Asymptotic scaling laws for the skin friction of zero pressure gradient boundary layers

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(Received 29 January 2023; accepted 4 August 2023; published 28 August 2023)

We propose a phenomenological description of the asymptotic near-wall momentum-exchange mechanism of flat plate zero-pressure gradient turbulent boundary-layer flows at the extreme Reynolds number regime, based on a simple model of attached eddies scaling with the location of the mesolayer, $\delta_m^+ \sim \mathrm{Re}_\tau^{1/2}$, and satisfying Kolmogorov's inertial similarity scaling for their turnover velocities. This yields an asymptotic power-law formula for the skin friction, $C_f \sim \mathrm{Re}_x^{-2/15}$, or, equivalently, $C_f \sim \mathrm{Re}_\delta^{-2/13}$. We also derive a formula for the asymptotic thickness of the boundary layer. We show that these asymptotic scaling laws are in excellent agreement with experimental data, and are consistent with classical semiempirical formulas. For moderately large Re number flows, we argue that the intermediate asymptotic Blasius scaling, $C_f \sim \mathrm{Re}_\delta^{-1/4}$, or, equivalently, $C_f \sim \mathrm{Re}_x^{-1/5}$, is more appropriate. The asymptotic model relies on an asymptotic analysis of the bulk normalized axial mean-momentum equation, and it is related to a phenomenology previously proposed for pipe and channel flows, which suggests the existence of a universal transition of the turbulent momentum-exchange mechanism of wall-bounded flows in the asymptotically large Re number regime.

DOI: 10.1103/PhysRevFluids.8.084607

I. INTRODUCTION

This work proposes a model for the momentum-exchange mechanism near the wall for asymptotically large Reynolds number turbulent boundary-layer flows that develop along a smooth flat plate with zero pressure gradient (ZPG TBL flows). As a consequence, we derive simple power-law formulas for the skin-friction factor, C_f, and the asymptotic thickness of boundary layers at the extreme Reynolds number regime.

All of the flows discussed in this work are either laminar or are assumed to be fully developed. For pipe and channel flows, this means that the streamwise derivative of all statistics is null, except for the constant mean pressure gradient. For ZPG TBL flows, the boundary layer continues to evolve along the streamwise direction, so that fully developed in this case means that all the remaining effects from the laminar to turbulent transition may affect only large-scale structures [1,2].

For fully developed pipe and channel flows, the Reynolds number is completely determined by two of the following three parameters: the pressure gradient, the flow rate, and δ, the channel half-height or pipe diameter. In contrast, for boundary-layer flows, any working definition of the local

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Reynolds number is such that it increases continuously along the flow. Some of these definitions are more appropriate than others depending on the information available. For example, the quantity $\mathrm{Re}_\delta = U_o\delta/\nu$ depends on the previous knowledge of the boundary-layer thickness, $\delta = \delta(x)$, which is generally ill-defined. A common definition of δ is the vertical distance from the wall at which the mean velocity achieves 99% of its free-stream value U_o, which, in practice, is hard to measure [1,3]. Moreover, the boundary-layer thickness relation $\delta(x)$ is not available *a priori*, thus Re_δ may not be a very useful input parameter for most engineering applications. Another common definition is $\mathrm{Re}_\theta = U_o\theta/\nu$, where θ is the momentum thickness [1,2]. Although θ is also difficult to be estimated *a priori*, Re_θ is more commonly used in the literature, since θ is related to the skin-friction factor, C_f, by the Kárman integral momentum equation as $C_f/2 := u_\tau^2/U_o^2 = d\theta/dx$, where $u_\tau = \sqrt{\tau_w/\rho}$ is the friction velocity. One may also define the friction Reynolds number $\mathrm{Re}_\tau = u_\tau\delta/\nu$, which does not directly involve the free-stream velocity. For more practical applications, a useful definition is given by the nominal downstream-distance-based Reynolds number, $\mathrm{Re}_x = U_o x/\nu$. In this work, we derive power-law friction formulas in terms of all the aforementioned Reynolds number definitions for the laminar, intermediate-Re, and extreme-Re regimes. For each regime, the formulas' deviations concerning available experimental data are shown to be very small.

Classical approaches to deriving friction factor relations usually involve the integration of the mean velocity profile (MVP), which is assumed to obey either a log-law or a power-law scaling in some domain of the flow [1]. The approach presented here does not involve any explicit assumption on the MVP, and it relies mainly on the relation of the wall shear stress with a new phenomenology of momentum exchanging eddies scaling with the location of the so-called mesolayer [4].

The friction formulas presented in this work are explicit and parsimonious (only one fitting constant for each regime), and for sufficiently high Reynolds number, skin-friction factor data obtained from different sources can be predicted essentially within experimental uncertainty by the models. We remark, nonetheless, that the main goal of this work is not to provide any practical improvement over the classical friction equations [1,2], which are either power-law formulas, with similar or worse accuracy than the ones presented here, or log-law formulas with two constants, but with good accuracy over a wider Re number domain. The main objective of this work is to show that there is a transition from an intermediate power-law scaling to an asymptotic power law for the friction factor, which is not obvious from the classical formulations. Moreover, we argue that this change of power-law scaling is related to a universal transition of the momentum-exchange mechanism scaling dominating the wall shear stress that is also present in other wall-bounded flows [5–7].

The understanding of the scaling transition reported in this work may be relevant for future research in the area of flow control since it has been reported that the effectiveness of some flow control strategies related to the near-wall structures is reduced for large Reynolds numbers [8–10], and because many of those strategies depend on the momentum-exchange mechanism near the wall. The relation between the proposed scaling transition and flow control is out of the scope of the present work.

It is common practice in the fluid dynamics community to extend the methods and the quantities used to describe turbulent pipe flows to the ZPG TBL problem. In his seminal work, for example, Barenblatt [11,12] proposed a power law for the velocity profile of turbulent pipe flows with an exponent depending on the Re number, by assuming an incomplete similarity hypothesis. Although the analysis brings no deeper understanding of the near-wall structure of the turbulent flow, the results compare well with experimental data. However, the extension to the ZPG TBL flows [13] faced difficulties in the definition of the characteristic length scale and its correlation to the counterpart scale in turbulent pipe flow, i.e., the diameter of the pipe. This theory also does not provide any further insight into the exponent of the power law, and experimental data is necessary for this purpose. More recently, following Barenblatt's steps, Dixit *et al.* [14] also proposed a friction semiempirical relation for turbulent pipe flows. To overcome the boundary-layer characteristic length problem, Dixit *et al.* [15] proposed an alternative set of dimensionless variables using the

TBL momentum rate instead of the main stream flow velocity. The results obtained by the authors are, nonetheless, still inconclusive and more research is necessary to confirm their main hypothesis.

