Towards quantum turbulence theory: A simple model with interaction of vortex loops

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This paper investigates quantized thin vortex rings with an internal structure. The quantization scheme of this dynamical system is based on an earlier approach proposed by the author. Both energy spectrum and circulation spectrum are calculated. Examples show that the set of permissible circulation values has a fractal structure. The suggested model allows us to describe the system of isolated vortex rings as well as the vortex rings with interaction. Furthermore, the application to the quantum turbulence theory is discussed. The general expression for the partition function of a turbulent flow is suggested.

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I. INTRODUCTION

The complexity of such a phenomenon as turbulence leads to the emergence of different approaches to its description. For example, any attempts to describe turbulent motion of a fluid using the Navier-Stokes equations lead to significant difficulties, even at the classical level. This fact, in particular, stimulates the search for other approaches to the description of this phenomenon. The description of such motion at the quantum level presents an even more complex problem. It is now an established fact that vortex structures play a primary role in the formation of turbulent flows of quantum fluids. A large number of works are devoted to this issue. Without reviewing the literature on this topic, we will mention some of the works [1–6]. It can be assumed that investigation of simplified models of quantum turbulence will be no less useful than, for example, simplified models in the quantum field theory. For example, such models could provide some progress in calculating the thermodynamic characteristics of a turbulent flow (entropy, Gibbs free energy, etc.)

In this paper, we propose a simple model of quantized vortices that demonstrates the following properties:

(1) Scale and Galilean invariance of the theory.

(2) A broad spectrum of energy and circulation values. In particular, the set of circulation values found has a fractal structure. In our opinion, this result is quite suitable for describing the random distribution of circulation in turbulent flows.

(3) The ability to describe the interaction of vortex loops, as well as the reconnecting of such a loop and resizing it.

Of course, the phenomenon of quantum turbulence is too complicated to be described completely in one paper. Here we consider a simple model that allows us to calculate the permissible values of energy, circulation, and certain other variables. For example, the suggested approach gives explicit formulas for quantized fluid velocity in some points of the fluid flow. We also describe the interaction of quantized vortex rings, creation and annihilation of the vortices included. Within the framework of our assumptions, the proposed theory makes it possible to write a general expression

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for the partition function of a turbulent flow. The author hopes that the subsequent development of the model, including the refinement of the resulting expression for the partition function, will be useful for some thermodynamic calculations.

As a starting point of our research, we consider the special configurations of the closed vortex filaments with an internal core structure. We suppose that the dynamics of such objects is restricted by the local induction approximation. Under certain assumptions [7], a vortex filament $\mathbf{r}(t, s)$ with a nonzero flow inside the core is described by the equation

$$\partial_t \mathbf{r}(t,s) = A\,\partial_s \mathbf{r}(t,s) \times \partial_s^2 \mathbf{r}(t,s) + B\big(\partial_s^3 \mathbf{r}(t,s) + \tfrac{3}{2}\,\big|\,\partial_s^2 \mathbf{r}(t,s)\big|^2 \partial_s \mathbf{r}(t,s)\big). \tag{1}$$

We use notations t and s for the time and the natural curve parameter correspondingly. Coefficients A and B are some dimensional coefficients which depend on circulation Γ, the radius a of the vortex core, and the components of the flow velocity in the core. Regarding the value a, we consider it finite and small enough. In general, all these values can vary from vortex to vortex in a turbulent flow.

The theory has three natural dimensional constants that are relevant to the physical system being described. These constants are the fluid's density ρ_0, the speed of sound in this fluid v_0, and the natural scale length R_0,

$$R_0 \in \{R{:}R = |\boldsymbol{r}_1 - \boldsymbol{r}_2|, \quad \boldsymbol{r}_1, \boldsymbol{r}_2 \in V\},$$

where symbol V denotes the domain where the investigated objects evolve. For example, the constant R_0 may be the radius of the pipe in which the fluid in question flows. Despite the fact that value $\tilde{\mu}_0 = \pi\rho_0 R_0^3$ is a natural parameter that determines the scale of the masses, we will also use the additional mass parameter μ_0. In our theory, this parameter denotes the central charge for central extension of the Galilei group $\widetilde{\mathscr{G}}_3$ (the appearance of the extended Galilei group in the considered approach was discussed in the author's work [8] in detail). Therefore, we have an additional dimensionless parameter here: $\alpha_{ph} = \mu_0/\tilde{\mu}_0$. We will clarify its role in our theory later. Along with constants ρ_0, v_0, R_0, we will use the auxiliary constants $t_0 = R_0/v_0$ and $\mathscr{E}_0 = \mu_0 v_0^2$.

