Letter

# Feasibility of the Obukhov-Bolgiano scaling in buoyancy affected turbulence

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The question of whether buoyancy forces arising in a turbulent flow in a stratified medium can influence the cascade in the inertial range was posed independently by Obukhov and Bolgiano more than 60 years ago, but has not been answered up to now. Here, we show that the Obukhov-Bolgiano scaling (OBs) is possible only if the injection rates of kinetic energy and thermal variance by force acting at a large scale, strictly correspond to the given buoyancy parameter. We found an analytical solution of energy balance equations which demonstrates both the OBs and Kolmogorov scaling as well as a transition between them. A new characteristic scale is introduced which characterizes the role of buoyancy in the spectral cascade of energy and determines the Bolgiano scale. The feasibility of the solution is illustrated by a fixed point of a shell model of turbulence. However, the quasistationary solutions obtained within the framework of essentially dynamic shell models do not reveal the OBs. We come to conclusion that the idealized conditions required to implement the OBs should be extended considering real turbulent flows.

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## I. INTRODUCTION

More than 60 years ago, Obukhov [1] and Bolgiano [2] almost simultaneously published papers, in which the problem of the influence of density variations on the cascade processes in small-scale turbulence was first considered. They concern turbulence, in which the Archimedes forces become significant on the scales related to the inertial range, in which, according to Kolmogorov's hypotheses (if the density is homogeneous), only inertial forces determine the turbulence dynamics, and the statistical properties of turbulence at a given scale are governed only by the kinetic energy dissipation rate $\varepsilon$ and the scale $l$ itself [3]. In a density stratified medium, the inertial range ceases to be such, since the buoyancy forces interfere in the process.

Assuming that the density variations on a given scale are due to the inhomogeneity of the temperature $\theta_l$, and intending to take into account the possibility of the influence of the Archimedes forces on the dynamics of velocity fluctuations in the inertial range, one should take into consideration the "buoyancy parameter" $\xi = g\alpha$ ($g$ is the acceleration of gravity, $\alpha$ is the coefficient of thermal expansion of the medium) and the rate of dissipation of the thermal variance (mean-square fluctuations of temperature) $\varepsilon_\theta$. It is expected that the buoyancy forces will disturb the spectral energy cascade and change the statistical properties of turbulence in the large-scale range $l_f > l > l_B$, where, $l_f$ is the forced (outer) scale of turbulence. For the small-scale boundary $l_B$, based on the analysis of the dimension of defining quantities, Obukhov found an estimate, which in terms of wave numbers, $k = 2\pi/l$, reads

$$k_{\mathrm{B}} = C_B \varepsilon^{-5/4} \varepsilon_\theta^{3/4} \xi^{3/2}, \tag{1}$$

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and evaluated this scale for the atmosphere boundary layer [1]. The scale, i.e., the wave number $k_B$, is known as the Bolgiano scale.

Bolgiano considered the turbulence in stably stratified atmosphere. He stated that the turbulent motion in a stratified fluid converts kinetic energy to potential energy, providing energy escape from the inertial range. He noted that these fluid motions generate also deviation of temperature (density) from the mean distribution, changing the spectral flux of the thermal variance. However, he supposed that $\varepsilon_\theta$ (in our notation) is a governing parameter for the spectral range under consideration. Assuming that the statistical properties in the range $k_f < k < k_B$ are determined by $\xi$, $\varepsilon_\theta$ and the wave number $k$, Bolgiano derived power laws for the spectral density of velocity and temperature fluctuations [2]