In this work, we pursue a different path. We start discussing pipe and channel flows in Sec. II, where we use the axial mean-momentum equation scaled in bulk coordinates to relate the friction factor to the action of the Reynolds stress at the top of the mesolayer. The presentation focuses on channel flows for the sake of simplicity. We then follow the steps in Refs. [5,7] to propose a closure model for the Reynolds' stress to obtain a power-law scaling in terms of Re_δ or Re_τ. In Sec. III, we use the boundary-layer streamwise mean-momentum equation to obtain a relation for the skin-friction factor as a function of the ratio of Re_x and Re_δ. Assuming a similar eddy phenomenology for ZPG TBL flows and for pipe/channel flows, we obtain a power-law scaling for the skin-friction factor in terms of Re_δ, which results in an expression for the skin friction in terms of Re_x, for every given streamwise location $x \approx X$. We show that both power-law scaling laws are asymptotically accurate. We also discuss the intermediate asymptotics for pipe, channel, and boundary-layer flows. We end Sec. III proposing power-law scaling laws for the thickness of the boundary layers, which implies an explicit relation between Re_x and Re_δ for each turbulent regime. We present our conclusions in Sec. IV.

II. FRICTION FACTOR FOR CHANNEL AND PIPE FLOWS

A. Bulk normalized axial mean-momentum equation

Let us begin with the analysis of the axial mean-momentum equation for channel flows, which reads

$$\nu \frac{d^2U}{dy^2} - \frac{d}{dy}\langle uv\rangle - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0. \tag{1}$$

This equation represents a balance of forces, or stress gradients, and we denote by $F_T := \frac{d}{dy}\langle uv\rangle$, the mean effect of turbulent inertia, by $F_\nu := \nu\frac{d^2U}{dy^2}$, the mean viscous force, and by $F_p := -\frac{1}{\rho}\frac{\partial p}{\partial x}$, the mean pressure gradient, so that $F_p = F_T - F_\nu$.

In Ref. [4], it was observed that turbulent wall flows can be divided into a four-layer structure, defined by the change of magnitude ordering of these mean forces. This analysis was later extended via the method of scaling patches (see Refs. [18–20]). The authors revealed the persistence of viscous effects from layer I, near the wall, until the top of the mesolayer or layer III, where the peak of Reynolds' stress is located, around $y^+ \sim \delta_m^+ := 2.6\,\mathrm{Re}_\tau^{1/2}$. In this work, we focus most of our analysis in the inertial/advection balance layer, or layer IV, located at $2.6\,\mathrm{Re}_\tau^{1/2} < y^+ \leqslant \mathrm{Re}_\tau$, where the magnitude ordering is $F_T \approx F_p \gg F_\nu$. Let us define the bulk normalized quantities:

$$\hat{y} = \frac{y}{\delta}, \quad \hat{x} = \frac{x}{L}, \quad \hat{U} = \frac{U}{\bar{U}}, \quad \hat{\tau}_T = \langle \hat{u}\hat{v}\rangle = \frac{\langle uv\rangle}{\bar{U}\,v_c}, \quad \hat{p} = \frac{p}{\Delta p}, \tag{2}$$

where v_c is a regime-dependent characteristic velocity, L is the length of the channel, δ is the half-height of the channel, $\bar{U}$ is the flow velocity averaged over the channel cross section, and Δp is the total pressure drop along the flow. Now, notice that

$$\frac{\delta}{\bar{U}^2}F_T = \frac{v_c}{\bar{U}}\frac{d\langle\hat{u}\hat{v}\rangle}{d\hat{y}}, \quad \frac{\delta}{\bar{U}^2}F_\nu = \frac{1}{\mathrm{Re}_\delta}\frac{\partial^2\hat{U}}{\partial\hat{y}^2}, \tag{3}$$

where $\mathrm{Re}_\delta := \rho\bar{U}\delta/\mu$ is the bulk Reynolds number. Because $\frac{d\hat{p}}{d\hat{x}} = 1$, it follows that

$$\frac{\delta}{\bar{U}^2}F_p = -\frac{\delta\Delta p}{L\rho\bar{U}^2}\frac{d\hat{p}}{d\hat{x}} = \frac{1}{2}f, \tag{4}$$

where $f = \frac{-2\delta\frac{dp}{dx}}{\rho\bar{U}^2} = \frac{-2\delta\Delta p}{\rho\bar{U}^2 L}$ is the friction factor for channel flows. In this work, we denote by f the friction factor for pipe and channel flows, and by C_f the skin-friction coefficient for boundary-layer

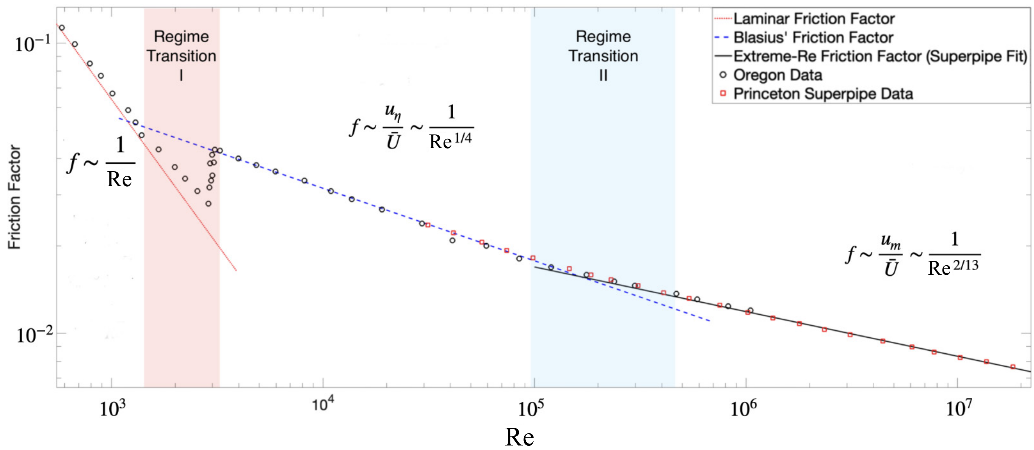

FIG. 1. Darcy friction factor data from the Princeton Superpipe [16] and Oregon [17]. Laminar friction equation: $f = \frac{64}{\mathrm{Re}}$; Blasius' friction equation: $f = \frac{3.16\times10^{-1}}{\mathrm{Re}^{1/4}}$; extreme Re friction equation: $f = \frac{9.946\times10^{-2}}{\mathrm{Re}^{2/13}}$.

flows. Let us denote

$$\hat{F}_v := \frac{\partial^2 \hat{U}}{\partial \hat{y}^2}, \quad \hat{F}_T := \frac{d\hat{\tau}_T}{d\hat{y}}, \quad \hat{F}_p := \frac{d\hat{p}}{d\hat{x}} = 1. \tag{5}$$

Multiplying Eq. (1) by $\frac{\delta}{\rho \bar{U}^2}$, one obtains the bulk normalized asymptotic axial mean-momentum equation,

$$\frac{2}{f}\frac{v_c}{\bar{U}}\hat{F}_T - \frac{2}{f}\frac{1}{\mathrm{Re}_\delta}\hat{F}_v = \hat{F}_p = 1. \tag{6}$$

The friction coefficient obeys three different scaling laws depending on the flow's regime [5,7] (see Fig. 1). We now present a discussion concerning these different regimes from the perspective of the bulk normalized axial mean-momentum equation (6). This is important for the extension of our arguments to obtain the new scaling laws for ZPG TBL flows.