The model under consideration allows us to consider a separate vortex ring as a particle with an internal degree of freedom. As a consequence, it becomes possible to use the standard tools of quantum many body theory to describe the interaction of such rings. For example, processes of creation and annihilation of the vortices in a fluid flow can be described. Note that the suggested approach is much simpler than using field string theory which is usually discussed in this context.

II. CLASSICAL DYNAMICS OF A SINGLE VORTEX

Let the symbol R denotes the arbitrary positive constant with the dimension of length. Along with the physical vectors $\mathbf{r}$, we will use a projective vector $\mathbf{r}/R$, denoting it with the same symbol. Further, to describe the considered vortex filament, we introduce the dimensionless parameters $\tau = t/t_0$ and $\xi = s/R$. As a consequence, Eq. (1) will be rewritten as

$$\partial_\tau \mathbf{r}(\tau,\xi) = \beta_1\big(\partial_\xi \mathbf{r}(\tau,\xi) \times \partial_\xi^2 \mathbf{r}(\tau,\xi)\big) + \beta_2\big(2\,\partial_\xi^3 \mathbf{r}(\tau,\xi) + 3\,\big|\,\partial_\xi^2 \mathbf{r}(\tau,\xi)\big|^2 \partial_\xi \mathbf{r}(\tau,\xi)\big). \tag{2}$$

The values β_1 and β_2 are dimensionless constants here.

Equation (2) has a certain solution that is of interest to the proposed model. This solution is

$$\mathbf{r}(\tau,\xi) = \left(\frac{q_x}{R} + \cos(\xi + \phi_0 + \beta_2\tau),\ \frac{q_y}{R} + \sin(\xi + \phi_0 + \beta_2\tau),\ \frac{q_z}{R} + \beta_1\tau\right), \tag{3}$$

where the angle $\phi_0 \in [0, 2\pi)$ and the coordinates q_x, q_y, q_z are some (time-independent) variables. This solution describes the vortex filament in the shape of a circle with a radius R. The filament moves along the axis $\mathbf{e} = \mathbf{e}_z$ with velocity $|\mathbf{u}_v| = \beta_1 R/t_0$ and rotates with the frequency β_2/t_0. The rotation simulates here the flow Φ inside the filament core. This flow is

$$\Phi = \beta_2 \pi \rho_0 \mathrm{a}^2 v_0 (R/R_0).$$

We will further consider only such solutions of Eq. (2). Thus, the set of possible vortex loops is reduced to the rings of an arbitrary radius and some fluid flow in the core here.

In addition to Eq. (2) that describes the evolution of the curve $\mathbf{r}(\cdot, \xi)$, we postulate the standard hydrodynamic formula [9] for the momentum $\tilde{\mathbf{p}}$:

$$\tilde{\mathbf{p}} = \frac{\rho_0}{2} \int \mathbf{r} \times \omega(\mathbf{r})\, dV. \tag{4}$$

The vector $\omega(\mathbf{r})$ stands for vorticity. In our model,

$$\omega(\mathbf{r}) = \omega_1(\mathbf{r}) + \omega_2(\mathbf{r}),$$

where the vorticity $\omega_1(\mathbf{r})$ is due to the fluid rotation around the filament and the vorticity $\omega_2(\mathbf{r})$ is due to the fluid flow in the filament core. Let us consider these summands separately. In cylindrical coordinates (ρ, φ, z), the fluid flow velocity $\mathbf{u}$ in the filament core is described by the formula

$$\mathbf{u}(\mathbf{r}) = \text{const} \times \delta(\rho - R)\delta(z)\mathbf{e}_\varphi(\varphi).$$

