PHYSICAL REVIEW FLUIDS **8**, L082601 (2023)

Letter

Dependence of the asymptotic energy dissipation on third-order velocity scaling

Kartik P. Iyer*

Department of Physics, Michigan Technological University, Houghton, Michigan 49931, USA and Department of Mechanical Engineering–Engineering Mechanics, Michigan Technological University, Houghton, Michigan 49931, USA

(Received 1 November 2022; accepted 21 July 2023; published 11 August 2023)

The asymptotic energy dissipation is connected to the third-order scaling of the longitudinal velocity increment magnitude in three-dimensional turbulence via the Kolmogorov 4/5 law. It is shown that the third-order longitudinal absolute velocity increment scaling should not exceed unity for anomalous dissipation to occur, that is for nonvanishing average dissipation in the inviscid limit—also known as the "zeroth law" of turbulence. Conversely, if the third-order longitudinal absolute velocity increment scaling exceeds unity, then the average dissipation must asymptotically vanish and the velocity increment field will becomes symmetric at least at the level of its skewness. This Letter highlights the importance of the third-order absolute velocity increment scaling in assessing the status of the zeroth law of turbulence.

DOI: 10.1103/PhysRevFluids.8.L082601

A surprising phenomenon in three-dimensional incompressible turbulence is that the average energy dissipation does not seem to decay with increasing Reynolds number. To account for this enhanced dissipation, Onsager asserted that if the spatial Hölder exponents of the velocity field are at most one-third, then the average dissipation can be nonzero in the inviscid limit [1]. Following a result by Eyink [2], Onsager's theorem was fully proved by Constantin *et al.* [3]. In particular, it was shown that if the spatial Hölder exponents exceeded one-third, then the average energy dissipation must vanish and energy will be conserved in the inviscid limit [3]. The Besov space formulation of Ref. [3] also meant that if the third-order exponent of the velocity increment magnitude moment exceeded unity, then the average dissipation must vanish asymptotically in the inviscid limit. Almost all such theoretical studies on the asymptotic dissipation starting from that of Onsager until now have related the energy dissipation to the scaling properties of the total velocity increment magnitude field [1–4]. However, due to practical considerations, it is the projections of the total velocity increments along the separation distances, known as the longitudinal velocity increments, that are routinely measured in experiments and simulations [5,6].

The purpose of this Letter is to connect the average energy dissipation to the third-order scaling exponent of the longitudinal velocity increment magnitude moment. This third-order connection between energy dissipation and the longitudinal absolute velocity increment scaling is contrasted to the already known connection between energy dissipation and the total absolute velocity increment

*kiyer@mtu.edu

scaling [3,4]. The implications of this result for the asymmetry of the small-scale velocity field in the inviscid limit are also discussed.

Consider the three-dimensional divergence-free velocity $\mathbf{u}^\nu := \mathbf{u}^\nu(\mathbf{x}, t)$ in a boundaryless domain such as the 3-torus $\mathbb{T}^3$. Here, ν is the kinematic viscosity of the fluid, $\mathbf{x}$ denotes position, and t denotes time. Define the Reynolds number $\mathrm{Re} := u'\ell_0/\nu$, where ℓ_0 is a (fixed) large length scale, $u' := \langle|\mathbf{u}^\nu|^3\rangle^{1/3}$ is the root mean cube of the velocity magnitude $|\mathbf{u}^\nu|$, and $\langle\cdot\rangle$ denote space-time averages. The average energy dissipation rate is given by

$$\epsilon^\nu := \nu\big\langle|\nabla\mathbf{u}^\nu|^2\big\rangle \geqslant 0. \tag{1}$$

Consider the total velocity increment $\boldsymbol{\delta}_\ell\mathbf{u}^\nu$ and its longitudinal component $\delta_\ell u_\parallel^\nu$ between two distinct points separated by vector $\boldsymbol{\ell} \in \mathbb{R}^3$ at distance $\ell = |\boldsymbol{\ell}| > 0 \in \mathbb{R}$, at given time t,

$$\boldsymbol{\delta}_\ell\mathbf{u}^\nu := \mathbf{u}^\nu(\mathbf{x}+\boldsymbol{\ell}, t) - \mathbf{u}^\nu(\mathbf{x}, t), \tag{2}$$

$$\delta_\ell u_\parallel^\nu := \boldsymbol{\delta}_\ell\mathbf{u}^\nu \cdot \boldsymbol{\ell}/|\boldsymbol{\ell}|. \tag{3}$$