B. Extreme-Re regime

Let $\delta_m = 2.6\delta/\mathrm{Re}_\tau^{1/2}$ denote the border of the mesolayer with the inertial/advection balance layer [4,18–20]. Inspired by arguments first presented in Refs. [21,22], and later expanded in Ref. [5], we assume that for extreme Re number flows, the Reynolds stress $\langle uv\rangle|_{y=\delta_m}$ results from the momentum exchange between the near-wall region up and down the inertial/advection balance layer through the work of the δ_m-scaling attached eddies [see Fig. 2(a) for a pictorial description]. This results in the main hypothesis for pipe and channel flows:

$$\langle uv\rangle|_{y=\delta_m} \sim -[u(y+s) - u(y-s)]v_m \sim -\bar{U} v_m, \tag{7}$$

where v_m is the eddy turnover velocity, and the momentum contrast is approximated with a fraction of $\bar{U}$. Based on Eq. (7), we define the characteristic velocity in (2) and (6) as

$$v_c := v_m := -\frac{\langle uv\rangle|_{y=\delta_m}}{\bar{U}}, \tag{8}$$

so that $\hat{\tau}_T := \langle uv\rangle / \langle uv\rangle|_{y=\delta_m}$ and that $2\frac{v_m}{f\bar{U}}\hat{F}_T - \frac{2}{f\,\mathrm{Re}_\delta}\hat{F}_v = 1$. Furthermore, because F_p is constant everywhere, the mean turbulent inertia $\frac{d}{dy}\langle uv\rangle$ is also approximately constant in the inertial/advection

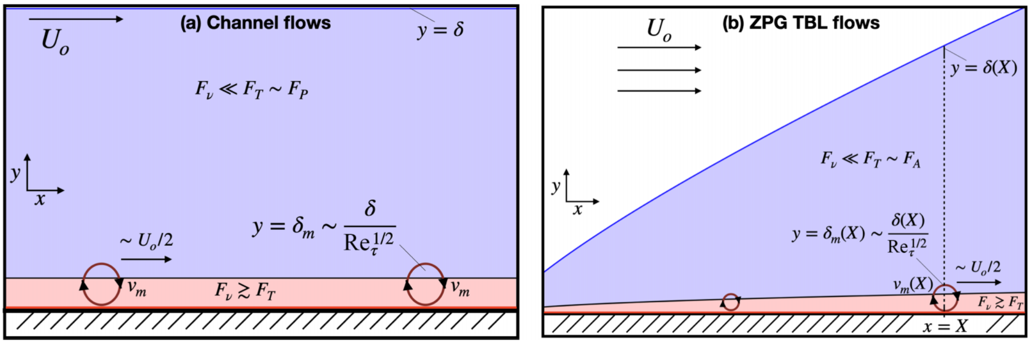

FIG. 2. Schematic of the model of an eddy that straddles the wet surface in the vicinity of the mesolayer that exchanges its streamwise mean momentum between adjacent layers. (a) Channel flows. (b) Flat plate ZPG TBL.

balance layer, L_{IV}, where F_ν is negligible [4]. This implies

$$-\langle uv\rangle|_{y=\delta_m} = \langle uv\rangle|_{y=\delta} - \langle uv\rangle|_{y=\delta_m} = \int_{L_{\mathrm{IV}}} \frac{d}{dy}\langle uv\rangle\, dy \approx \delta\left(1 - \frac{\delta_m}{\delta}\right)\frac{d}{dy}\langle uv\rangle. \tag{9}$$

This yields the asymptotic limit $\hat{F}_T = \frac{\delta}{\langle uv\rangle|_{y=\delta_m}}\frac{d}{dy}\langle uv\rangle \sim O(1)$, as $\mathrm{Re}_\tau \to \infty$, for y in the inertial/advection balance layer.

We illustrate in Fig. 3(a) this asymptotic behavior with the analysis of direct numerical simulation (DNS) data of high Reynolds number channel flows [23], where one can observe that $\hat{F}_T \approx \delta F_T/\langle uv\rangle|_{y=\delta_m} \approx -1$, and that $\hat{F}_\nu \ll O(1)$, for sufficiently large y/δ. Because in the turbulent regime, the friction factor satisfies $f > 1/\mathrm{Re}_\delta$, this implies that the viscous term is negligible in (6), for sufficiently large y/δ. Therefore, (6) reduces to $f \to \frac{v_m}{\bar{U}}$, as $\mathrm{Re}_\tau \to \infty$, for $y/\delta \sim O(1)$ in the inertial/advection balance layer.

As argued in Ref. [5], based on evidence of self-similar behavior for eddies with wall-normal length scales spanning more than a decade in pipe flows [24], we assume Kolmogorov's self-similarity scaling $\frac{v_m}{\bar{U}} \propto (\frac{\delta_m}{\delta})^{1/3}$ for the eddy turnover velocity. It is noted that this is not an assumption on the MVP. This implies that the friction factor for extreme-Re number flows, f_E, satisfies

$$f_E \sim \frac{v_m}{\bar{U}} \sim \left(\frac{\delta_m}{\delta}\right)^{1/3} \sim \frac{1}{\mathrm{Re}_\tau^{1/6}} \sim \frac{1}{\mathrm{Re}_\delta^{2/13}}, \tag{10}$$

where we used that $\delta_m/\delta \sim 1/\mathrm{Re}_\tau^{1/2}$.