$$E(k) = C_1 \varepsilon_\theta^{2/5} \xi^{4/5} k^{-11/5}, \tag{2}$$

$$E_\theta(k) = C_2 \varepsilon_\theta^{4/5} \xi^{-2/5} k^{-7/5}. \tag{3}$$

Actually, the search for spectral laws (2) in real flows started with the development of measuring and computing technology in the 1990s. Fragments of spectra with the slopes similar to (2) were detected both in the laboratory [4,5], and numerical experiments [6–8]. Nevertheless, no reliable confirmation of the existence of the Obukhov-Bolgiano scaling (OBs) has been found. The identification problems of the OBs and the possibility of its manifestation were discussed in detail in the review [9]. The authors of Ref. [10] noted a fundamental difference between the stably and unstably stratified fluid and studied the possibility of realizing OBs in stably stratified convection using a shell model of turbulence. However, the condition for the balance between the work done by the buoyancy force and the nonlinear transfer of the energy of the velocity fluctuations, which is a necessary condition for the existence of the OBs, was not fulfilled. The observed slopes of the spectrum, close to the “-11/5” and “-7/5” laws, can be explained by the transition from the undeveloped inertial range to the dissipative one [11].

In this paper, we return to the question of feasibility of the extended range of scales with OBs in fully developed turbulence. Particular attention is paid to the fulfillment of the scale-by-scale balance of spectral fluxes of the kinetic energy and thermal variance. We show that OBs is possible only when the temperature fluctuations are anticorrelated with the velocity fluctuation (equivalent to Bolgiano’s suggestion that the buoyancy forces exhaust the kinetic energy, converting it to potential one) and if the injection rates of kinetic energy and thermal variance, acting at the forcing scale, are strongly linked to the given buoyancy parameter. We check the realizability of the spectral distributions (2) and (3) using the shell model of turbulence.

## II. PROBLEM STATEMENT AND PHENOMENOLOGY

We consider a stationary forced turbulence under the influence of thermogravitational convection. The scale-by-scale energy transfer and interactions in Fourier space are convenient to analyze using the balance equation for the spectral energy flux $d\Pi(k)/dk = \mathcal{I}(k) - \mathcal{D}(k)$, where $\mathcal{I}(k)$ is the energy injection rate by external or internal forces and $\mathcal{D}(k)$ is the dissipation rate at wave number $k$. The basic relations for the spectral transfer analysis can be found in Ref. [12] and the particular set of equations for the buoyant flows are derived in some self-contained papers [13,14]. In the frame of our problem, we assume that external forces act at a large scale, denoted by the wave number $k_f$, and provide constant injection rates of kinetic energy $I$ and thermal variance $I_\theta$. The spectral densities of the kinetic energy $E(k)$ and thermal variance $E_\theta(k)$ are related to the corresponding spectral fluxes $\Pi(k)$ and $\Pi_\theta(k)$ by the balance equations

$$\frac{\mathrm{d}\Pi(k)}{\mathrm{d}k} + 2\nu k^2 E(k) = I\delta(k-k_f) - 2\xi[E(k)E_\theta(k)]^{1/2}, \tag{4}$$

$$\frac{\mathrm{d}\Pi_\theta(k)}{\mathrm{d}k} + 2\chi k^2 E_\theta(k) = I_\theta \delta(k-k_f). \tag{5}$$

Here, there are two important simplifications which have been implemented in order to reproduce Bolgiano's scenario. Firstly, the buoyancy term (last one) in Eq. (4) is written such that the fluctuations of velocity and temperature are fully anticorrelated. It is important to note that this assumption is a commonplace, explicitly or implicitly made to obtain the OBs. Let us refer, as an example, to the work of Yakhot, who derived a generalized form of Kolmogorov "4/5" law for convective turbulence [15]. This generalized equation directly includes the cross correlation of velocity and temperature fluctuations (Eq. (10) in Ref. [15]) and restores the Kolmogorov equation if the correlations vanish. The subsequent dimensional analysis of this equation gave the OBs, but implicitly involved the assumption that velocity and temperature fluctuations are strictly correlated. Secondly, any stratification is neglected in Eq. (5). It means that the spectrum of temperature fluctuations results from the large-scale source of fluctuations $I_\theta$ (artificially anticorrelated to velocity fluctuations) and the turbulent cascade described by $\Pi_\theta(k)$.