Therefore,

$$\mathbf{r} \times \omega_2(\mathbf{r}) = \mathbf{r} \times (\nabla \times \mathbf{u}(\mathbf{r})) = C(\rho, z)\mathbf{e}_\varphi(\varphi).$$

Because the equality $\int_0^{2\pi} \mathbf{e}_\varphi(\varphi) d\varphi = 0$ holds, the value $\omega_2(\mathbf{r})$ does not contribute to the integral Eq. (4). Regarding the first summand $\omega_1(\mathbf{r})$, the vorticity of the closed vortex filament is calculated by means of the formula (see, for example, Ref. [7])

$$\omega_1(\mathbf{r}) = \Gamma \int_0^{2\pi} \delta(\mathbf{r} - \mathbf{r}(\xi))\partial_\xi \mathbf{r}(\xi) d\xi, \tag{5}$$

where the symbol Γ stands for circulation.

Taking into account the formulas Eqs. (4) and (5), we deduce the following expression for the canonical momentum:

$$\tilde{\mathbf{p}} = \frac{\rho_0 \Gamma R^2}{2} \iint_0^{2\pi} [\xi - \eta]\, \partial_\eta \mathbf{r}(\tau, \eta) \times \partial_\xi \mathbf{r}(\tau, \xi) d\xi d\eta. \tag{6}$$

The notation $[x]$ means the integer part of the number $x/2\pi$ here.

For our solution Eq. (3), the integral on the right-hand side of the formula Eq. (6) is easily calculated. Therefore,

$$\tilde{\mathbf{p}} = \pi \rho_0 R^2 \Gamma \mathbf{e}, \quad |\mathbf{e}| = 1, \tag{7}$$

where constant unit vector $\mathbf{e}$ defines the axis of the rotating ring Eq. (3).

Earlier, a new approach to the Hamiltonian description and quantization of a single vortex loop was proposed by the author [8]. In this paper, we modify the suggested approach for our purposes. Moreover we perform the reduction of the considered dynamical system to the finite number of degrees of freedom.

As can be seen from the formulas Eqs. (3) and (4), the natural variables that parametrize our dynamical system are variables

$$\mathbf{q} = (q_x, \quad q_y, \quad q_z), \quad R, \quad \phi = \phi(\tau) = \phi_0 + \beta_2 \tau, \quad \Gamma, \quad \mathbf{e}, \tag{8}$$

where $|\mathbf{e}| = 1$. It must be noted that we describe the vortex ring as an abstract closed curve which evolves in accordance with Eq. (2). The variable as velocity of the fluid will not be needed for this purpose, and still we intend to take into account the dynamics of the surrounding fluid in a minimal way: we declare the value Γ as a dynamic variable, in addition to variables $\mathbf{q}$, R, ϕ, and $\mathbf{e}$.

The variables Eqs. (8), for example, enable us to determine the core radius a in our model. Indeed, if $\mathrm{a} \simeq 0$, the following formula for the vortex ring velocity $\mathbf{u}_v$ takes place [10]:

$$|\mathbf{u}_v| \simeq \frac{\Gamma}{4\pi R} \ln \frac{8R}{\mathrm{a}}.$$

In our model, $|\mathbf{u}_v| = \beta_1 R/t_0$; therefore, the following expression is true for the value a:

$$\mathrm{a} \simeq 8R \exp\left(-\frac{4\pi\beta_1 R^2}{t_0\Gamma}\right). \tag{9}$$

As the next step, we replace the natural set of variables Eqs. (8) by another one which is more convenient. Let us define the variables

$$\varpi = \frac{R}{R_0}\cos(\phi_0 + \beta_2\tau), \quad \chi = \frac{R}{R_0}\sin(\phi_0 + \beta_2\tau).$$

Dynamical equations for these variables are canonical Hamiltonian equations for a harmonic oscillator:

$$\partial_\tau\varpi = -\beta_2\chi,$$
$$\partial_\tau\chi = \beta_2\varpi.$$

Next, we introduce a vector $\mathbf{p} = \alpha_{ph}\tilde{\mathbf{p}}$ instead the canonical momentum $\tilde{\mathbf{p}}$. The formula Eq. (7) is than rewritten as follows:

$$\mathbf{p} = \pi\alpha_{ph}\rho_0 {R_0}^2\Gamma(\varpi^2 + \chi^2)\mathbf{e}, \quad |\mathbf{e}| = 1. \tag{10}$$