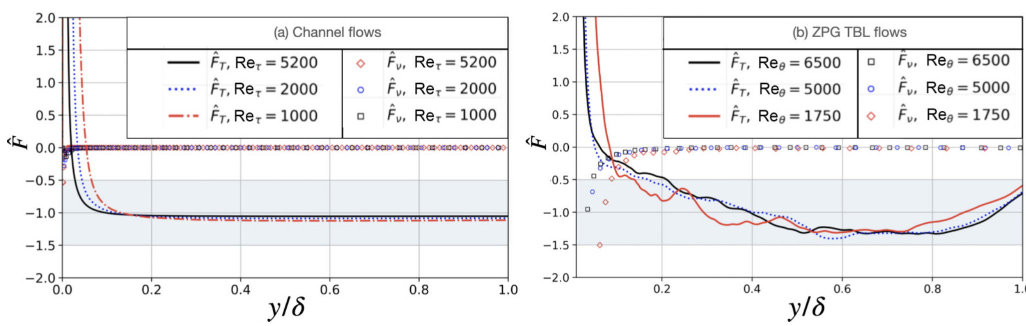

FIG. 3. Bulk normalized turbulent stress $\hat{F}_T := \frac{d\langle \hat{u}\hat{v}\rangle}{d\hat{y}}$ and bulk normalized viscous force $\hat{F}_\nu := \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}$. (a) DNS data for channel flows [25]. (b) DNS data for flat plate ZPG TBL flows [23].

It is important to underscore that employing external (inviscid) quantities does not signify the dismissal of viscous effects within the mesolayer. In accordance with the classical turbulence theory, it is broadly acknowledged that energy acquired at larger scales undergoes an energy-conserving cascade process towards smaller scales, commonly referred to as the Richardson energy cascade. This cascade mechanism ensures that energy is ultimately dissipated at the finest scales, thereby linking the larger and smaller scales through energy conservation principles. Consequently, to characterize the inner scales in terms of pertinent engineering parameters, the incorporation of larger scales becomes an indispensable step. Figure 1 shows that $f = (9.946 \times 10^{-2})/\mathrm{Re}_\delta^{2/13}$ yields a very good approximation to experimental data at the extreme range. The colored regions in Fig. 1 mark regime transitions, where the deviation from power-law formulas to experimental data is higher than the experimental uncertainty. Because $f = 2\tau_w/\rho\bar{U}^2$, we remark that it follows from (10) that the wall shear stress, τ_w, scales as $\tau_w \sim -\rho\langle uv\rangle|_{y=\delta_m} \sim \rho\bar{U}v_m$ asymptotically.

Moreover, taking into account the studies by Hoyas *et al.* in Refs. [26,27], which use direct numerical simulation of channel flow at a friction Reynolds number of 10 000, we derive two pivotal conclusions about channel flows. Firstly, the maximum of the intensity of the streamwise velocity increases with the Reynolds number. Secondly, the extension of the mean streamwise velocity's logarithmic layer surpasses earlier assumptions, spanning from 400 to 2500 wall units. These significant findings echo the transitional behavior examined in our current study.

C. Laminar flow and intermediate asymptotics

In the laminar regime, by definition, F_T is negligible throughout the flow. Assuming that $\hat{F}_\nu = \frac{d^2\hat{U}}{d\hat{y}^2} = O(1)$ for $y/\delta \sim O(1)$, one obtains from (6) the well-known scaling laws for the friction factor of laminar flows: $f_L \sim \frac{1}{\mathrm{Re}_\delta}$.

For the intermediate asymptotics' Reynolds number range, where the viscous term in (6) is negligible, but the wall shear stress is not dominated by the momentum-exchange mechanism described in the previous section, one observes the emergence of an intermediate power-law scaling $f_I \sim \mathrm{Re}^{-1/4}$, known as the Blasius correlation. As first noticed in Refs. [21,22], one can derive such relation by replacing the characteristic velocity v_c in (6) by $v_c \sim u_\eta$, where u_η is the $K41$'s dissipative velocity, satisfying $u_\eta/\bar{U} \sim \mathrm{Re}^{-1/4} \sim f$. An inspection of Fig. 1 shows that Blasius' correlation yields an excellent approximation of the friction factor in the intermediate range. In Refs. [5,28], it is suggested that at lower turbulent Reynolds numbers, wall-incoherent motions made up of Kolmogorov-type fine scales and other wall-detached motions are dominant for friction, but a complete description of the flow structure underlying Blasius' relation remains elusive. It is noted, nonetheless, that previous works [29–31] by the authors generalized Blasius' formula for a large family of purely viscous non-Newtonian fluid flows by generalizing the definition of Kolmogorov's dissipative velocity scale, u_η, for each rheology and by extending the relation $\tau_w \sim \rho\bar{U}v_n \sim \rho\bar{U}u_\eta$.

III. SKIN-FRICTION FACTOR FOR FLAT PLATE ZPG TBL FLOWS

A. Bulk normalized streamwise mean-momentum equation

In this section, we extend the theory developed for pipes and channels in the previous section to flat plate ZPG TBL flows. Consider the streamwise mean-momentum equation

$$\nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}\langle uv\rangle = \left[U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right]. \tag{11}$$

This equation represents a balance of forces, or stress gradients, and we denote by $F_T := \frac{d}{dy}\langle uv\rangle$, the mean effect of turbulent inertia, by $F_\nu := \nu\frac{d^2U}{dy^2}$, the mean viscous force, and by $F_A := [U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}]$, the mean advective force, so that $F_\nu - F_T = F_A$. We now define the bulk normalized quantities for boundary-layer flows at a given $x \approx X$:

$$\hat{y} = \frac{y}{\delta(X)}, \quad \hat{x} = \frac{x}{X}, \quad \hat{U} = \frac{U}{U_o}, \quad \hat{\tau}_T = \langle \hat{uv} \rangle = \frac{\langle uv \rangle}{U_o v_c}, \quad \hat{V} = \frac{XV}{U_o \delta(X)}, \tag{12}$$

where v_c is a regime-dependent characteristic velocity, U_o is the free-stream mean velocity, and δ is the thickness of the boundary layer at $x = X$. We remark that X is fixed, but arbitrary, and the analysis will be restricted to $\hat{x} \approx 1$, for each X. The scaling in $\hat{V}$ follows from the mean continuity equation. It is noted that

$$\frac{\delta(X)}{U_o^2} F_T = \frac{v_c}{U_o} \frac{d\langle \hat{uv} \rangle}{d\hat{y}}, \quad \frac{\delta(X)}{U_o^2} F_\nu = \frac{1}{\mathrm{Re}_\delta} \frac{\partial^2 \hat{U}}{\partial \hat{y}^2}, \tag{13}$$

and that

$$\left[\delta(X)/U_o^2\right] F_A = \frac{\delta(X)}{X} \left(\hat{U} \frac{\partial \hat{U}}{\partial \hat{x}} + \hat{V} \frac{\partial \hat{U}}{\partial \hat{y}} \right). \tag{14}$$