To close the problem, we assume that spectral transfers are dominated by local interactions, which implies that the spectral fluxes $\Pi(k)$ and $\Pi_\theta(k)$ can be expressed in terms of energies $E(k)$ and $E_\theta(k)$. Then, the dimensional analysis gives

$$E(k) = C\Pi(k)^{2/3}k^{-5/3}, \tag{6}$$

$$E_\theta(k) = C_\theta \Pi(k)^{-1/3}\Pi_\theta(k)k^{-5/3}. \tag{7}$$

These relations admit any spectral law, but tend to Kolmogorov's law, if the spectral fluxes are constant in a certain range of scales.

For the considered forcing, the spectral fluxes are defined at the largest scale $\Pi(k_f) = I$, and $\Pi_\theta(k_f) = I_\theta$. Then, in the dissipation-free case, i.e., $\nu = \chi = 0$, Eqs. (4) and (5) for $k > k_f$ have the following solution:

$$\Pi(k) = I(1 - (k_\mathrm{b}/k_f)^{2/3} + (k_\mathrm{b}/k)^{2/3})^{6/5}, \tag{8}$$

$$\Pi_\theta(k) = I_\theta = \varepsilon_\theta, \tag{9}$$

where a new characteristic scale (let us call it the outer buoyancy scale) occurs

$$k_\mathrm{b} = (25CC_\theta/4)^{3/4}I^{-5/4}I_\theta^{3/4}\xi^{3/2}. \tag{10}$$

One can see that $k_\mathrm{b}$ scales with governing parameters similar to the Bolgiano scale $k_\mathrm{B}$, but instead of the kinetic energy dissipation rate $\varepsilon$ it includes the kinetic energy injection rate $I$. The latter point is crucial because $I$ is actually the governing parameter of the problem. The outer buoyancy scale can vary in the range $0 \leqslant k_\mathrm{b} \leqslant k_f$, because at $k_\mathrm{b} > k_f$ the spectral flux (8) can be negative which is beyond the scope of the considered model.

Let us consider two extreme forms of the solution (8). If the buoyancy forces vanish, $k_\mathrm{b} \to 0$ (this can result from a relatively strong $I$, a weak $I_\theta$, or a small $\xi$), $\Pi(k)$ becomes constant, i.e., $\Pi(k) = I = \varepsilon$, and the Kolmogorov scaling (Ks) is established

$$E(k) = C\varepsilon^{2/3}k^{-5/3}, \quad E_\theta(k) = C_\theta\varepsilon^{-1/3}\varepsilon_\theta k^{-5/3}. \tag{11}$$

If the outer buoyancy scale matches the forcing scale, i.e., $k_\mathrm{b} = k_f$, then Eq. (8) reads

$$\Pi(k) = I(k_f/k)^{4/5} \tag{12}$$

and Eqs. (6) and (7) reproduce the OBs, Eqs. (2) and (3) with prefactors

$$C_1 = (5/2)^{4/5}C^{7/5}C_\theta^{2/5}, \quad C_2 = (2/5)^{2/5}C^{-1/5}C_\theta^{4/5}.$$

Note, that the OBs takes place over the whole range of scales only at a certain relation of the governing parameters, namely

$$CC_\theta(5\xi/2)^2 I_\theta = k_f^{4/3}I^{5/3}. \tag{13}$$

A limited buoyancy affected spectral range can appear if the outer buoyancy scale is larger than the injection scale, i.e., $k_{\mathrm{b}} < k_f$. Then, Ks and OBs occur with a transition at the Bolgiano scale $k_{\mathrm{B}}$ provided by the relation

$$(k_{\mathrm{b}}/k_{\mathrm{B}})^{2/3} = 1 - (k_{\mathrm{b}}/k_f)^{2/3}, \tag{14}$$

which yields

$$k_{\mathrm{B}} = k_{\mathrm{b}}\sigma^{-3/2}, \tag{15}$$

where the parameter $\sigma \equiv 1 - (k_{\mathrm{b}}/k_f)^{2/3}$ shows how far $k_{\mathrm{B}}$ is from $k_{\mathrm{b}}$. This solution becomes equivalent to Eq. (1) when the kinetic energy dissipation rate is determined by the spectral flux $\Pi(k)$ in the Kolmogorov's range, formally, as the limit of (8) at $k \to \infty$, i.e.,