At order three, the longitudinal velocity increment moment and the longitudinal absolute velocity increment moment, also known as the longitudinal structure function and the longitudinal absolute structure function, respectively, are defined for any separation ℓ as

$$S_3^\parallel(\ell) := \big\langle(\delta_\ell u_\parallel^\nu)^3\big\rangle, \quad A_3^\parallel(\ell) := \big\langle|\delta_\ell u_\parallel^\nu|^3\big\rangle. \tag{4}$$

We note for later use that at any scale ℓ the two structure functions are related by the triangle inequality as

$$S_3^\parallel(\ell) \leqslant |S_3^\parallel(\ell)| \leqslant A_3^\parallel(\ell). \tag{5}$$

In what follows, we nondimensionalize all physical quantities using ℓ_0, u', and ℓ_0/u' as the relevant length scales, velocity scales, and timescales, respectively. The dimensionless viscosity becomes the inverse Reynolds number $\nu = 1/\mathrm{Re}$, and the asymptotic limit $\mathrm{Re} \to \infty$ is equivalent to the inviscid limit $\nu \to 0$. We note that, in the following analysis, $\langle|\mathbf{u}^\nu|^3\rangle = 1$ due to the above nondimensionalization.

For $\ell > 0$ we start with the trivial identity

$$\frac{4}{5}\epsilon^\nu = -\frac{S_3^\parallel(\ell)}{\ell} + \frac{4}{5}\epsilon^\nu + \frac{S_3^\parallel(\ell)}{\ell}. \tag{6}$$

Using the triangle inequality and (5) we can bound the left-hand side of (6) as

$$\frac{4}{5}\epsilon^\nu \leqslant \frac{A_3^\parallel(\ell)}{\ell} + \left|\frac{4}{5}\epsilon^\nu + \frac{S_3^\parallel(\ell)}{\ell}\right|. \tag{7}$$

Since it follows from (3) that the longitudinal velocity increment magnitude cannot exceed the total velocity increment magnitude at any ℓ, i.e., $|\delta_\ell u_\parallel^\nu| \leqslant |\boldsymbol{\delta}_\ell\mathbf{u}^\nu|$, their corresponding third-order moments are related as

$$A_3^\parallel(\ell) \leqslant \big\langle|\boldsymbol{\delta}_\ell\mathbf{u}^\nu|^3\big\rangle \leqslant 8, \tag{8}$$

where the last inequality follows from Minkowski's inequality. We now assume the following power-law bound,

$$A_3^\parallel(\ell) \leqslant 8\ell^{\xi_{3,\parallel}}, \tag{9}$$

where $\xi_{3,\parallel} > 0$ is the third-order longitudinal absolute scaling exponent, $A_3^\parallel(\ell) \propto \ell^{\xi_{3,\parallel}}$. Our principal assumption here is that bound (9) holds uniformly in viscosity $\nu \in (0, 1]$. Substituting the upper bound (9) into right-hand side of (7), we get

$$\frac{4}{5}\epsilon^{\nu} \leqslant 8\ell^{(\xi_{3,\parallel}-1)} + \left|\frac{4}{5}\epsilon^{\nu} + \frac{S_3^{\parallel}(\ell)}{\ell}\right|. \tag{10}$$

In order to obtain the asymptotic dissipation, we first send $\nu \to 0$ and then send $\ell \to 0$ in (10). Since the left-hand side in (10) is ℓ independent and $\xi_{3,\parallel}$ is assumed to be ν independent, we get

$$\frac{4}{5}\limsup_{\nu\to 0}\epsilon^{\nu} \leqslant 8\lim_{\ell\to 0}\ell^{(\xi_{3,\parallel}-1)} + \lim_{\ell\to 0}\lim_{\nu\to 0}\left|\frac{4}{5}\epsilon^{\nu} + \frac{S_3^{\parallel}(\ell)}{\ell}\right|. \tag{11}$$

Since the precise formulation of the Kolmogorov $4/5$ law is given as [7–11]

$$\lim_{\ell\to 0}\lim_{\nu\to 0}\frac{S_3^{\parallel}(\ell)}{\ell} = -\frac{4}{5}\lim_{\nu\to 0}\epsilon^{\nu}, \tag{12}$$

the second term on the right-hand side in (11) vanishes to give

$$\limsup_{\nu\to 0}\epsilon^{\nu} \leqslant 10\lim_{\ell\to 0}\ell^{(\xi_{3,\parallel}-1)}. \tag{13}$$