Let us denote

$$\hat{F}_\nu := \frac{\partial^2 \hat{U}}{\partial \hat{y}^2}, \quad \hat{F}_T = \frac{d\hat{\tau}_T}{d\hat{y}} \tag{15}$$

and

$$\hat{F}_A := \hat{U} \frac{\partial \hat{U}}{\partial \hat{x}} + \hat{V} \frac{\partial \hat{U}}{\partial \hat{y}}. \tag{16}$$

Multiplying Eq. (11) by (δ/U_o^2), one obtains the equivalent of relation (6) for ZPG TBL flows:

$$\frac{1}{\mathrm{Re}_\delta} \hat{F}_\nu - \frac{v_c}{U_o} \hat{F}_T = \frac{\delta(X)}{X} \hat{F}_A. \tag{17}$$

B. Extreme-Re flows

For turbulent flows, we may assume that the viscosity does not play a direct role in the advective forces in the inertial/advection balance layer, and therefore the scaling $F_A(y; x) \sim \frac{U_o^2}{X} \Phi(\frac{y}{\delta(X)}) \sim O(\frac{U_o^2}{X})$ holds for $\frac{y}{\delta(X)} \sim O(1)$ in the inertial/advection balance layer, and $\hat{x} \approx 1$. This implies that $\hat{F}_A(y; x) = \frac{X}{U_o^2} F_A \sim O(1)$ within this region of the flow. Let $\delta_m(X) = 2.6\,\delta/\mathrm{Re}_\tau(X)^{1/2}$ denote the border of the mesolayer with the inertial/advection balance layer at $x = X$ [19]. As for pipe and channel flows, we assume that for extreme Re number flows, the Reynolds stress $\langle uv \rangle|_{y=\delta_m}$ results from momentum exchange between the near-wall region up and down the inertial/advection balance layer through the work of the δ_m-scaling attached eddies [see Fig. 2(b) for a pictorial description]. The main difference between pipe and channel flows is that for boundary-layer flows, the characteristics of the attached eddy are x dependent since both the boundary-layer and the mesolayer thicknesses evolve with x. This results in our hypothesis

$$\langle uv \rangle|_{y=\delta_m(x)} \sim -[u(y+s) - u(y-s)] v_m \sim -U_o v_m, \tag{18}$$

where $v_m = v_m(x)$ is the eddy turnover velocity at x, and the momentum contrast is approximated with a fraction of U_o, which is compatible with the discussion in Ref. [4]. We now define the characteristic velocity in (12) and (17) as

$$v_c := v_m := -\langle uv \rangle|_{y=\delta_m}/U_o, \tag{19}$$

so that $\hat{\tau}_T := \langle uv \rangle / \langle uv \rangle|_{y=\delta_m(x)}$. In Fig. 3(b), we illustrate that, far from the wall, $\hat{F}_T(y; x) = \frac{\delta(X)}{U_o v_m} F_T \to O(1)$ and $\hat{F}_\nu \ll O(1)$ for data obtained from DNS of boundary-layer flows at high

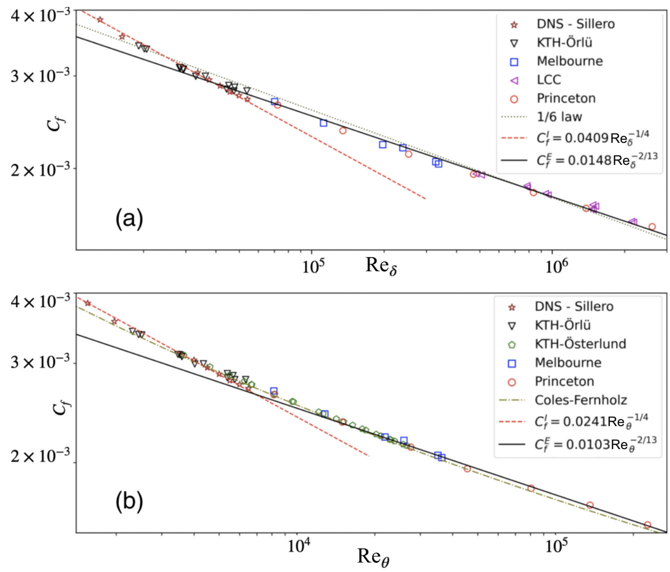

FIG. 4. Variation of skin-friction coefficient for ZPG TBL flows as a function of (a) Re_δ and (b) Re_θ. Intermediate-Re power law (C_f^I): Eq. (28) and Extreme-Re power law (C_f^E): Eq. (21); 1/6 power law [1] ($C_f = 0.02\mathrm{Re}_\delta^{-1/6}$) and modified Coles-Fernholz [32,33] ($C_f = 2[1/(0.384)\ln \mathrm{Re}_\theta + 4.127]^{-2}$), compared to DNS data from Ref. [23] and experimental data from KTH [34,35], Princeton [3], Melbourne [36], and LCC [37].

Reynolds number [23]. Inspired by the friction relation for channel and pipe flows, we assume that $\tau_w \sim -\rho\langle uv\rangle|_{y=\delta_m(x)} = \rho U_o v_m$.

This implies that the Gioia and Chakraborty momentum-exchange mechanism hypothesis, initially proposed for pipe/channel flow, can be expanded to encompass flat plate boundary-layer flows. The fundamental premise of this hypothesis is rooted in the observation that the asymptotic structures of confined flows and turbulent boundary layers on flat plates exhibit notable similarities [4]. As a result, it is reasonable to postulate that the momentum-exchange mechanisms within the wet layer also share resemblances. Thus,

$$C_f = \frac{2\tau_w}{\rho U_o^2} \sim \frac{v_m}{U_o} \sim \frac{1}{\mathrm{Re}_\delta^{2/13}}. \tag{20}$$

If one assumes that for turbulent flows, $\theta \sim \delta$, one can write

$$C_f^E = \frac{d_e}{\mathrm{Re}_\delta^{2/13}} = \frac{\tilde{d}_e}{\mathrm{Re}_\theta^{2/13}}. \tag{21}$$

Figure 4 shows that Eqs. (21) result in very good approximations with $d_e = 0.0148$ and $\tilde{d}_e = 0.103$. Now, because $\frac{\delta(X)}{X} = \frac{\mathrm{Re}_\delta}{\mathrm{Re}_X}$, we obtain

$$\frac{\mathrm{Re}_X}{\mathrm{Re}_\delta^2}\frac{\hat{F}_v}{\hat{F}_A} - C_f\frac{\mathrm{Re}_X}{\mathrm{Re}_\delta}\frac{\hat{F}_T}{\hat{F}_A} = 1. \tag{22}$$

