Universal scaling regimes in rotating fluid turbulence

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(Received 1 December 2022; accepted 19 May 2023; published 22 June 2023)

We analyze the scaling properties of the energy spectra in fully developed incompressible turbulence in forced, rotating fluids in three dimensions (3D), which are believed to be characterized by *universal scaling exponents* in the inertial range. To elucidate the scaling regimes, we set up a scaling analysis of the 3D Navier–Stokes equation for a rotating fluid that is driven by large-scale external forces. We use scaling arguments to extract the scaling exponents, which characterize the different scaling regimes of the energy spectra. We speculate on the intriguing possibility of two-dimensionalization of 3D rotating turbulence within our scaling theory. Our results can be tested in large-scale simulations and relevant laboratory-based experiments.

DOI: 10.1103/PhysRevFluids.8.064611

I. INTRODUCTION

Nonequilibrium systems are described by the appropriate equations of motion for the relevant dynamical variables and exhibit much richer universal behavior than usually observed in equilibrium critical dynamics [1]. Hydrodynamic turbulence in fluids, described by the Navier–Stokes equation [2,3] for the evolution of the velocity field $\mathbf{v}$, is a prime example of an out-of-equilibrium system due to the external drive acting on the fluids. Interestingly, fully developed fluid turbulence in three dimensions (3D) and two dimensions (2D) show markedly different behavior. In 3D, the energy spectra follow the well-known K41 result for homogeneous and isotropic 3D hydrodynamics turbulence where the one-dimensional energy spectrum $E(k) \sim k^{-5/3}$ (hereafter K41) in the inertial range, where k is a wave vector [4]. This K41 result is quite robust and universal and found in wide-ranging natural systems, e.g., shear flows [5], viscoelastic fluids [6] and jet flows [7]. In contrast, 2D turbulence is characterized by an inverse cascade of energy at very large length scales with $E(k) \sim k^{-5/3}$, and forward cascade of enstrophy with $E(k) \sim k^{-3}$ at intermediate scales [8–12].

Rotating turbulence, i.e., turbulence in a rotating fluid, is a naturally occurring phenomenon in many astrophysical and geophysical flows, as well as in laboratory-based engineering fluid flows. The presence of the Coriolis forces is the distinctive feature of rotating turbulence, which should affect the scaling properties of rotating turbulence differently at different length scales. In spite of extensive studies, there is still no good agreement on the scaling of the energy spectra in rotating turbulence, in particular how the Coriolis forces affect the scaling of the energy spectrum at very large scales. Varieties of analytical, numerical, or experimental investigations of rotating turbulent fluids suggest that the kinetic-energy spectra in the rotation-dominated small-k regions should scale as $E(k) \sim k^{-m}$, $m \in (2, 3)$ [13–27]. A trend towards two-dimensionalization in rotating turbulence was detected in a study reported in Ref. [28], highlighting the anisotropic nature of the turbulent

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state. Recent perturbative studies indicate that the one-dimensional kinetic-energy spectra made out of the velocity component parallel to the rotation axis scales as $k^{-5/3}$, indistinguishable from the K41 prediction. In contrast, the one-dimensional kinetic-energy spectra made out of the velocity components lying in a plane normal to the rotation axis scales as k^{-3}, different from the K41 scaling [29]. The precise forms of the scaling of the energy spectra in rotating turbulence however is still not well settled.

There is a degree of formal similarity between the (linearized) equations of motion of rotating turbulence, which is nothing but the Navier–Stokes equation in a rotating frame (see below) and the equations for magnetohydrodynamic (MHD) turbulence in the presence of a mean magnetic field B_0 [30,31]. In the former case, the Coriolis forces lead to oscillatory modes, whereas in the MHD case, a nonzero B_0 gives rise to propagating Alfv́en waves. Strong Alfv́en waves are known to make the energy spectra in MHD anisotropic, and change the scaling as well [32]. In the same vein, strong Coriolis forces should make the scaling of energy spectra in rotating turbulence anisotropic and also *different* from its isotropic counterpart (i.e., the K41 scaling).

In this work, we revisit the universal scaling of energy spectrum in forced, statistically steady rotating turbulence in its inertial range. To this end, we have set up a scaling theory to study the scaling of the energy spectrum in the inertial range. We cover both the weak and strong rotation limits. In the former case, unsurprisingly, the K41 result is obtained. With stronger rotation, anisotropic scaling with different exponents ensues. In particular, in the wave vector region $k_\perp \gg k_\parallel$, our scaling theory gives the scaling of the 2D spectra $E(k_\perp, k_\parallel)$, where $\mathbf{k}_\perp$ and $k_\parallel$ are the components of the wave vector $\mathbf{k}$ in the plane perpendicular to the rotation axis (here the $\hat{z}$ axis) and along the rotation axis, respectively. We find $E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}$ for $k_\perp \gg k_\parallel$, which agrees with the Kuznetsov–Zakharov–Kolmogorov spectra predicted by the weak inertial-wave turbulence theory for the rotating fluids [19]. This implies partial two-dimensionalization. We show that this result is unaffected by nonlinear fluctuation corrections at the one-loop order. We further demonstrate that this result could be obtained by demanding that the cascade of the kinetic-energy flux is hindered by a nonzero helicity, which is naturally present in a rotating fluid. In the opposite limit of $k_\perp \ll k_\parallel$, we get $E(k_\perp, k_\parallel) \sim k_\perp^{-1} k_\parallel^{-2}$. We also show perturbatively in the rotation Ω that the kinetic-energy flux is indeed reduced by it. The remainder of the article is organized as follows: In Sec. II, we set up the forced Navier–Stokes equation in a rotating fluid. Then in Sec. III we set up the scaling arguments. Then in Sec. III A, we revisit the K41 scaling scaling in an isotropic, nonrotating fluid turbulence and show how our scaling theory reproduces it. Next, in Sec. III B, we show that, for weak rotation, the energy spectra again show the K41 scaling. Then in Sec. III C we study the 2D anisotropic energy spectra in the opposite limit of large Ω. After that, in Sec. IV, we discuss and speculate on the energy spectrum in decaying rotating turbulence. In Sec. V we discuss and summarize our results. We provide some technical results, including a perturbative demonstration of the reduction of the kinetic-energy flux by helicity, in the Appendix for interested readers.