Denoting the limit superior of the normalized dissipation as follows,

$$\limsup_{\nu\to 0}\epsilon^{\nu} := \epsilon^{*}, \tag{14}$$

we can finally write (13) as

$$\epsilon^{*} \leqslant 10\lim_{\ell\to 0}\ell^{(\xi_{3,\parallel}-1)}. \tag{15}$$

From (15) it follows that if $\xi_{3,\parallel} > 1$, then $\epsilon^{*} = 0$ and so the limit of the nondimensional dissipation exists and is zero. That is,

$$\text{If }\ \xi_{3,\parallel} > 1 \Rightarrow \epsilon^{*} = \lim_{\nu\to 0}\epsilon^{\nu} = 0. \tag{16}$$

In this case, the normalized dissipation vanishes and energy is conserved in the asymptotic limit $\nu \to 0$. It follows from (16) that a necessary (but not sufficient) condition for ϵ^{*} to be nonzero is that $\xi_{3,\parallel} \leqslant 1$. We note that although (15) clarifies the fate of the asymptotic dissipation ϵ^{*}, it does not provide a conditional decay rate for dissipation. Such a conditional dissipation decay rate is provided in Ref. [4] in terms of the third-order total absolute structure function exponent ξ_3, where $\langle|\boldsymbol{\delta}_{\ell}\mathbf{u}^{\nu}|^{3}\rangle \propto \ell^{\xi_3}$.

Furthermore, in isotropic turbulence the integral scale $\ell_{\rm int}$ is typically defined as [12]

$$\ell_{\rm int} := \frac{3}{2}\frac{\pi}{\langle|\mathbf{u}^{\nu}|^{2}\rangle}\int_0^{\infty}\frac{E(\kappa)}{\kappa}d\kappa \leqslant 1, \tag{17}$$

where κ is the wave-number magnitude and $E(\kappa)$ is the three-dimensional energy spectrum. If the asymptotic dissipation ϵ^{*} vanishes, that is, if (16) holds, then the asymptotic normalized dissipation defined using $\ell_{\rm int}$ must also vanish, since (17) implies

$$\lim_{\nu\to 0}\epsilon^{\nu}\ell_{\rm int} \leqslant \epsilon^{*}. \tag{18}$$

The upper bound (18) is especially useful in direct numerical simulations (DNS), where the evolution of $\ell_{\rm int}$ is often undercut by limited domain sizes ($\sim\ell_0^3$) [13]. In such a scenario, an examination of ϵ^{ν} rather than that of $\epsilon^{\nu}\ell_{\rm int}$ can be more insightful, since if the former vanishes, then the latter must also disappear due to (18).

In the above analysis we have chosen $\langle|\mathbf{u}^{\nu}|^{3}\rangle^{1/3}$ as the relevant velocity scale, since $8\langle|\mathbf{u}^{\nu}|^{3}\rangle$ is an upper bound of the third-order velocity structure functions as given in (8), hence such a normalization is physically relevant to the connection between the mean dissipation and the third-order absolute velocity structure functions. Equivalently, one could choose the root-mean-square (rms) velocity $\langle|\mathbf{u}^\nu|^2\rangle^{1/2}$ as the velocity scale and arrive at the same result (16) for the normalized dissipation $\epsilon^\nu/\langle|\mathbf{u}^\nu|^3\rangle$.

Discussion. The asymptotic behavior of turbulent dissipation is not only a problem of fundamental importance but it is also relevant to energy considerations in modeling turbulent drag, in applications such as aerodynamics and fluid transport in pipelines [14–16]. Until now the asymptotic dissipation has been connected to the third-order total absolute scaling exponent ξ_3, which is seldom measured in empirical studies. In this Letter, we have shown that under the assumption of the Kolmogorov 4/5 law, the normalized dissipation $\epsilon^\nu/\langle|\mathbf{u}^\nu|^3\rangle$ must vanish in the asymptotic limit $\nu \to 0$, if the third-order longitudinal absolute exponent $\xi_{3,\parallel} > 1$. Such a normalization is in contrast to the more traditional normalization of $\epsilon^\nu/\langle|\mathbf{u}^\nu|^2\rangle^{3/2}$, which uses the rms velocity instead [17]. A natural question that arises here is whether the result (16) extends to dissipation normalized using other large-scale velocity statistics as well.