If we consider this equation within the inertial/advection balance layer, where $\frac{\hat{F}_T}{\hat{F}_A} \sim O(1)$ and $\frac{\hat{F}_v}{\hat{F}_A} \ll O(1)$, and because for turbulent flows, $C_f > 1/\mathrm{Re}_\delta$, we obtain the asymptotic relation

$$C_f \sim \frac{\mathrm{Re}_\delta}{\mathrm{Re}_X} \sim \frac{\delta(X)}{X}. \tag{23}$$

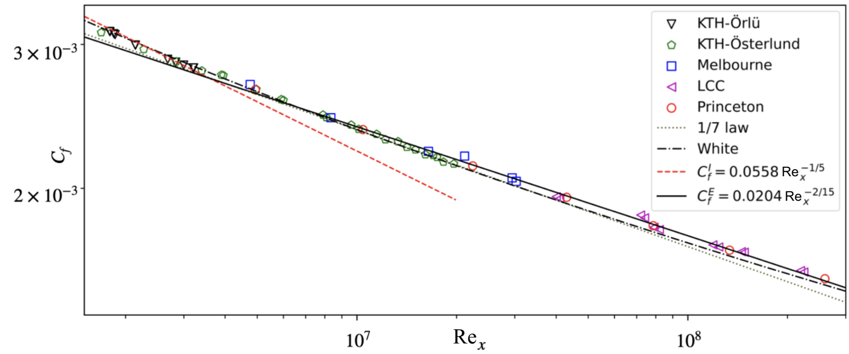

FIG. 5. (a) Variation of skin-friction coefficient for ZPG TBL flows as a function of Re_x. Intermediate-Re power law (C_f^I): Eq. (28) and extreme-Re power-law (C_f^E): Eq. (24); modified White's formula ($C_f = [0.4177\ \ln(0.06\mathrm{Re}_x)]^{-2}$) and 1/7 law ($C_f = 0.023\,58\mathrm{Re}_x^{-1/7}$) from Ref. [33], compared to experimental data.

Because $\mathrm{Re}_\delta \sim C_f^{-13/2}$, this implies $C_f^{1+(13/2)} \sim \frac{1}{\mathrm{Re}_X}$. Since $x = X$ is arbitrary within the extreme Re range, we may write

$$C_f^E = \frac{c_e}{\mathrm{Re}_x^{2/15}} \sim \frac{\delta_E(x)}{x}, \tag{24}$$

where C_f^E denotes the skin-friction coefficient at the extreme-Re regime. Figures 5 and 6 show that Eq. (24) yields a good approximation with $c_e = 0.0204$.

C. Laminar flows and intermediate asymptotics

In the laminar regime, F_T is negligible throughout the flow, so that $\frac{1}{\mathrm{Re}_\delta}\hat{F}_\nu \approx \frac{\delta(X)}{X}\hat{F}_A$. Because $\hat{F}_\nu \sim \hat{F}_A \sim O(1)$ for $\hat{y} \sim O(1)$ and $\hat{x} \approx 1$, one obtains $\frac{1}{\mathrm{Re}_\delta} \sim \frac{\delta(X)}{X}$. Then

$$C_f^L = \frac{\mu \frac{dU}{dy}|_{y=0}}{\frac{1}{2}\rho U_o^2} \sim \frac{\mu \frac{U_o}{\delta}}{\frac{1}{2}\rho U_o^2} \sim \frac{1}{\mathrm{Re}_\delta}, \tag{25}$$

where C_f^L denotes the skin-friction coefficient in the laminar regime. Because $\frac{1}{\mathrm{Re}_\delta} \sim \frac{\delta(X)}{X} = \frac{\mathrm{Re}_\delta}{\mathrm{Re}_X}$, we have $\mathrm{Re}_\delta^2 \sim \mathrm{Re}_X$. This implies the well-known relation [1]

$$C_f^L \sim \frac{\delta(X)}{X} \sim \frac{1}{\mathrm{Re}_\delta} \sim \frac{1}{\mathrm{Re}_x^{1/2}}. \tag{26}$$

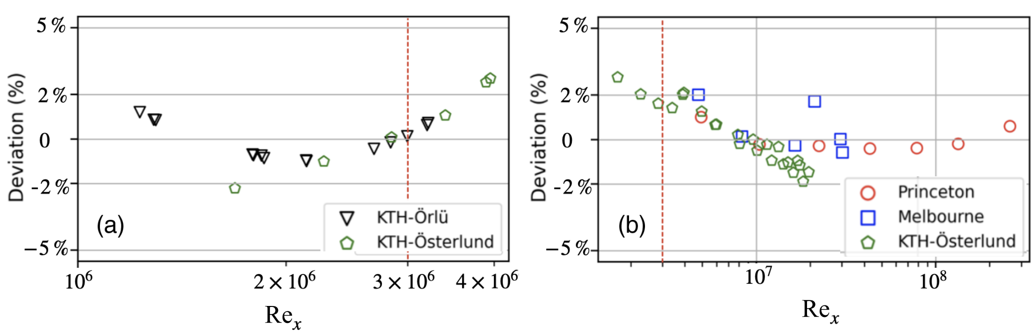

FIG. 6. (a) Deviation of friction factor from experimental data to intermediate Re_x power-law formula (28). (b) Deviation of friction factor from experimental data to extreme Re_x power-law formula (24).

As for pipe and channel flows, one also observes the emergence of an intermediate power-law scaling:

$$C_f \sim \mathrm{Re}_\delta^{-1/4} \sim \mathrm{Re}_\theta^{-1/4} \tag{27}$$

(see Fig. 4). This relation can be derived by replacing the characteristic velocity $v_c \sim u_\eta$ in $C_f \sim v_c/U_o$, where u_η is the $K41$'s dissipative velocity satisfying $u_\eta/\bar{U} \sim \mathrm{Re}_\delta^{-1/4}$. Therefore, replacing $\mathrm{Re}_\delta \sim C_f^{-4}$ in relation $C_f \sim \frac{\mathrm{Re}_\delta}{\mathrm{Re}_X}$, one obtains $C_f \mathrm{Re}_X \sim C_f^{-4}$. Thus,

$$C_f^I = \frac{c_b}{\mathrm{Re}_x^{1/5}}, \tag{28}$$

where C_f^I denotes the skin-friction coefficient in the intermediate regime. Figures 5 and 6 demonstrate that Eq. (28) provides an excellent approximation with $c_b = 0.0558$. Furthermore, it is noteworthy that the deviation between the current theory and experimental data, as displayed in Figs. 5 and 6, is smaller than the experimental error reported in the references. Consequently, any speculation concerning the higher-order behavior of the friction factor carries limited scientific relevance. Nonetheless, it is crucial to anticipate larger discrepancies in the transition region for both equations, (24) and (28), as neither of them is entirely valid within that domain.