II. TURBULENCE IN A ROTATING FLUID

The Navier–Stokes equation for the velocity field $\mathbf{v}(\mathbf{r}, t)$ in a rotating frame with rotation $\mathbf{\Omega} = \omega\hat{z}$ is given by

$$\frac{\partial \mathbf{v}}{\partial t} + 2(\mathbf{\Omega} \times \mathbf{v}) + \lambda(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p^*}{\rho} + \nu\nabla^2\mathbf{v} + \mathbf{f}, \tag{1}$$

where $p^* = p + \frac{1}{2}|\mathbf{\Omega} \times \mathbf{v}|^2$ is the effective pressure. We assume $\mathbf{\Omega} = \Omega\hat{z}$, i.e., the rotation is about the z axis; see Fig. 1.

In this case, (1) may be written in terms of components as

$$\frac{\partial v_z}{\partial t} + \lambda(\mathbf{v} \cdot \nabla)v_z = -\frac{\partial_z p^*}{\rho} + \nu\nabla^2 v_z + f_z, \tag{2}$$

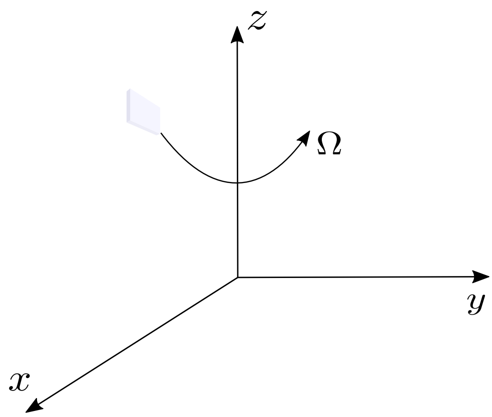

FIG. 1. Geometry of the rotating fluid. We assume the rotation to be about the z axis. In this coordinate system, $\mathbf{k}_\perp = (k_x, k_y)$ and $k_\parallel = k_z$.

$$\frac{\partial v_x}{\partial t} - 2\Omega v_y + \lambda(\mathbf{v}\cdot\nabla)v_x = -\frac{\partial_x p^*}{\rho} + \nu\nabla^2 v_x + f_x, \tag{3}$$

$$\frac{\partial v_y}{\partial t} + 2\Omega v_x + \lambda(\mathbf{v}\cdot\nabla)v_y = -\frac{\partial_y p^*}{\rho} + \nu\nabla^2 v_y + f_y. \tag{4}$$

Here, $\lambda = 1$. The equations (2)–(4) in 3D admit two conserved quantities in the inviscid limit: (i) kinetic energy $E = \int d^3x\rho\, v^2/2$ and (ii) helicity $H = \int d^3x\, \mathbf{v}\cdot\nabla\times\mathbf{v} = \int d^3x\, \mathbf{v}\cdot\boldsymbol{\omega}$, where $\boldsymbol{\omega} \equiv \nabla\times\mathbf{v}$ is the local vorticity. In the viscous steady states, in a Kolmogorov-like picture neglecting intermittency, E and H should have constant (i.e., scale-independent) fluxes. Clearly, E/H has the dimension of a length, which allows us to define a length-scale $l^* = E/H$. We assume the external forces to be nonhelical, i.e., no helicity injection by the forces. Thus helicity is generated in the bulk only by the global rotation. We consider the incompressible limit, i.e., the mass density $\rho =$ const., or, equivalently, $\nabla\cdot\mathbf{v} = 0$. At this stage, it is useful to set up the notations. Below we use $\tilde{\omega}$ and $\tilde{\Omega}$ to denote Fourier frequencies, while Ω and $\boldsymbol{\omega}$ represent the global rotation frequency and vorticity, respectively.

III. SCALING ANALYSIS

To classify the scaling regimes, we first define the following dimensionless numbers:

(i) Rossby number $R_o = U/(2\Omega L)$,

(ii) Reynolds number $R_e = \frac{LU}{\nu}$, and

(iii) Ekman number $Ek = R_o/R_e = \nu/(2\Omega L^2)$,

where L is the linear system size, and U is a typical velocity. We expect to find two distinct scaling regimes as characterized by R_o (or Ω):

(i) weak rotation $\Omega \to 0$, or $R_o \to \infty$,

(ii) large rotation $\Omega \to \infty$, or $R_o \to 0$.