In order to examine the specific choice of the normalized dissipation, we note that the probability density function (PDF) of the velocity field $\mathbf{u}^\nu$ is dominated by nonuniversal large-scale structures; hence its PDF need not be universal. However, all available data of the standardized velocity component $\mathbf{u}^\nu_\alpha/\langle|\mathbf{u}^\nu_\alpha|^2\rangle^{1/2}$ (with zero mean and unity variance) strongly suggest that it has a (slightly) sub-Gaussian PDF core, which is independent of the Reynolds number [17–24]. That is, low-amplitude velocity events $|\mathbf{u}^\nu_\alpha|/\langle|\mathbf{u}^\nu_\alpha|^2\rangle^{1/2} \lessapprox 4$, which tend to dominate low-order normalized moments (at least up until order four), assume a ν-independent, self-similar distribution [18–23]. This low-order, sub-Gaussian, ν-independent behavior is verified in Fig. 1, for the third-order velocity component magnitude ratio $\langle|\mathbf{u}^\nu_\alpha|^3\rangle/\langle|\mathbf{u}^\nu_\alpha|^2\rangle^{3/2}$ in isotropic DNS, which appear to be tightly bounded (from above) by the corresponding Gaussian moment [25]. Furthermore, the velocity magnitude ratios at any order $p \geqslant 1$ are bounded by the corresponding component ratios in isotropic turbulence, $\langle|\mathbf{u}^\nu|^p\rangle/\langle|\mathbf{u}^\nu|^2\rangle^{p/2} \leqslant \langle|\mathbf{u}^\nu_\alpha|^p\rangle/\langle|\mathbf{u}^\nu_\alpha|^2\rangle^{p/2}$, and display a similar Reynolds number independence for $p = 3$, as shown in Fig. 1. Also shown is the corresponding third-order moment ratio for a divergence-free, three-dimensional, Gaussian random field (GRF) with the same two-point velocity correlation as the DNS [26]. The ν-independent GRF third moment closely envelopes the third-order velocity magnitude moment ratio, in a manner that is consistent with the low-order sub-Gaussianity of $\mathbf{u}^\nu$. The upshot is that, due to observed low-order self-similarity of $\mathbf{u}^\nu$, the normalized absolute velocity moment $\langle|\mathbf{u}^\nu|^p\rangle/\langle|\mathbf{u}^\nu|^2\rangle^{p/2}$ for $p \leqslant 4$ remains ν independent and hence result (16) can be extended to other low-order normalizations of dissipation as well. Specifically, the vanishing asymptotic limit (16) can be empirically extended to the pathologically normalized dissipation $\epsilon^\nu/\langle|\mathbf{u}^\nu|^2\rangle^{3/2}$.

Having empirically reconciled the different normalization choices for dissipation, we turn to the significance of this work in the following. At order three, the longitudinal absolute exponent $\xi_{3,\parallel}$ is expected to be larger than the total absolute exponent, i.e., $\xi_{3,\parallel} \geqslant \xi_3$. This is because $\langle|\boldsymbol{\delta}_\ell \mathbf{u}^\nu|^3\rangle$ includes the transverse velocity difference component, which is known to be more intermittent, with a smaller associated exponent, even at order three [27]. It then follows from this work that the asymptotic dissipation must vanish even if the total absolute exponent $\xi_3 \leqslant 1$, as long as the Kolmogorov 4/5 law is valid and $\xi_{3,\parallel} > 1$. Accordingly, it follows on empirical grounds that result (16) is sharper than that of Refs. [3,4].

Another implication of this work is for the asymptotic longitudinal velocity difference field. If $\xi_{3,\parallel} > 1$ and $\epsilon^* = 0$, it then follows from the exact Kolmogorov 4/5 law that asymptotically, $S_3^\parallel(\ell)/\ell \to 0$ [10,11]. Since the longitudinal velocity increment is known to scale linearly, $S_3^\parallel(\ell) \propto \ell^1$ in the inertial range [20,25], it must follow that, if $\xi_{3,\parallel} > 1$, then the third-order longitudinal structure function $S_3^\parallel(\ell) \to 0$ as $\nu \to 0$, due to cancellations in its power-law prefactor. This implies that, if $\xi_{3,\parallel} > 1$, the velocity increment field will asymptotically become symmetric, at least at the level of its skewness. In the Lagrangian context, this symmetrization will result in the asymptotic mean-square particle displacement becoming time reversible [28]. This space-time