It should also be noted that, for boundary layers, the transition to the extreme Reynolds regime occurs around $\mathrm{Re}_\delta \approx 3\text{–}4 \times 10^4$, while for pipe flows, it is around $\mathrm{Re}_\delta \approx 1.0 \times 10^5$.

D. The thickness of the boundary layer

We showed that the scaling laws

$$C_f \sim \frac{1}{\mathrm{Re}_\tau^{\alpha_r}} \sim \frac{1}{\mathrm{Re}_\delta^{\beta_r}} \sim \frac{1}{\mathrm{Re}_x^{\gamma_r}} \tag{29}$$

yield excellent approximations for each regime r, where $\alpha_L = 2$, $\beta_L = 1$, $\gamma_L = 1/2$ for the laminar regime, $\alpha_I = \frac{2}{7}$, $\beta_I = \frac{1}{4}$, $\gamma_I = \frac{1}{5}$ for the intermediate regime, and $\alpha_e = \frac{1}{6}$, $\beta_e = \frac{2}{13}$, $\gamma_e = \frac{2}{15}$ for the extreme Reynolds number regime. Because $\delta(x) \sim x C_f \sim x \frac{1}{\mathrm{Re}_x^\gamma}$, one obtains the following relation for the thickness of the boundary layer:

$$\delta(x) \sim x^{1-\gamma} \left(\frac{\nu}{U_o}\right)^\gamma. \tag{30}$$

For the laminar regime, this yields the well-known scaling [1]

$$\delta_L(x) \sim \left(\frac{\nu}{U_o}\right)^{1/2} x^{1/2}, \tag{31}$$

or $\mathrm{Re}_\delta \sim \mathrm{Re}_x^{1/2}$ in dimensionless form. For turbulent flows in the intermediate regime, this implies

$$\delta_I(x) \sim \left(\frac{\nu}{U_o}\right)^{1/5} x^{4/5}, \tag{32}$$

or, equivalently, $\mathrm{Re}_\delta \sim \mathrm{Re}_x^{4/5}$. For the extreme Reynolds regime, one obtains

$$\delta_e(x) \sim \left(\frac{\nu}{U_o}\right)^{2/15} x^{13/15}, \tag{33}$$

or $\mathrm{Re}_\delta \sim \mathrm{Re}_x^{13/15}$. Figure 7 shows that the proposed scaling laws result in good approximations to empirical data. We also present these laws by replacing the boundary-layer thickness δ by the momentum thickness θ. Table I sums up all derived scaling laws.

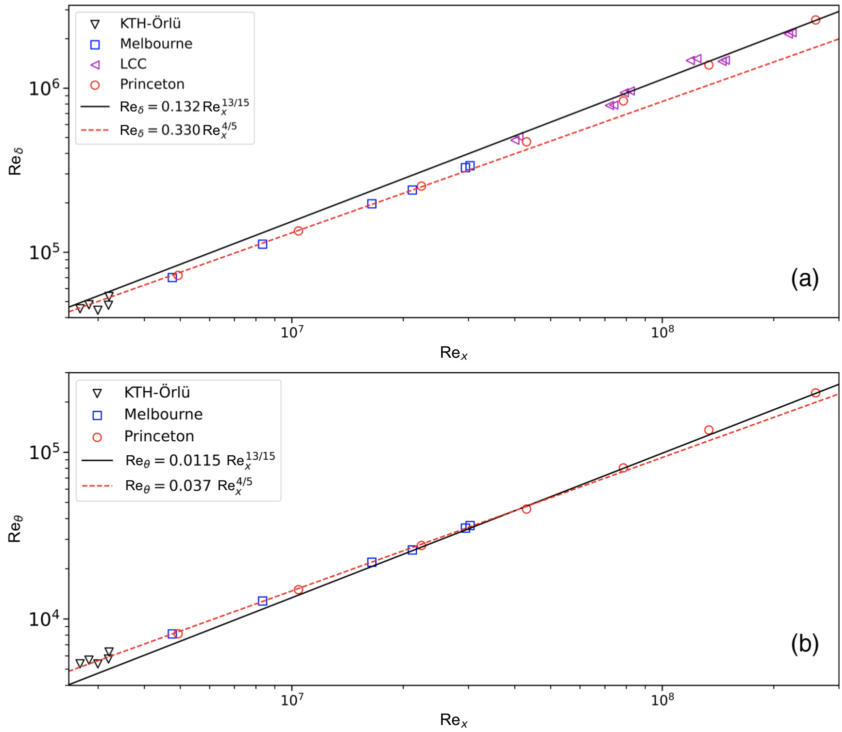

FIG. 7. (a) Variation of bulk Reynolds number, Re_δ, with that based on streamwise distance Re_x. (b) Variation of Reynolds number based on momentum thickness, Re_θ, with Re_x. Comparison with experimental data.

IV. CONCLUSIONS

In this work, we derived power-law formulations for the friction factor of boundary-layer flows as functions of different Reynolds number definitions dependent on the flow's regime. The formulas for the extreme-Re range result from a phenomenological description for the momentum-exchange mechanism at the local mesolayer of the flow, and scaling considerations of the axial mean-momentum equation normalized by bulk quantities. Unlike most formulations for the friction factor in the literature, our proposed formulation does not follow from log-law or power-law similarity assumptions on the mean velocity profile.

The empirical data suggest that the exchange mechanism proposed in this work appears valid for an extensive range of extreme Re_x flows, as shown in Figs. 5 and 6. In the intermediate turbulent regime, the classical $\mathrm{Re}_\delta^{-1/4}$ (or $\mathrm{Re}_x^{-1/5}$) scaling represents an excellent approximation. However, an open issue is the lack of a compatible description of the flow structure underlying the wall shear stress within this range. It is noted that we are proposing a description with two different power-law

TABLE I. Scaling laws for different flow regimes.

Flow regime	Boundary-layer thickness	Friction factor
Laminar	$\mathrm{Re}_\delta \sim \mathrm{Re}_x^{1/2}$	$C_f^L \sim \frac{1}{\mathrm{Re}_\delta} \sim \frac{1}{\mathrm{Re}_x^{1/2}}$
Intermediate-Re	$\mathrm{Re}_\delta \sim \mathrm{Re}_x^{4/5}$	$C_f^I \sim \frac{1}{\mathrm{Re}_\delta^{1/4}} \sim \frac{1}{\mathrm{Re}_x^{1/5}}$
Extreme-Re	$\mathrm{Re}_\delta \sim \mathrm{Re}_x^{13/15}$	$C_f^E \sim \frac{1}{\mathrm{Re}_\delta^{2/13}} \sim \frac{1}{\mathrm{Re}_x^{2/15}}$

formulations with only one free parameter each, with the transition between regimes starting around $\mathrm{Re}_x \approx 3\text{–}4 \times 10^6$. In contrast, log-law formulations depend on two free parameters, but with a wider region of validity as displayed in Figs. 4 and 5.