Since the rotation picks up a direction (the axis of rotation, here the z axis), system is generally anisotropic. We therefore construct an anisotropic scaling theory of the system: we assume scaling under the transformations

$$\mathbf{r}_\perp \to l_\perp \mathbf{r}_\perp, \quad z \to l_\perp^{\xi} z, \quad \mathbf{v}_\perp \to l_\perp^{a_\perp}\mathbf{v}_\perp, \quad v_z \to l_\perp^{a_z} v_z. \tag{5}$$

Here, $\mathbf{r}_\perp \equiv (x,\ y)$, $\mathbf{v}_\perp \equiv (v_x,\ v_y)$. In a general anisotropic situation, $\xi \neq 1$. We also allow for the possibility $a_\perp \neq a_z$, i.e., $\mathbf{v}_\perp$ and v_z may not scale in the same way under spatial rescaling. We further define timescale t to scale as

$$t \sim l_\perp^{\tilde{z}}, \tag{6}$$

where $\tilde{z}$ is a dynamic exponent. Furthermore, we define a phenomenological dimensionless constant $\tilde{\Omega}$ by

$$\hat{\Omega} = \frac{[\boldsymbol{\Omega}]}{[\boldsymbol{\omega}]}, \tag{7}$$

Here, $[\ldots]$ implies "in a dimensional sense" [32]. Clearly, the two limiting cases $\hat{\Omega} \to 0$ and $\hat{\Omega} \to \infty$ phenomenologically correspond to $R_o \to \infty$ and $R_o \to 0$.

We note that, by balancing the Coriolis force terms against the advective nonlinear terms, we can extract a length-scale L_o. In a scaling sense, we set

$$\Omega v \sim \frac{v^2}{L_0}, \tag{8}$$

giving $v \sim \Omega L_0$. Dimensionally speaking, energy dissipation

$$\epsilon \sim \frac{v^3}{L_0} \sim \Omega^3 L_0^2. \tag{9}$$

This gives

$$L_0 \sim \sqrt{\frac{\epsilon}{\Omega^3}}. \tag{10}$$

The corresponding wave vector $k_0 \equiv 2\pi/L_0$ is the *Zeman wave vector* [33]. For length scales $L \gg L_0$ (or equivalently, for wave vector $k \ll k_0$), we naïvely expect Coriolis force terms to be important relative to the nonlinear interactions, effects of the rotation should be strong, and hence non-K41 spectra should follow. This is in analogy with the Ozmidov scale in stably stratified turbulence [34,35]. The role of K_0 in freely decaying rotating turbulence together with the associated nature of the anisotropy was systematically investigated in Ref. [36] by direct numerical simulations (DNS). In a forced rotating turbulence, energy dissipation ϵ is directly related to the forcing amplitude. Thus, the Zeman wave vector gives a comparison between the forcing amplitude and the Coriolis force, and should be significant in analyzing forced rotating turbulence [37]; see also Ref. [38] for general illustrating discussions on this topic. In the opposite limit of $L \ll L_0$ (or $k \gg k_0$), Coriolis forces should be irrelevant, and hence K41 scaling should follow. Thus a dual scaling is believed to exist [14,24,33]. In terms of the dimensionless numbers, we are interested in $R_e \to \infty$ for fully developed turbulence. Together with $R_e \to \infty$ (implying fully developed turbulence) and $Ek \to 0$ (implying Coriolis forces dominating over the viscous damping at large scales), we can have two situations: (i) $R_o \to \infty$ for weak rotation, and (ii) $R_o \to 0$ for strong rotation. Lastly, one has the dissipation scale η_d, such that for length scales smaller than η_d, the dissipation range ensues. Then in terms of the length scales defined above, we can have the following scenarios: In a sufficiently large system, there should be adequate scale separations, such that $\eta_d \ll L_0$, i.e., L_0 should belong to the inertial range. This should allow for both the scaling regimes, viz. K41 and non-Kolmogorov scaling regimes to be observed.

Below we set up our scaling analysis by following the line of arguments used in Refs. [32,35]. Similar scaling analysis for 3D MHD turbulence [32] and turbulence in stably stratified fluid turbulence [35] revealed the presence of anisotropic scaling and the possibility of weak dynamic scaling.

A. Nonrotating isotropic case

For a nonrotating, isotropic fluid, $\Omega = 0$ in (3) and (4) gives the usual 3D isotropic Navier–Stokes equation. Let us briefly revisit the extraction of the K41 scaling by applying the scaling arguments on the usual 3D Navier–Stokes equation first. Due to the isotropy of the system, we expect $\xi = 1$ strictly, and make no distinction between $l_\perp$ and $l_\parallel$, the rescaling factors of $\mathbf{r}_\perp$ and z respectively: $l_\perp \sim l_\parallel \sim l$. Demanding scale invariance [32], we find

$$\frac{\partial \mathbf{v}}{\partial t} \sim \mathbf{v} \cdot \nabla \mathbf{v} \Rightarrow l^{a-\tilde{z}} = l^{2a-1} \Rightarrow a = 1 - \tilde{z}. \tag{11}$$

Next, in a mean-field like approach, we assume the kinetic-energy flux or the kinetic-energy dissipation per unit mass is scale invariant in the inertial range. This gives

$$\frac{\partial v^2}{\partial t} \sim l^0 \Rightarrow 2a = \tilde{z}. \tag{12}$$

Combining then, we get $a = 1/3$, $\tilde{z} = 2/3$. This corresponds to a one-dimensional (1D) kinetic energy spectra $E(k) \sim k^{-5/3}$, the expected K41 result.