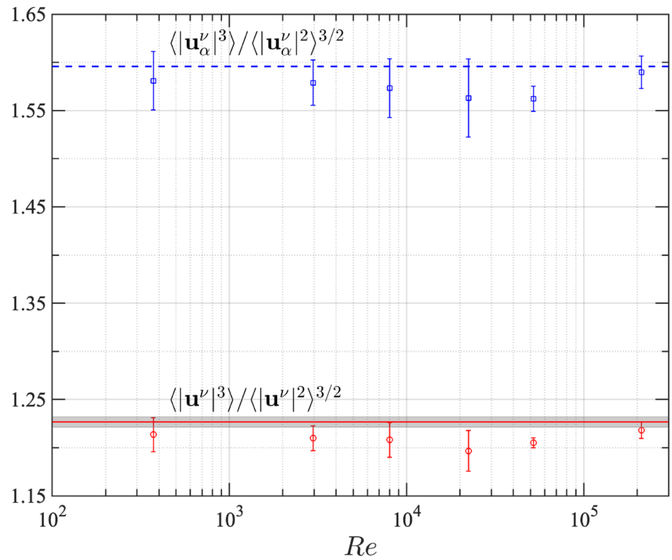

FIG. 1. Third-order moments of (squares) velocity component magnitude $|\mathbf{u}_\alpha^\nu|$ and (circles) velocity vector magnitude $|\mathbf{u}^\nu|$, normalized by the respective second-order moments, plotted against Reynolds number $\mathrm{Re} := \langle|\mathbf{u}^\nu|^3\rangle^{1/3}\ell_0/\nu$ on log-lin scales, from DNS of three-dimensional, homogeneous isotropic turbulence on periodic cubes with side length $2\ell_0 = 2\pi$; ν is the kinematic viscosity [25]; statistics of $\mathbf{u}_\alpha^\nu$ are averaged over the three Cartesian directions. Error bars correspond to the standard deviation of the temporal fluctuations of the normalized moments in the statistically steady state. The dashed line at $\pi^{1/4}\Gamma(2)/\Gamma(1.5)^{3/2}$ is the normalized third-order absolute Gaussian moment, where $\Gamma(\cdot)$ is the gamma function; the solid line is the normalized third-order velocity magnitude moment for a three-dimensional, synthetic, Gaussian random field (GRF); the GRF is constructed on a 8192^3 grid with the same two-point correlation as that for the 8192^3 DNS at $\mathrm{Re} = 2.1 \times 10^5$ [26]. The shaded region is the statistical error bar (standard deviation) for the normalized absolute third-order GRF moment.

recovery of reversibility, at least up to order three, in the vanishing dissipation case can have important consequences in modeling the asymptotic limit $\nu \to 0$.

In the alternate scenario, where $\xi_{3,\|} \leqslant 1$ and $\epsilon^* > 0$, the small-scale asymmetry will persist at all nontrivial orders. In particular, it will follow from the Kolmogorov 4/5 law that the velocity increment field will have nonvanishing (negative) skewness in the asymptotic limit, $S_3^{\|}(\ell) < 0$ as $\nu \to 0$. The nonzero asymptotic dissipation will manifest in temporal asymmetry of the kinetic energy of particles in the asymptotic limit, $\nu \to 0$, that is different short-time particle dispersion in forward and backward time, in the Lagrangian setting [28].

Finally, a few remarks about the Reynolds number scaling of the asymptotic dissipation from experiments and simulations are in order. A majority of the empirical studies with few exceptions have observed a nontrivial independence of the normalized turbulent energy dissipation on the Reynolds number—this phenomenon, known as dissipative anomaly, has been accorded the status of the "zeroth law" of turbulence [29–34]. A direct assessment of dissipation scaling is challenging because of the large timescales of the quantities involved. This translates into longer statistically steady-state run times at ever increasing Reynolds numbers for both experiments and simulations, a prohibitively expensive proposition.