Apparently, the set of the variables $\mathbf{p}$, $\mathbf{q}$, ϖ, and χ adequately describes our dynamical system Eq. (3) as a structured 3D particle with an internal degree of the freedom. The formula Eqs. (10) together with the definition of the variables ϖ and χ provides one-to-one correspondence between the set Eqs. (8) and the new set ($\mathbf{p}$, $\mathbf{q}$, ϖ, χ). Note that the variables ϖ and χ are invariants under Galilean and scale transformations of space E_3. It might be appropriate to remember Lord Kelvin's old idea [11] about interpretation of vortices as some structured particles. This idea is still being discussed [12].

The next step in the development of our model is the Hamiltonian description of the considered dynamical system. Pursuant to the Dirac's prescriptions about the primacy of the Hamiltonian structure, we define such structure axiomatically here. The relevant definitions are given below.

(1) Phase space $\mathscr{H} = \mathscr{H}_{pq} \times \mathscr{H}_b$. The space $\mathscr{H}_{pq}$ is the phase space of a 3D free structureless particle. It is parametrized by the variables $\mathbf{q}$ and $\mathbf{p}$. The space $\mathscr{H}_b$ is a phase space for one-dimensional harmonic oscillators.

(2) Poisson structure:

$$\{p_i, q_j\} = \delta_{ij} \quad (i, j = x, y, z), \quad \{\varpi, \chi\} = \frac{1}{\mathscr{E}_0 t_0}. \tag{11}$$

All other brackets vanish.

(3) Hamiltonian:

$$H = \frac{\mathbf{p}^2}{2\mu_0} + \frac{\beta_2\mathscr{E}_0}{2}(\varpi^2 + \chi^2). \tag{12}$$

One of the main questions here is how we can describe the energy of the vortex rings under consideration. It is a well-known fact that the canonical formula [10]

$$\mathscr{E} = \frac{1}{8\pi}\iint\frac{\omega(\mathbf{r})\omega(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\,dV\,dV' = \frac{\Gamma^2}{8\pi}\iint\frac{\partial_\xi\mathbf{r}(\xi)\partial_\xi\mathbf{r}(\xi')}{|\mathbf{r}(\xi) - \mathbf{r}(\xi')|}\,d\xi\,d\xi'$$

leads to the unsatisfactory result for thin filaments. Indeed, the integral in this formula diverges for the filament with the core radius $\mathrm{a} \to 0$. The standard approach to solving this problem is to use various regularization methods. In our case, where the value a depends on dynamic variables of the theory [see Eq. (9)], such a procedure would look somewhat ambiguous. There exists yet another method proposed in Ref. [8]: the energy of an arbitrary closed vortex filament is considered from a group-theoretic point of view there. This approach is based on the fact that Lee algebra of the group $\mathscr{G}_3$ has three Cazimir functions:

$$\hat{C}_1 = \mu_0 \hat{I}, \quad \hat{C}_2 = \left(\hat{M}_i - \sum_{k,j=x,y,z} \epsilon_{ijk} \hat{P}_j \hat{B}_k \right)^2 \quad \hat{C}_3 = \hat{H} - \frac{1}{2\mu_0} \sum_{i=x,y,z} \hat{P}_i^2,$$

where $\hat{I}$ is the unit operator, $\hat{M}_i$, $\hat{H}$, $\hat{P}_i$, and $\hat{B}_i$ ($i = x, y, z$) are the respective generators of rotations, time and space translations, and Galilean boosts. Traditionally, the function $\hat{C}_3$ can be interpreted as an internal energy of the particle. In our case, it is a natural postulate that

$$C_3 = \beta_2 \mathscr{E}_0 |\, b \,|^2, \quad b = \frac{\chi + i\varpi}{\sqrt{2}}.$$

Therefore, the identification $E = H$ is justified in our approach.