B. Weak rotation effects: $k \gg 2\pi/L_0$

To study the effects of weak rotation on the scaling of the energy spectra, we consider the limit $R_o \to \infty$, or $\Omega \to 0$. Equivalently, we consider length scales $L \ll L_0$, with the understanding that $L \gg \eta_r$, the dissipation scale. In this case, the Coriolis force is unimportant. Hence $\langle H \rangle \approx 0$, where $\langle \ldots \rangle$ implies averages over the statistical steady states. Thus, the flux of E is the relevant (in the Kolmogorov sense) flux. Then, proceeding as in Ref. [32], we unsurprisingly recover the K41 scaling:

$$a_\perp = a_z = 1/3, \quad \tilde{z} = 2/3, \quad \xi = 1. \tag{13}$$

The last of the above results in (13) naturally means isotropic scaling (although geometry remains anisotropic). Furthermore, if we let $R_o \sim l^\eta$ and demand scale invariance of all the terms (including the Coriolis force terms), we find

$$\eta = \tilde{z} = 2/3. \tag{14}$$

Thus $R_o \to \infty$ as $l \to \infty$, which corresponds to nonrotating and isotropic fully developed turbulence. We have assumed that the nonlinear coupling constant does not scale under spatial rescaling, which is consistent with the nonrenormalization of λ due to the Galilean invariance of the Navier–Stokes equation. The scaling of the viscosity is controlled by the dynamic exponent $\tilde{z}$.

C. Strong rotation effects: $k \ll 2\pi/L_0$

We next consider the large rotation case, i.e., when $R_o \leqslant O(1)$, or $\Omega \geqslant O(1)$. Two distinct features are expected for strong rotations. First of all, the energy spectrum should be spatially anisotropic, i.e., the dependence on $k_\perp$ and $k_\parallel$ should be different. Second, the energy content in the in-plane component $\mathbf{v}_\perp$ may be different from v_z, the component along the rotation axis. This is an effective *reduction* in components of the flow field, and is different from spatial anisotropy. Below we analyze the scaling in several equivalent ways.

Balancing different terms of (3) and (4) we find

$$\tilde{z} = 0, \quad a_\perp = 1, \quad a_z = \xi. \tag{15}$$

To proceed further, we allow for the possibility that not only the spatial scaling may be anisotropic, there may be different dynamic exponents for $\mathbf{v}_\perp$ and v_z, with $\tilde{z}$ being identified as the dynamic exponent $\tilde{z}_\perp$ of $\mathbf{v}_\perp$. To study this, we separately consider the contribution to the kinetic energy from the in-plane velocity $\mathbf{v}_\perp$ and normal component of the velocity v_z. Interestingly, the exponents in (15) mean that the flux of the "in-plane kinetic energy" $E_\perp \equiv \int d^3x\, v_\perp^2$ cannot be scale-independent! Let us now consider the kinetic energy $E_z \equiv \int d^3x\, v_z^2$ flux of v_z. If we assume $\tilde{z} = 0$ is the dynamic exponent of v_z also, then $a_z = 0 = \xi$ can actually keep the flux of E_z scale independent. However,

$a_z = 0$ is unexpected because it means v_z *does not* scale with l at all. Assume the dynamics $z_\parallel$ of v_z be nonzero: $z_\parallel > 0$. Now consider Eq. (2) and balance

$$\frac{\partial v_z}{\partial t} \sim v_z \partial_z v_z \Rightarrow a_z = 1 - z_\parallel. \tag{16}$$

Next, demanding scale-invariance of the flux of E_z gives

$$2a_z = \tilde{z}_\parallel \Rightarrow a_z = \tfrac{1}{3}, \quad \tilde{z}_\parallel = \tfrac{2}{3}. \tag{17}$$

This further means $\xi = a_z = 1/3$. Notice that with $a_\perp = 1$, $\xi = 1/3$, $(\mathbf{v}_\perp \cdot \nabla_\perp) v_z$, and $v_z \partial_z v_z$ scale the same way. Since we get $a_\perp > a_z$, we should have $\langle v_\perp^2 \rangle \gg \langle v_z^2 \rangle$ in the long-wavelength limit, suggesting concentration of the kinetic energy in a plane normal to the rotation axis [39]. This implies an effective *two-componentization* of the turbulent velocity. See Refs. [28,39,40] for related discussions. On the other hand, $\tilde{z} < \tilde{z}_\parallel$ implies $\mathbf{v}_\perp \ll v_z$ in the long-time limit, a conclusion contradictory to our above inference. In fact, this alternative scenario implies a type of dimensional reduction, where most of the energy is confined to the z direction. We are unable to conclusively predict which of these two scenarios actually holds. Numerical studies should be useful in this regard.

Does the ensuing flow field in the limit $R_o \to \infty$ truly resemble 2D turbulence? In our opinion, the answer is no. First of all, the flow remains overall 3D incompressible. This means the effective 2D flow field might be *2D compressible*, which is an interesting possibility. Second, it is not clear whether the direction of the kinetic-energy cascade becomes backward, a hallmark of pure 2D turbulence. Third, enstrophy is a conserved quantity in the inviscid limit of pure 2D turbulence, whereas it is not expected to be so in the 3D rotating case even in the limit of high rotation. Therefore, notwithstanding the dominance of $v_\perp$ over v_z, the resulting flow field should be fundamentally different from pure 2D nonrotating turbulence. Lastly, since $z_\perp \neq z_\parallel$, we find weak dynamic scaling [32,41]. We now calculate the scaling of the two-dimensional kinetic-energy spectra $E_\perp(k_\perp, k_\parallel)$ and $E_z(k_\perp, k_z)$, such that total kinetic energy $E_{\text{tot}} = \int dk_\perp dk_z [E_\perp(k_\perp, k_z) + E_z(k_\perp, k_z)]$. The scaling of $E_\perp(k_\perp, k_z)$ and $E_z(k_\perp, k_z)$ can be obtained as follows: We use the general definition to write in a dimensional or scaling sense,

$$\begin{aligned} v_m(\mathbf{k}, \tilde{\omega}) &\sim \int v_m(\mathbf{x}, t) \exp(i\mathbf{k}_\perp \cdot \mathbf{x}_\perp) \exp(ik_\parallel z) \exp(i\tilde{\omega} t) d^2 x_\perp dz dt \\ &\sim l_\perp^{a_m} l_\perp^2 l_\parallel \sim \frac{1}{k_\perp^{a_m + 2}} \frac{1}{k_\parallel}. \end{aligned} \tag{18}$$