In contrast, probing the validity of the zeroth law using the third-order longitudinal absolute scaling exponent is more favorable due to the following reasons. First, inertial range moments evolve over shorter timescales than large-scale quantities, which means that experiments and simulations require shorter run times to capture their temporal evolution [35]. Second, third-order moments have less stringent resolution requirements than higher-order moments, hence they can be measured with greater accuracy. Lastly, longitudinal velocity differences are one-dimensional cuts from the total velocity difference tensor, hence are more feasible to measure in experiments. Despite these advantages, the third-order longitudinal absolute scaling exponent has largely been overlooked, with some exceptions [20,23,36–40]. Nevertheless, almost none of the empirical studies so far have connected the third-order absolute exponents to the phenomenon of dissipative anomaly.

Consequently, in this Letter we have highlighted the importance of the third-order absolute exponents to the Reynolds number scaling of turbulent dissipation. Under the assumption of the Kolmogorov 4/5 law, we have shown that, if $\xi_{3,\|} > 1$, the mean turbulent energy dissipation must vanish in the infinite Reynolds number limit, that is, the zeroth law of turbulence will be violated. Alternatively, if the third-order longitudinal absolute exponent $\xi_{3,\|} \leqslant 1$, then dissipative anomaly can hold strictly. An examination of the Reynolds number evolution of the third-order scaling of both longitudinal and total velocity increment magnitudes might be pivotal in asserting the fate of the zeroth law of turbulence. Such a study over a wide range of Reynolds number will be reported as future work.

I am grateful to T. D. Drivas, G. L. Eyink, K. R. Sreenivasan, and P. K. Yeung for helpful discussions. This work used Stampede2 at Texas Advanced Computing Center through allocation PHY200084 from the Advanced Cyberinfrastructure Coordination Ecosystem: Services & Support (ACCESS) program, which is supported by National Science Foundation Grants No. 2138259, No. 2138286, No. 2138307, No. 2137603, and No. 2138296.

[1] L. Onsager, Statistical hydrodynamics, Nuovo Cimento **6**, 279 (1949).
[2] G. L. Eyink, Energy dissipation without viscosity in ideal hydrodynamics I. Fourier analysis and local energy transfer, Physica D **78**, 222 (1994).
[3] P. Constantin, W. E, and E. S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, Commun. Math. Phys. **165**, 207 (1994).
[4] T. D. Drivas and G. L. Eyink, An Onsager singularity theorem for Leray solutions of incompressible Navier–Stokes, Nonlinearity **32**, 4465 (2019).
[5] K. R. Sreenivasan and R. A. Antonia, The phenomenology of small-scale turbulence, Annu. Rev. Fluid Mech. **29**, 435 (1997).
[6] T. Ishihara, T. Gotoh, and Y. Kaneda, Study of high-Reynolds number isotropic turbulence by direct numerical simulation, Annu. Rev. Fluid Mech. **41**, 165 (2009).
[7] A. N. Kolmogorov, Dissipation of energy in locally isotropic turbulence, Dokl. Akad. Nauk. SSSR **32**, 16 (1941).
[8] Q. Nie and S. Tanveer, A note on the third-order structure functions in turbulence, Proc. R. Soc. London, Ser. A **455**, 1615 (1999).
[9] G. L. Eyink, Local 4/5-law and energy dissipation anomaly in turbulence, Nonlinearity **16**, 137 (2003).
[10] J. Bedrossian, M. C. Zelati, S. Punshon-Smith, and F. Weber, A sufficient condition for the Kolmogorov 4/5 law for stationary Martingale solutions to the 3D Navier–Stokes equations, Commun. Math. Phys. **367**, 1045 (2019).
[11] T. D. Drivas, Self-regularization in turbulence from the Kolmogorov 4/5-law and alignment, Philos. Trans. R. Soc. A **380**, 20210033 (2022).
[12] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, UK, 2000).
[13] S. M. de Bruyn Kops and J. J. Riley, Direct numerical simulation of laboratory experiments in isotropic turbulence, Phys. Fluids **10**, 2125 (1998).
[14] G. L. Eyink and K. R. Sreenivasan, Onsager and the theory of hydrodynamic turbulence, Rev. Mod. Phys. **78**, 87 (2006).