$$\varepsilon = I(1 - (k_{\mathrm{b}}/k_f)^{2/3})^{6/5} = I\sigma^{6/5} \tag{16}$$

and $C_{\mathrm{B}} = (5/2)^{3/2}(CC_\theta)^{3/4}$. Obviously, one can expect the turbulence to be affected by buoyancy only if $k_{\mathrm{B}} > k_f$, so that the OBs is observable for a rather narrow range of the control parameter

$$2^{-2/3}k_f < k_{\mathrm{b}} \leqslant k_f. \tag{17}$$

Next, let us consider finite $\nu$ and $\chi$ which prescribe the corresponding dissipation wave lengths $k_\nu$ and $k_\chi$. On the assumption that $k_\nu, k_\chi \gg k_{\mathrm{B}}$, meaning that the Ks establishes at large wave numbers, the Kolmogorov estimate of the dissipation scale can be written as

$$k_\nu \approx \varepsilon^{1/4}\nu^{-3/4} \approx I^{1/4}\sigma^{3/10}\nu^{-3/4},$$

which shows that a smaller $\sigma$ (more extended OBs range) provides a stronger sink of kinetic energy, which leads to an increase of the viscous scale.

We analyze the numerical solutions of Eqs. (4) and (5) for fixed $k_f = 1, I = 1, I_\theta = 1, C = C_\theta = 1$ and varying $\nu$, $\chi$ and $\sigma$. Figure 1 shows the results for given $\nu = \chi = 10^{-10}$ and different values of $\sigma = 0.5, 0.05, 0.005$, which gradually approaches zero extending the OBs range. We remind that the value of $\sigma^{6/5}$ defines the portion of injected kinetic energy that remains in the spectral flux at $k > k_{\mathrm{B}}$. If the OBs range takes three decades, this portion reduces to less than 1%. Also one can see that the transition of power law from OBs to Ks nearby $k_{\mathrm{B}}$ does not occur abruptly and covers two decades of scales.

At a very small $\sigma$ or large dissipation the OBs may end at $k < k_{\mathrm{B}}$. Assuming that $\nu \geqslant \chi$ one gets for the upper wave number limit of the OB range the following estimate:

$$k_\nu \approx \varepsilon_\theta^{1/8}\xi^{1/4}\nu^{-5/8}.$$

If $\nu < \chi$ the upper wave number limit of the OB range is defined by

$$k_\chi \approx \varepsilon_\theta^{1/8}\xi^{1/4}\chi^{-5/8}.$$

Figures 2 and 3 show the results of numerical solutions of Eqs. (4) and (5) with $\sigma = 0$ and different $\nu$ and $\chi$. For $\nu < \chi$ the Ks is restored at $k > k_\chi$.

## III. NUMERICAL SIMULATIONS USING SHELL MODELS OF TURBULENCE

Shell models are an appropriate tool for studying spectral cascades in fully developed turbulence (see for a review, see Ref. [16]). There are a number of works in which shell models were used to simulate the buoyancy driven turbulence [10,17–20]. We employ shell models to validate our theoretical results.