Here, $m = \perp, \parallel$. Furthermore,

$$\langle \mathbf{v}_\perp(\mathbf{k}_1) \cdot \mathbf{v}_\perp(\mathbf{k}_2) \rangle = F_\perp(k_1) \delta(\mathbf{k}_1 + \mathbf{k}_2), \tag{19}$$

$$\langle v_z(\mathbf{k}_1) v_z(\mathbf{k}_2) \rangle = F_\parallel(k_1) \delta(\mathbf{k}_1 + \mathbf{k}_2). \tag{20}$$

Dimensionally then,

$$F_\perp \sim \frac{1}{k_\perp^{a_\perp + 2} k_\parallel}, \quad F_\parallel \sim \frac{1}{k_\perp^{a_\parallel + 2} k_\parallel}. \tag{21}$$

This gives for the two-dimensional energy spectra

$$E_\perp(k_\perp, k_\parallel) \sim k_\perp^{-3}, \quad E_z(k_\perp, k_z) \sim k_\perp^{-5/3}. \tag{22}$$

If we ignore anisotropy, we can define two corresponding one-dimensional energy spectra $E_\perp(k)$ and $E_z(k)$ from (22) by $E_{\text{tot}} = \int dk [E_\perp(k) + E_z(k)]$. Notice that, neglecting anisotropy, the one-dimensional spectra corresponding to $E_\perp(k) \sim k_\perp^{-3}$ should scale as k^{-2} as argued above, in agreement with Refs. [14,24,33]; see also Ref. [40] for a similar result in an asymptotic quasinormal Markovian model for wave turbulence in rapidly rotating flows. Nonetheless, in spite of this agreement, we notice that our results (22) appear to suggest that $E_\perp(k_\perp, k_z)$ and $E_z(k_\perp, k_z)$ have *no* k_z dependence, which should be unphysical. We try to rectify this below.

First of all, for large Ω, the scaling should be dominated by the Coriolis forces. The vorticity $\boldsymbol{\omega}$ satisfies [19]

$$\partial_t \boldsymbol{\omega}(\mathbf{k}) = -2\Omega \frac{k_\parallel \hat{e}_\mathbf{k} \times \boldsymbol{\omega}_\mathbf{k}}{k}, \tag{23}$$

giving timescale $\tau \sim k/k_\parallel \sim k_\perp/k_\parallel$ for $k_\perp \gg k_\parallel$.

It is thus reasonable to assume that τ as defined above is the relevant timescale when $R_o \to 0$. In what follows below, we do not make any distinction between $v_\perp$ and v_z. We now impose the scale-independence of the kinetic-energy flux Π. The energy flux may be calculated from the Navier–Stokes equations (1). We find

$$\begin{aligned}\Pi = -2\lambda^2 \int_{\mathbf{k},\mathbf{q},\tilde{\omega},\tilde{\Omega}} &[M_{imn}(\mathbf{k}) M_{ijp}(-\mathbf{k}) \langle v_m(-k+q, -\tilde{\omega}+\tilde{\Omega}) v_p(\mathbf{k}-\mathbf{q}, \tilde{\omega}-\tilde{\Omega})\rangle \\ &+ M_{jmn}(\mathbf{k}) M_{ijp}(\mathbf{q}) \langle v_i(-k, -\tilde{\omega}) v_n(\mathbf{k}, \tilde{\omega})\rangle \langle v_m(-\mathbf{k}+\mathbf{q}, -\tilde{\omega}+\tilde{\Omega}) v_p(\mathbf{k}-\mathbf{q}, \tilde{\omega}-\tilde{\Omega})\rangle \\ &+ M_{jmn}(\mathbf{k}-\mathbf{q}) M_{ijp}(\mathbf{k}) \langle v_i(-\mathbf{k}, -\tilde{\omega}) v_n(\mathbf{k}, \tilde{\omega})\rangle \langle v_j(\mathbf{q}, \tilde{\Omega}) v_m(-\mathbf{q}, -\tilde{\Omega})\rangle].\end{aligned} \tag{24}$$