[15] J. C. Vassilicos, Dissipation in turbulent flows, Annu. Rev. Fluid Mech. **47**, 95 (2015).
[16] B. Dubrulle, Beyond Kolmogorov cascades, J. Fluid Mech. **867**, P1 (2019).
[17] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, UK, 1953).
[18] J. Jiménez, A. Wray, P. Saffman, and R. Rogallo, The structure of intense vorticity in isotropic turbulence, J. Fluid Mech. **255**, 65 (1993).
[19] A. Noullez, G. Wallace, W. Lempert, R. Miles, and U. Frisch, Transverse velocity increments in turbulent flow using the RELIEF technique, J. Fluid Mech. **339**, 287 (1997).
[20] K. R. Sreenivasan and B. Dhruva, Is there scaling in high-Reynolds-number turbulence? Prog. Theor. Phys. Suppl. **130**, 103 (1998).
[21] J. Jiménez, Turbulent velocity fluctuations need not be Gaussian, J. Fluid Mech. **376**, 139 (1998).
[22] G. S. Lewis and H. L. Swinney, Velocity structure functions, scaling, and transitions in high-Reynolds-number Couette-Taylor flow, Phys. Rev. E **59**, 5457 (1999).
[23] T. Gotoh, D. Fukayama, and T. Nakano, Velocity field statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation, Phys. Fluids **14**, 1065 (2002).
[24] H. Mouri, M. Takaoka, A. Hori, and Y. Kawashima, Probability density function of turbulent velocity fuctuations in a rough-wall boundary layer, Phys. Rev. E **68**, 036311 (2003).
[25] K. P. Iyer, K. R. Sreenivasan, and P. K. Yeung, Scaling exponents saturate in three-dimensional isotropic turbulence, Phys. Rev. Fluids **5**, 054605 (2020).
[26] K. P. Iyer, S. S. Bharadwaj, and K. R. Sreenivasan, The area rule for circulation in three-dimensional turbulence, Proc. Natl. Acad. Sci. USA **118**, e2114679118 (2021).
[27] B. Dhruva, Y. Tsuji, and K. R. Sreenivasan, Transverse structure functions in high-Reynolds-number turbulence, Phys. Rev. E **56**, R4928 (1997).
[28] T. D. Drivas, Turbulent cascade direction and Lagrangian time-asymmetry, J. Nonlinear Sci. **29**, 65 (2019).
[29] K. R. Sreenivasan, On the scaling of the turbulence energy dissipation rate, Phys. Fluids **27**, 1048 (1984).
[30] G. Zocchi, P. Tabeling, J. Maurer, and H. Willaime, Measurement of the scaling of the dissipation at high Reynolds numbers, Phys. Rev. E **50**, 3693 (1994).
[31] K. R. Sreenivasan, An update on the energy dissipation rate in isotropic turbulence, Phys. Fluids **10**, 528 (1998).
[32] B. R. Pearson, P.-Å. Krogstad, and W. van de Water, Measurements of the turbulent energy dissipation rate, Phys. Fluids **14**, 1288 (2002).
[33] S. Goto and J. C. Vassilicos, The dissipation rate coefficient of turbulence is not universal and depends on the internal stagnation point structure, Phys. Fluids **21**, 035104 (2009).
[34] B. Saint-Michel, E. Herbert, J. Salort, C. Baudet, M. Bon Mardion, P. Bonnay, M. Bourgoin, B. Castaing, L. Chevillard, F. Daviaud, P. Diribarne, B. Dubrulle, Y. Gagne, M. Gibert, A. Girard, B. Hébral, T. Lehner, and B. Rousset, Probing quantum and classical turbulence analogy in von Kármán liquid helium, nitrogen, and water experiments, Phys. Fluids **26**, 125109 (2014).
[35] P. K. Yeung and K. Ravikumar, Advancing understanding of turbulence through extreme-scale computation: Intermittency and simulations at large problem sizes, Phys. Rev. Fluids **5**, 110517 (2020).
[36] S. I. Vainshtein and K. R. Sreenivasan, Kolmogorov's 4/5th Law and Intermittency in Turbulence, Phys. Rev. Lett. **73**, 3085 (1994).
[37] F. Belin, P. Tabeling, and H. Willaime, Exponents of the structure functions in a low temperature helium experiment, Physica D **93**, 52 (1996).
[38] B. R. Dhruva, An experimental study of high Reynolds number turbulence in the atmosphere, Ph.D. thesis, Yale University, 2000.
[39] D. Geneste, H. Faller, F. Nguyen, V. Shukla, J.-P. Laval, F. Daviaud, E.-W. Saw, and B. Dubrulle, About universality and thermodynamics of turbulence, Entropy **21**, 326 (2019).
[40] K. R. Sreenivasan, K. P. Iyer, and A. Vinodh, Asymmetry of velocity increments in turbulence, Phys. Rev. Res. **4**, L042002 (2022).