13**, 309 (1916).

[2] S. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, UK, 2000).

[3] F. White, *Fluid Mechanics* (McGraw-Hill, New York, 2011).

[4] R. Chhabra and J. Richardson, *Non-Newtonian Flow and Applied Rheology: Engineering Applications* (Butterworth-Heinemann, Oxford, UK, 2011).

[5] G. S. Barenblatt, *Scaling*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, UK, 2003).

[6] G. Sanfins, H. Anbarlooei, D. Cruz, and F. Ramos, Complete and incomplete similarity for the mean velocity profile of turbulent pipe and channel flows at extreme Reynolds number, Phys. Fluids **33**, 085118 (2021).

[7] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (CRC Press, Boca Raton, FL, 2018).

[8] P. Swamee and N. Aggarwal, Explicit equations for laminar flow of Bingham plastic fluids, J. Pet. Sci. Eng. **76**, 178 (2011).

[9] R. Thompson and E. Soares, Viscoplastic dimensionless numbers, J. Non-Newtonian Fluid Mech. **238**, 57 (2016).

[10] S. L. D. Kfuri, E. J. Soares, R. L. Thompson, and R. N. Siqueira, Friction coefficients for Bingham and power-law fluids in abrupt contractions and expansions, J. Fluids Eng. **139**, 021203 (2016).

[11] H. Anbarlooei, D. Cruz, F. Ramos, C. Santos, and A. Silva Freire, Phenomenological friction equation for turbulent flow of Bingham fluids, Phys. Rev. E **96**, 023107 (2017).

[12] J. Darby and J. Melson, How to predict the friction factor for flow of Bingham plastics, Chem. Eng. **88**, 59 (1981).

[13] C. Nouar and A. Bottaro, Stability of the flow of a Bingham fluid in a channel: Eigenvalue sensitivity, minimal defects and scaling laws of transition, J. Fluid Mech. **642**, 349 (2010).