Here, $M_{ijp}(\mathbf{k}) = P_{ij}(\mathbf{k}) k_p + P_{ip}(\mathbf{k}) k_j$. Since we are interested in the scaling, it suffices to consider the scaling of Π, suppressing indices and wave vector labels. At this one-loop order, suppressing indices and wave vector labels, and assuming $k_\perp \gg k_\parallel$,

$$\Pi \sim \int d^2k_\perp d^2p_\perp dk_\parallel dp_\parallel \int dt \int d^2k_\perp\, k_\perp^2 C^2 \sim \text{const.}, \tag{25}$$

where $C \equiv \langle v\, v\rangle$ is the correlation function, again suppressing indices and wave vector labels. This gives

$$C \sim k_\perp^{-7/2} k_\parallel^{-1/2}, \tag{26}$$

where we have used $\int dt \sim \tau$. This implies for the two-dimensional kinetic-energy spectra

$$E_\perp(k_\perp, k_\parallel) \sim C k_\perp \sim k_\perp^{-5/2} k_\parallel^{-1/2}, \tag{27}$$

as in the wave turbulence theory. It now behooves us to show that the scaling of τ with wave vector does not get renormalized at the one-loop order. We restrict ourselves here to a scaling-level demonstration. As shown in the Appendix, the one-loop self-energy $\Sigma_{ik}(\mathbf{k}, \tilde{\omega})$ has the form

$$\Sigma_{ij}(\mathbf{k}, \tilde{\omega}) \sim \int d^3q d\tilde{\Omega}\, M_{imn}(\mathbf{k}) \langle v_m(\mathbf{q}, \tilde{\Omega}) v_r(-\mathbf{q}, -\tilde{\Omega})\rangle G_{ns}(\mathbf{k}-\mathbf{q}, \tilde{\omega}-\tilde{\Omega}) M_{srj}(\mathbf{k}-\mathbf{q}). \tag{28}$$

Here, $G_{ns}(\mathbf{k}, \tilde{\omega})$ is the propagator defined via

$$G_{ns}(\mathbf{k}, \tilde{\omega}) \equiv \left\langle \frac{\delta v_n(\mathbf{k}, \tilde{\omega})}{\delta f_s(\mathbf{k}, \tilde{\omega})} \right\rangle. \tag{29}$$

Considering the one-loop self-energy Σ and suppressing indices and wave vector labels (see also the Appendix), we obtain

$$\Sigma(\mathbf{k}, \omega = 0) \sim k_\perp^2 \int d^2q_\perp dq_z \int dt\, G(t) C(t), \tag{30}$$

where G is a propagator. Assuming the dominant timescale in the above time integral is given by τ, we get (in a scaling sense)

$$k_\perp^2 \int d^2q_\perp dq_z \frac{q_\perp}{q_z} q_\perp^{-7/2} k_\parallel^{-1/2} \sim [k_\perp]^{3/2} [k_\parallel]^{1/2}, \tag{31}$$

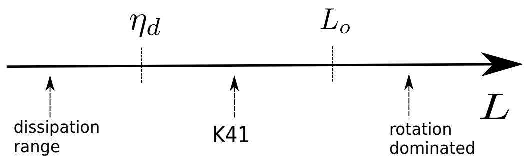

FIG. 2. Schematic diagram illustrating the scaling regimes at different length scales. Dissipation range for small scales, K41 scaling regime at the intermediate scales and rotation-dominated scaling regimes at the largest scales are shown.

which is less singular than the bare form of τ. Thus our scaling results on the 2D kinetic-energy spectra remain unaffected by the advective nonlinearities in the asymptotic long-wavelength limit.

Interestingly, we can also derive the above results by using phenomenological arguments of suppression of the kinetic-energy flux by the helicity generated by the rotation, which are similar to the arguments set up in Ref. [32] for scaling of the energy spectra in the presence of a strong mean magnetic field in magnetohydrodynamic turbulence. It is known that the predominant role of a (large) nonzero helicity flux is to hinder the cascade of the kinetic-energy flux [42]. In fact, it is easy to see from (3) and (4) that a large Ω should suppress the nonlinear effects, relative to the Coriolis force terms. Since the nonlinear terms are responsible for the cascade phenomena, we expect the flux of $E_\perp$ to be suppressed by a large Ω. This is similar to the suppression of the energy flux by a strong mean magnetic field in fully developed magnetohydrodynamic turbulence [32]; see also similar treatment for turbulence in a stably stratified fluid [35]. We write

$$\left[\frac{\partial v_\perp^2}{\partial t}\right]\left[\frac{\omega^2}{\Omega^2}\right] \sim l^0 \tag{32}$$

as the condition of the flux being scale independent. Since dimensionally, $[\omega^2] \sim [v_\perp^2/l^2]$, we get

$$v_\perp \sim l_\perp^{3/4} l_\parallel^{-1/4}. \tag{33}$$

This gives

$$E_\perp(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}, \tag{34}$$

in agreement with the conclusion from the wave turbulence theory approaches [19]. It is easy to get the spectrum in the opposite limit $k_\parallel \gg k_\perp$. In this limit, $\tau^{-1} \sim k_\perp^0 k_\parallel^0$. At this one-loop order, suppressing indices and wave vector labels, and assuming $k_\perp \ll k_z$,

$$\Pi \sim \int d^2k_\perp d^2p_\perp dk_z dp_\parallel \int d^2k_\perp\, k_z^2 C^2 \sim \text{const.} \tag{35}$$

Proceeding as before, we find

$$E_\perp(k_\perp, k_\parallel) \sim k_\perp^{-1} k_\parallel^{-2}. \tag{36}$$

In each of these cases, the corresponding one-dimensional spectra, without making any distinction between $k_\perp$ and $k_\parallel$, scale as k^{-2}, consistent with the recent shell-model studies on rotating turbulence [43].

A pictorial summary of the scaling regimes are shown in Fig. 2.

In the above, we have implicitly assumed that the kinetic-energy flux E is the relevant flux (in the Kolmogorov sense). As we have discussed above this holds for length scales $\gg l^*$. An interesting case may arise if $l^* \gg L_0$, in which case in the window between L_0 and l^*, helicity flux H dominates over E. If this indeed holds, the scaling of the energy spectra might change within this window. We do not discuss this further here.