We remark that empirical power-law approximations for the skin-friction factor are classical subjects. Several expressions have been proposed in the literature [1,33], for example, the $1/6$ law as a function of Re_δ [in contrast to the $1/6.5$ law proposed in Eq. (20)], or the $1/7$ law as a function of Re_x [in contrast to the $1/7.5$ law proposed in Eq. (24)]. It is noted, nonetheless, that besides being obtained from a physical description of the flow, the power-law formulations presented in this work yield smaller deviations from recent available experimental data, as observed in Figs. 4 and 5. The intermediate-Re power-law formulas (27) and (28) are also commonly found in the literature, but it is generally regarded as a purely empirical approximation, and not as a transitory regime.

Concerning the thickness of the boundary layer, the intermediate $4/5$ law is usually derived in textbooks [1] using a power-law approximation to the MVP, and it is also regarded as an empirical approximation. In Ref. [38], the authors obtain the $4/5$ law from spectral considerations and Kolmogorov's K41 scaling. Figure 7 shows, however, that this law should be considered valid only in the intermediate regime.

Scaling transition for the friction factor has also been observed for boundary layers in Ref. [39], where the authors adjusted empirical power-law fits. They proposed that the friction factor scales as $C_f \sim \mathrm{Re}_\delta^{-0.2209}$ for $\mathrm{Re}_\delta \lesssim 2 \times 10^5$ (in contrast to $\mathrm{Re}_\delta^{-0.25}$), and as $C_f \sim \mathrm{Re}_\delta^{-0.1363}$ for higher Reynolds number flows (whereas $2/13 = 0.1538\ldots$).

We also remark that the proposed power-law relation for the extreme Reynolds regime, $\mathrm{Re}_\theta \sim \mathrm{Re}_\delta \sim \mathrm{Re}_x^{13/15}$, is very close to the one obtained empirically by Nagib *et al.* in Ref. [33], where the authors propose $\mathrm{Re}_\theta \sim \mathrm{Re}_x^{0.8659}$ (whereas $13/15 = 0.866\ldots$). It is noted that in this work we have considered $\theta \sim \delta$, for turbulent flows, but in Ref. [39], the authors suggest that $\theta/\delta \sim \mathrm{Re}_\delta^{-0.0228}$ for $\mathrm{Re}_\delta \leqslant 2 \times 10^5$, and that $\theta/\delta \sim \mathrm{Re}_\delta^{-0.0855}$ for larger Reynolds numbers. It is noted, however, that direct and reliable measures of δ and θ, independent of friction, are difficult to obtain and that for very large Reynolds number flows, the velocity profile becomes ever more affected by roughness effects. Regardless of the relation between θ and δ, the skin friction and the boundary-layer thickness can be reliably approximated by the formulas proposed in this work in terms of Re_δ, as well as a function of Re_θ. More importantly, the asymptotic errors for the skin friction in terms of the more practical parameter, Re_x, displayed in Fig. 6, are within experimental uncertainty.

A critical observation in the present study involves the employment of a mixed scale in the derived scaling, with the mesolayer scaling proportional to $\sqrt{\delta\delta_\nu}$. Mixed scales have played a pivotal role in recent advancements in turbulent boundary-layer research. For example, DeGraaf and Eaton [40] assert that the maximum value, $K_{\max}$, of turbulent kinetic energy (TKE) scales with the mixed scale $u_\tau U_\infty$. More recently, Wei [41] performed a novel dimensional analysis to identify an appropriate scaling for TKE and its dissipation. The author establishes that the controlling parameters in the near-wall region encompass the kinematic viscosity and the TKE dissipation at the wall, $\epsilon_{k,w}$. As a result, a suitable inner velocity scale constitutes the Kolmogorov wall velocity, u_ϵ, while the appropriate length scale is the Kolmogorov wall length, ν/u_ϵ. Wei also presents a new mixed scale for $k_{\max}$, defined as $\nu\epsilon_{k,w}/u_\tau^2$, which is substantiated by the inclusion of a controlling parameter in an innovative dimensional analysis approach.

It is also worth mentioning that in [42,43], the authors utilize the maximum Reynolds shear stress location to determine the appropriate scales for the outer region ofTBLs under adverse pressure gradient (APG) conditions. This approach yields a novel scaling of the mean-momentum equation for the outer region of APG TBLs.

Pursuant to Ref. [6], we are concurrently investigating the possibility of a power-law scaling for the friction factor being linked to a corresponding power law for the MVP in specific flow regions within the extreme Reynolds number regime.

It is essential to note that since the early 1930s, there has been an ongoing debate concerning the behavior of turbulent flow friction factors in pipes and boundary layers. Early models adopted the

Blasius friction relation for pipes and a $\mathrm{Re}_x^{1/5}$ power law for boundary-layer flows. However, this assumption was disproven for higher Reynolds numbers, and a satisfactory theoretical explanation remained elusive. Subsequently, the log law became the standard formulation. The work of Barenblatt as well as of Gioia and Chakraborty, has rekindled interest in this subject. It is also noteworthy that Prandtl proposed two power-law equations for flat plate turbulent boundary friction: $f \sim \mathrm{Re}_x^{1/5}$ and $f \sim \mathrm{Re}_x^{1/7}$. Both equations are currently employed by engineers in practical applications such as aeronautical flows, demonstrating their effectiveness in producing accurate results, as evidenced in [1].

The analysis herein naturally yields the $1/5$ relation when restricted to the Blasius regime. Furthermore, the proposed formulation approximates the $1/7$ law, which is refined by the $2/15$ law proposed for the extreme-Re regime. This study contends that the transition in the momentum-exchange mechanism primarily accounts for the coexistence of these two empirical power laws, frequently utilized in various applications. Furthermore, it provides a more robust explanation for the emergence of the $1/5$ law, beyond being a mere curve fitting.

Given the accuracy of the power-law correlations presented in this study, which are comparable to the more intricate logarithmic models within their respective regimes of validity, and considering the consistent application of the power law over several decades, we argue that the two power-law correlations presented herein, each with a single constant, constitute suitable physical models. Consequently, they can serve as parsimonious and straightforward alternatives for predicting the friction factor of flat plate turbulent boundary-layer flow.

ACKNOWLEDGMENTS

The authors acknowledge the support of Agência Nacional de Petróleo (ANP), National Council for Scientific and Technological Development (CNPq), and Petrogal Brasil.

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