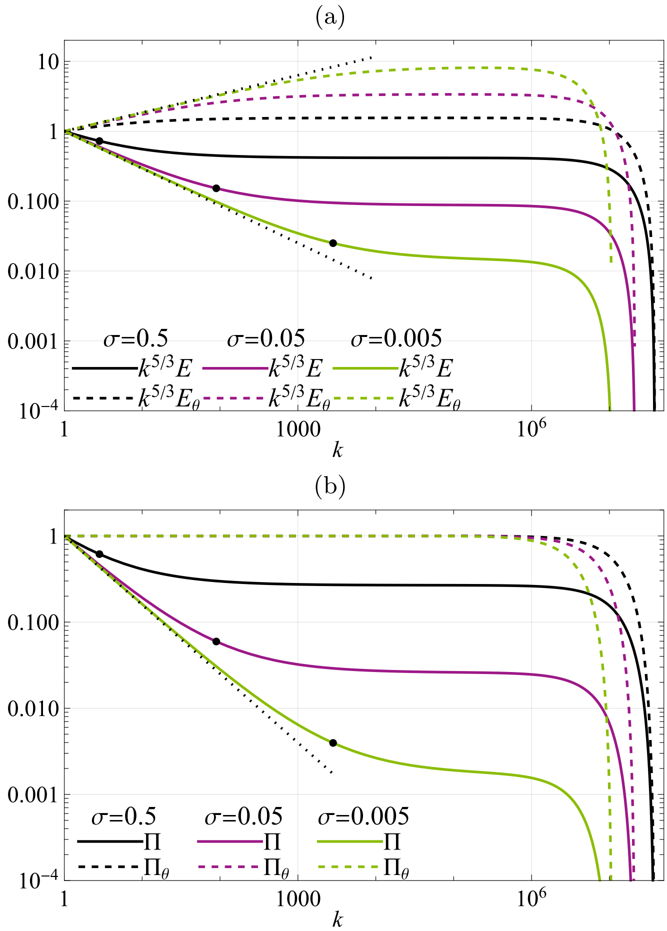


FIG. 1. Kinetic energy and thermal variance spectra, compensated by the Ks (a) and corresponding spectral fluxes (b), obtained by numerical solution of Eqs. (4) and (5) for fixed $\nu = \chi = 10^{-10}$ and varying $\sigma$. Dotted lines correspond to BOs. Large black dots show the Bolgiano scale $k_B$ for each $\sigma$.

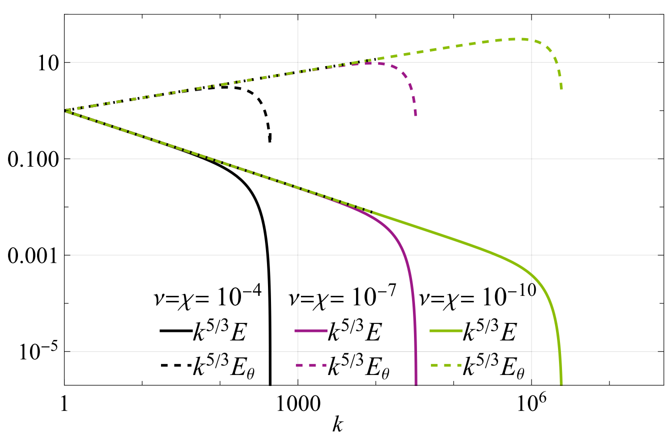


FIG. 2. Compensated spectra (similar to the upper panel in Fig. 1) for $\sigma = 0$ and varying $\chi = \nu$.

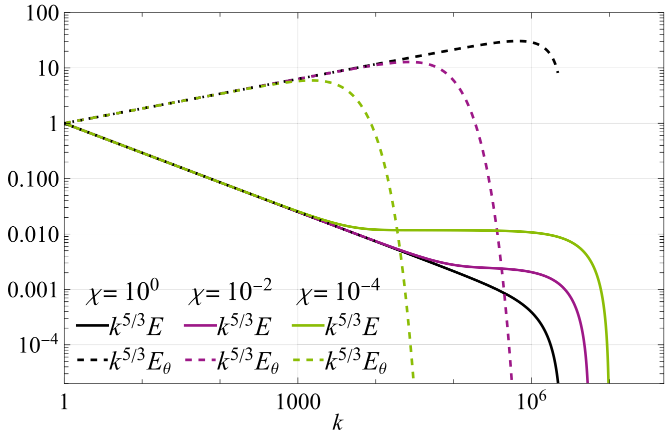


FIG. 3. The same, as in Fig. 2, but for fixed $\nu = 10^{-10}$ and varying $\chi > \nu$.