IV. DECAYING ROTATING TURBULENCE

How to extend the above scaling arguments to decaying rotating turbulence remains an important issue. Decaying turbulence is fundamentally different from forced, steady turbulence due to the former not being in a steady state. All physical quantities in decaying turbulence decay in time. It is believed that for a large enough initial Reynolds number with a K41-type initial energy spectrum, there should be an intermediate time window in which the energy spectrum retains its power-law form, but has a time-dependent (decaying) amplitude in the form of a power law in time [44]. In the asymptotic large limit of time, the energy spectrum should, however, decay faster than a power law, presumably exponentially, being controlled by the viscous dissipation range. Recent numerical studies [26] revealed that in decaying turbulence, the anisotropic energy transfer is predominantly towards the equator, which is distinct from forced rotating turbulence, where the energy transfer is equatorward only for the scales large than the forcing scales, whereas for smaller scales it is poleward. Our above scaling analysis for forced, steady rotating turbulence has opened the possibilities of either $|v_\perp| \gg |v_z|$ or $|v_\perp| \ll |v_z|$, although cannot uniquely determine which one holds. We speculate these are reflections of the anisotropic energy transfer to either equatorward or poleward in forced rotating turbulence. More work is necessary to firmly establish this connection. Furthermore, in the case of decaying turbulence, all quantities are time dependent. Assuming a rapid, non-power-law form of temporal decay, there is no dynamic scaling. Nonetheless, assuming spatial scaling and following our line of arguments, we get $a_\perp = 1$, $a_z = \xi$. Assuming $\xi < 1$, a reasonable expectation, we see that under rescaling, the system effectively flattens, leading to some sort of two-dimensionalization. We believe this observation should be connected to the numerical results in Ref. [26]. Lastly, there is another important issue concerning the scaling of the energy spectrum in decaying turbulence. We have argued above that in steady forced turbulence, a non-K41 scaling should be visible for very large length scales bigger than the Zeman scale L_0, whereas in an intermediate window of length scales that is smaller than L_0 but bigger than the dissipation scale, K41 scaling should be displayed. Thus, a pertinent question is whether $L_0 = 2\pi/k_0$ grows or decays in time if the energy supply to a forced rotating turbulent fluid is suddenly switched off. We can get a hint to this question from the definition (10) of L_0. In (10), consider ϵ as the time-dependent energy dissipation rate (as opposed to a temporally constant one, as assumed in the scaling arguments employed in case of steady turbulence). In a decaying situation, ϵ should be a decaying function of time, as the total energy is depleted. This means, as time grows, L_0 should get reduced, or, equivalently, the Zeman wave vector k_0 should grow. Thus, to observe the K41 scaling one needs to go to smaller and smaller length scales (or larger and larger wave vectors) as time grows. In other words, the range of length scales showing the K41-type energy spectrum should reduce in time; see Fig. 3 for a schematic picture. This is not surprising because, with a decaying ϵ, kinetic-energy flux decreases relative to the Coriolis forces, ultimately narrowing the range of the K41 spectrum. We thus expect that, in an intermediate time window, which itself can be fairly large for a large enough system and Reynolds number, the dual scaling of the energy spectrum changes in a way to extend the range of the non-K41 scaling for small k and shorten the range of the K41 scaling for intermediate values of k, with an overall amplitude that decreases in time. We hope our work here will provide impetus for further studies along these lines.

V. SUMMARY AND OUTLOOK

In this work we have developed a scaling theory for fully developed incompressible hydrodynamic turbulence in a rotating fluid in the inertial range. We have studied the scaling of the energy spectrum in the inertial range for weak and strong rotations, i.e., for small and large Rossby number R_o. We argue that for wave vectors smaller than the Zeman wave vector rotation is important, whereas in the opposite limit, rotation is unimportant. It is therefore expected that in the former regime the scaling may be different from the K41 scaling, but, in the other regime, K41 scaling should ensue. The scaling theory that we developed here bears this out.

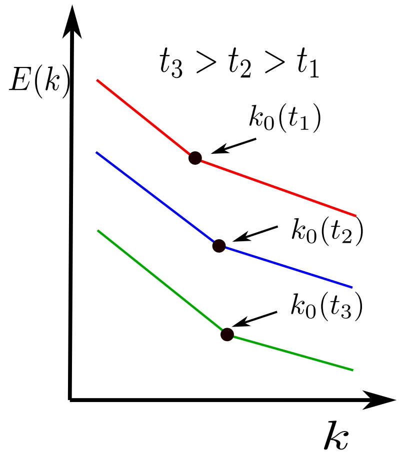

FIG. 3. Schematic time-evolution of the dual scaling of the one-dimensional energy spectrum $E(k)$ versus k in a log-log plot in decaying turbulence. The K41 scaling $\approx k^{-5/3}$ is seen for $k > k_0$, the Zeman wave vector or crossover wave vector, whereas the non-K41 one-dimensional spectrum $\approx k^{-2}$ (we do not consider anisotropy here for simplicity) is observed for $k < k_0$. As k_0 (the Zeman wave vector) grows in time, with t_1 being earliest time and t_3 being the latest, the range of wave vectors showing K41 scaling shrinks and the range showing non-K41 scaling grows. Unsurprisingly, the overall amplitude of $E(k)$ decays in time.

Our scaling theory reveals that in the rotation dominated regime, not only the scaling itself is anisotropic (i.e., different dependence on $k_\perp$ and $k_\parallel$), the scaling of $v_\perp$ and v_z are different. Even the dynamic exponents of $v_\perp$ and v_z are different, indicating weak dynamic scaling. This suggests that in the limit of a large rotation, the flow fields are dominated by only some of the velocity components. However, our theory cannot conclusively predict whether $v_\perp$ or v_z will be the dominant part. We further speculate that in a decaying rotating turbulence, the wave vector range of the non-K41 scaling should increase in time, whereas the range of the K41 scaling should decrease. Numerical simulations should be useful to make more quantitative conclusions.

We have throughout assumed that the kinetic-energy flux is the relevant flux (in the Kolmogorov sense), neglecting the helicity flux. However, for sufficiently large rotation, there may be a window of length scales where the helicity flux is the dominant flux. In this regime, the scaling of the energy spectra should be different. This will be discussed elsewhere.

ACKNOWLEDGMENT

A.B. thanks the SERB, DST (India) for partial financial support through the MATRICS scheme (file no. MTR/2020/000406).

APPENDIX: REDUCTION OF THE KINETIC ENERGY FLUX IN ROTATING TURBULENCE: PERTURBATION THEORY

We now calculate the kinetic-energy flux Π in the perturbation theory. We derive (24) and calculate Π to lowest nontrivial order in Ω. We start by eliminating pressure from the Navier–Stokes equation (1) in a rotating frame. We obtain

$$(-i\tilde{\omega} + \nu k^2)v_i + 2P_{im}(\mathbf{k})\Omega\epsilon_{mzp}v_p(\mathbf{k},\tilde{\omega}) + i\frac{\lambda}{2}M_{ijp}(\mathbf{k})\sum_{\mathbf{q},\tilde{\omega}} v_j(\mathbf{q},\tilde{\omega})v_p(\mathbf{k}-\mathbf{q},\tilde{\omega}-\tilde{\Omega}) = f_i. \quad \text{(A1)}$$

To lowest order in Ω, we expect Π to depend on Ω quadratically, since the energy cascade should be independent of the sense of rotation around the z axis, i.e., should be the same for clockwise and anticlockwise rotations. To this order in Ω, it suffices to expand (A1) to $O(\Omega)$ and construct an effective equation:

$$(-i\tilde{\omega}+\nu k^2)v_i+\frac{i\lambda}{2}M_{ijp}(\mathbf{k})\sum_{\mathbf{q},\tilde{\omega}}v_j(\mathbf{q},\tilde{\Omega})v_p(\mathbf{k}-\mathbf{q},\tilde{\omega}-\tilde{\Omega})=f_i-2P_{im}(\mathbf{k})\epsilon_{mzp}\Omega\frac{f_p}{-i\tilde{\omega}+\nu k^2}. \tag{A2}$$

Clearly, the last term on the right-hand side of (A2), an effective noise, is the dominant noise in the hydrodynamic limit. We use (A2) to calculate the kinetic-energy flux Π to $O(\Omega^2)$. The flux Π follows:

$$\begin{aligned}\frac{d}{dt}\langle v^2\rangle=-\frac{i\lambda}{2}\int_{\mathbf{k},\tilde{\omega}}\Bigg\langle\Bigg[&M_{ijp}(\mathbf{k})\sum_{\mathbf{q},\tilde{\omega}}v_i(-\mathbf{k},-\tilde{\omega})v_j(\mathbf{q},\tilde{\Omega})v_p(\mathbf{k}-\mathbf{q},\tilde{\omega}-\tilde{\Omega})\\&+M_{ijp}(-\mathbf{k})\sum_{\mathbf{q},\tilde{\omega}}v_i(\mathbf{k},\tilde{\omega})v_j(\mathbf{q},\tilde{\Omega})v_p(-\mathbf{k}-\mathbf{q},-\tilde{\omega}-\tilde{\Omega})\Bigg]\Bigg\rangle.\end{aligned} \tag{A3}$$

Next we iterate and expand the right-hand side of (A3) up to the one-loop order, which gives (24) above. Now, to the linear order in Ω,

$$\begin{aligned}\langle v_j(\mathbf{q},\tilde{\Omega})v_n(-\mathbf{q},-\tilde{\Omega})\rangle=&\frac{\langle f_j(\mathbf{q},\tilde{\Omega})f_n(-\mathbf{q},-\tilde{\Omega})\rangle}{\tilde{\Omega}^2+\nu^2q^4}-2P_{js}(\mathbf{q})\epsilon_{szp}G_0^2(q,\tilde{\Omega})\Omega\frac{\langle f_p(\mathbf{q},\tilde{\Omega})f_n(-\mathbf{q},-\tilde{\Omega})\rangle}{i\tilde{\Omega}+\nu q^2}\\&-2P_{ns}(\mathbf{q})\epsilon_{szp}G_0^2(-q,-\tilde{\Omega})\tilde{\Omega}\frac{\langle f_p(-\mathbf{q},-\tilde{\Omega})f_j(\mathbf{q},\tilde{\Omega})\rangle}{-i\tilde{\Omega}+\nu q^2}.\end{aligned} \tag{A4}$$

Substituting in (24) and evaluating, we find $\Pi(\Omega)<\Pi(\Omega=0)$, indicating reduction of the kinetic-energy flux by rotation. Since the helicity generated by the rotation scales with Ω, we conclude that with a rising helicity, the kinetic-energy flux is suppressed.

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