The shell model approach is based on the idea to sample the spectral densities $E(k)$ and $E_\theta(k)$ using a series of variables $U_n$ and $T_n$. They are supposed to represent the kinetic energy and thermal variance in each shell in the vicinity of the corresponding wave number $k_n = \lambda^n$

$$\frac{U_n^2}{2} = \int_{k_n}^{k_{n+1}} E(k)dk, \tag{18}$$

$$\frac{T_n^2}{2} = \int_{k_n}^{k_{n+1}} E_\theta(k)dk, \tag{19}$$

where $\lambda$ is the shell thickness in logarithmic scale.

The shell model equations for buoyancy affected flows can be written as

$$\frac{\mathrm{d}U_n}{\mathrm{d}t} = \tilde{C}W_n(U,U) - \nu k_n^2 U_n + f_n^U - \xi T_n, \tag{20}$$

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} = \tilde{C}_\theta W_n(U,T) - \chi k_n^2 T_n + f_n^T, \tag{21}$$

where $W_n$ are the nonlinear terms, the second terms correspond to dissipation, third terms describe forcing, and last term in Eq. (20) is the buoyancy force.

Several forms have been suggested for the nonlinear term $W_n$. Without going into detail, we only note that $W_n$ should necessarily obey the same conservation laws as the corresponding hydrodynamical equations in the dissipationless limit.

First of all we consider the shell model, introduced in the pioneer work of Desnianskii and Novikov [21]

$$W_n(X,Y) = k_n(X_{n-1}Y_{n-1} - \lambda X_n Y_{n+1}). \tag{22}$$

In the limit $\nu = 0$, $\chi = 0$ and $\xi = 0$, Eqs. (20) and (21) conserve the kinetic energy $\frac{1}{2}\sum U_n^2$ and thermal variance $\frac{1}{2}\sum T_n^2$. This shell model is the simplest one, because it operates with one real variable per shell and therefore does not allow additional conservative quantities such as helicity. We choose it because this model allows for steady solutions.

There is an important technical question which is usually disregarded in shell model studies. It concerns the definition of energy and spectral flux for a given shell. The fact is that each shell covers the wave number range $k_n < k < k_{n+1}$ and the shell energy should be associated with an inner point

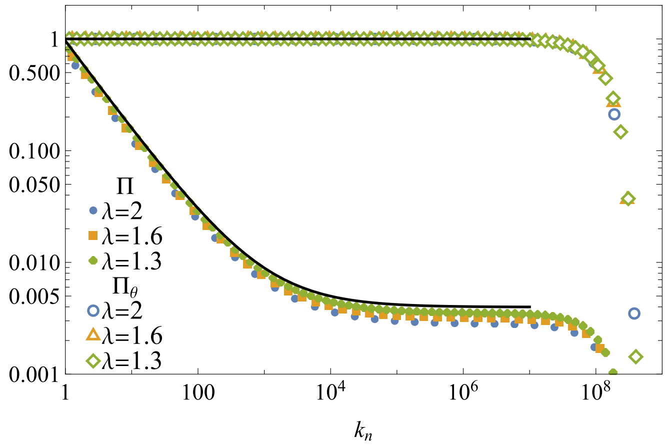


FIG. 4. Spectral fluxes found from numerical solution of shell model for fixed $\nu = \chi = 10^{-10}$, $\sigma = 0.01$ and different λ. Solid black line corresponds to the analytical solution.

of this range, while the spectral fluxes, derived from Eqs. (20) and (21) as

$$\Pi_n^U = \tilde{C} k_{n+1} U_n^2 U_{n+1}, \tag{23}$$

$$\Pi_n^T = \tilde{C}_\theta k_{n+1} U_n T_n T_{n+1}, \tag{24}$$

are associated with the boundary $k = k_n$. Thus, comparing shell variables with spectral densities (6) and (7) and fluxes (8) and (9) we get

$$E(k_{n+1/2}) \approx \frac{U_n^2}{2k_{n+1/2}\ln\lambda}, \quad \Pi(k_n) \approx \Pi_n^U, \tag{25}$$

$$E_\theta(k_{n+1/2}) \approx \frac{T_n^2}{2k_{n+1/2}\ln\lambda}, \quad \Pi_\theta(k_n) \approx \Pi_n^T. \tag{26}$$

The coefficients $\tilde{C}$ and $\tilde{C}_\theta$ in Eqs. (20) and (21) can be related to the coefficients $C$ and $C_\theta$, introduced in Eqs. (6) and (7),

$$\tilde{C} = (2\ln\lambda)^{-3/2} C^{-3/2} \lambda^{-2/3}, \tag{27}$$

$$\tilde{C}_\theta = (2\ln\lambda)^{-3/2} C^{-1/2} C_\theta^{-1} \lambda^{-2/3}. \tag{28}$$

Then, the Ks ($U_n = U_0 k_n^{-1/3}$, $T_n = T_0 k_n^{-1/3}$) appears at $\xi = 0$ in the inertial range of scales. The OBs ($U_n = U_0 k_n^{-3/5}$, $T_n = T_0 k_n^{-1/5}$) can be found under the condition that

$$CC_\theta \left( \frac{2\xi\lambda^{29/45}\ln\lambda}{\lambda^{4/5} - 1} \right)^2 I_\theta = k_f^{4/3} I^{5/3}. \tag{29}$$

This relation converges to the analytically derived Eq. (13) in the limit $\lambda \to 1$.

Numerical simulations of shell equations (20)–(22) confirm that a steady solution exits at low and moderate $\sigma$. Figure 4 shows that the numerical solution at $\sigma = 0.01$ (corresponding to $k_\mathrm{b} = 0.985 k_f$) approaches the analytical solution in the limit $\lambda \to 1$. A further decrease in $\sigma$ results in the loss of stability.

This shell model reveals a stable OBs at $\sigma \geqslant 0.005$. The value $\sigma = 0.005$ corresponds to the Bolgiano scale $k_B \approx 3000$. If the parameter $\sigma$ decreases up to the critical value, the instability evolves nearby the Bolgiano scale. Shell variables $U_n$, $T_n$ in the neighbor of this scale start to fluctuate with a growing amplitude. Subsequently, the perturbations propagate throughout the spectrum. The solution becomes chaotic and this leads, in particular, to a change in the sign of temperature

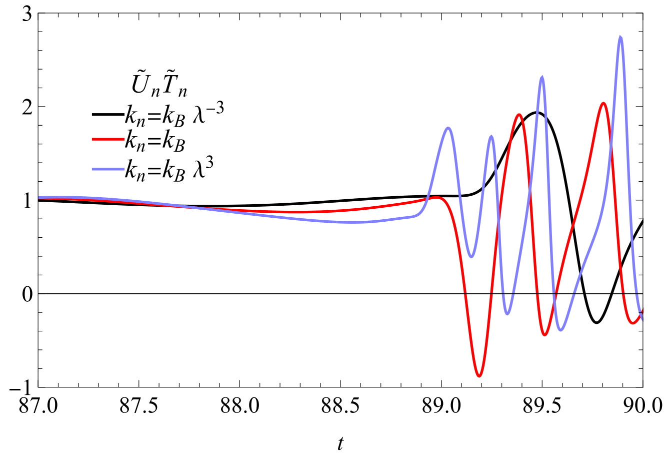


FIG. 5. Evolution of the normalized kinetic energy production by buoyancy $\tilde{U}_n\tilde{T}_n = (U_nT_n)/(U_0T_0k^{-4/5})$ for three shells near the Bolgiano scale for $\sigma < 0.005$, $\nu = \chi = 10^{-10}$.

fluctuations, thereby violating the assumption of anticorrelated fluctuations of temperature and velocity. This process is illustrated in Fig. 5, which shows the time dependence of the normalized kinetic energy production by the buoyancy term at scales, close to the Bolgiano scale $k_B$.

Note that the attempts to reproduce the OBs in shell models have been made before, using models with real or complex variables [10,11,17–19]. A systematic search for the OBs was undertaken in Ref. [11] based on a complex shell model of helical turbulence which satisfied all of the known conservation laws. In that work, it was unambiguously concluded that in all examined cases of convective turbulence considered (under both stable and unstable background stratification of the medium), the OBs does not appear in the inertial range, where the Ks is established, and the temperature behaves like a passive scalar. We have revisited the problem using the same complex model. It was found that in all cases shown in Fig. 1 the solution becomes unstable and degenerates into a quasistationary state with Ks.

## IV. CONCLUSIONS

The possibility of a stable OBs over a wide range of scales in buoyancy-affected turbulence is studied. From the theoretical point of view, the OBs requires two opposite conditions to be satisfied. On the one hand, the constant spectral flux of thermal variance implies either negligible stratification or vanishing correlation of velocity and temperature fluctuations. On the other hand, a permanent kinetic energy sink due to buoyancy is associated with anticorrelated velocity and temperature fluctuations, which usually result from a stable stratification. We circumvent these contradictory statements by considering the forced nonstratified turbulence. But in addition to a kinetic energy injection, the large-scale force generates temperature fluctuations which are anticorrelated with velocity fluctuations. Then we obtained the analytical solution and showed that the OBs is possible only if the governing parameters $I$, $I_\theta$, $k_f$, and $\xi$ are related by Eq. (13).

We introduced a characteristic scale $k_\mathrm{b}$, named the outer buoyancy scale, which highlights the role of buoyancy in the kinetic energy cascade. $k_\mathrm{b}$ increases with the buoyancy parameter and temperature fluctuation injection rate, and decreases with the growth of the kinetic energy injection rate. At $k_\mathrm{b} = 0$ the buoyancy effects vanish and the Kolmogorov scaling is established in the whole range of available scales, while $k_\mathrm{b} = k_f$ provides the OBs. For intermediate $k_\mathrm{b}$ both scalings can take place. We found that the Bolgiano scale, which characterizes the transition to the Ks, depends on $k_\mathrm{b}$ and $k_f$. From the physical point of view, only $k_\mathrm{B} > k_f$ is of practical interest, which is equivalent to $k_\mathrm{b} > k_f/2^{2/3}$.

Since the OBs is realized only at a precise balance of several factors, each of which may have a random component in a real (laboratory or, moreover, natural) turbulent flow, the physical feasibility of solution (8) should be verified. We used the shell model (generalized Novikov-Desniankii model), which provides a fixed point with Ks at vanishing buoyancy parameter. In numerical simulations we succeeded to get an extended stable OBs range with $k_\mathrm{B} \approx 3000$. A further increase of the buoyancy effect ($k_\mathrm{b} \to k_f$ or $\sigma \to 0$) results in the loss of stability.

We attempted to obtain OBs using shell models which give intrinsically a quasistationary solution. We found that in these models the OB regime is unstable and after the transition the system goes into a chaotic state, the average characteristics of which correspond to the Kolmogorov cascade of kinetic energy, in which temperature fluctuations behave as a passive scalar. So the appearance of OBs in dynamic systems and moreover in real turbulent flows is highly questionable. This skepticism is supported by the long-standing experience of identifying OBs in numerical and laboratory studies. This conclusion most likely indicates that Bolgiano's contradictory statements should be modified to some extent. Then, the obvious way is to abandon the requirement of constancy of spectral flux $\Pi_\theta$. In fact most studies that pretend to obtain OBs are showing very approximately constant $\Pi_\theta$ in a short wave number range or even they do not check it out at all. The theoretical possibility of OBs with variable $\Pi_\theta$ is yet not demonstrated. In Ref. [14], authors revisited OBs considering the total energy flux $\Pi + \Pi_\theta$ to be constant. They found OBs as an asymptotic solution without transition to Ks. Another option for a wider view on OB regime is that the contribution of buoyancy can be reversed, i.e., it can produce kinetic energy. Specifically, such a scenario was suggested in Ref. [18], where OBs developed under the imposed inverse energy cascade.

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