Finally, the proposed approach enables us to consider the vortex as a point particle with coordinates $\mathbf{q}$ and momentum $\mathbf{p}$. Each such particle has an internal degree of freedom which is described by oscillator variables χ and ϖ. These variables define the radius R of the vortex and the flow in the vortex core. We assume that the coordinates $\mathbf{q}$ are the coordinates of the center of the vortex ring. Therefore, taking into account the definition of circulation, the following expression for the fluid velocity $\mathbf{u}_f$ takes place in this point:

$$\mathbf{u}_f = \mathbf{u}_v + \frac{\Gamma}{2\pi R}\mathbf{e} = \left(\frac{\beta_1 R}{t_0} + \frac{\Gamma}{2\pi R} \right)\mathbf{e}, \quad \mathbf{e} = \frac{\mathbf{p}}{|\mathbf{p}|}. \tag{13}$$

III. QUANTIZATION

The constructed Hamiltonian structure defines the way for quantization of the vortex ring being studied. First, we must define a Hilbert space $\mathbf{H}_1$ of the quantum states of our dynamical system. The structure of the phase space $\mathscr{H}$ leads to the following natural structure of the space $\mathbf{H}_1$,

$$\mathbf{H}_1 = \mathbf{H}_{pq} \otimes \mathbf{H}_b, \tag{14}$$

where the symbol $\mathbf{H}_{pq}$ denotes the Hilbert space of a free structureless 3D particle [space $L^2(R_3)$, for example] and the symbol $\mathbf{H}_b$ denotes the Hilbert space of the quantum states for the harmonic oscillator. The creation and annihilation operators $\hat{b}^+$, $\hat{b}$ as well as the standard orthonormal basis $|\, n\rangle$ in the space $\mathbf{H}_b$ are defined by well-known formulas

$$[\hat{b}, \hat{b}^+] = \hat{I}_b, \quad \hat{b}|\, 0_b\rangle = 0, \quad |\, n\rangle = \frac{1}{\sqrt{n!}} (\hat{b}^+)^n |\, 0_b\rangle \quad |\, 0_b\rangle \in \mathbf{H}_b,$$

where the operator $\hat{I}_b$ is a unit operator in the space $\mathbf{H}_b$.

Let us quantize our theory. For certainty, we will consider the case when $\mathbf{H}_{pq} = L^2(R_3)$ and $\mathbf{H}_b = L^2(R)$. According to the classical quantization scheme, we must construct the function $A \to \hat{A}$, where symbol A denotes some classical variable and symbol $\hat{A}$ denotes some operator in the space $\mathbf{H}_1$. For fundamental Hamiltonian variables, the relation

$$[\hat{A}, \hat{B}] = -i\hbar \widehat{\{A, B\}}$$

must be satisfied. This equality can possess some anomalous terms if the observables A, B are the functions of the fundamental variables. These terms depend on the ordering rule of noncommuting operators. We will not discuss these issues here [13]. Thus, our postulate of quantization is as follows:

$$q_{x,y,z} \to q_{x,y,z} \otimes \hat{I}_b, \quad p_{x,y,z} \to -i\hbar \frac{\partial}{\partial q_{x,y,z}} \otimes \hat{I}_b, \quad b \to \sqrt{\frac{\hbar}{t_0 \mathscr{E}_0}}\, (\hat{I}_{pq} \otimes \hat{b}),$$

where operator $\hat{I}_{pq}$ is a unit operator in the space $\mathbf{H}_{pq}$. We will not subsequently write the constructions $(\cdots \otimes \hat{I}_b)$ and $(\hat{I}_{pq} \otimes \dots)$ explicitly, hoping that this will not lead to misunderstandings. In accordance with our quantization postulates, the Hamiltonian is defined by the operator

$$\hat{H} = \frac{\hbar^2}{2\mu_0}\Delta + \frac{\beta_2 \hbar}{t_0}\Big(\hat{b}^+\hat{b} + \frac{1}{2}\Big). \tag{15}$$

To find possible values of circulation Γ, let's square the equality Eqs. (10). After quantization, we have the following equation:

$$\big(\hbar^2\Delta + \pi^2\alpha_{ph}^2\rho_0^2\Gamma^2 R_0^4\big[(\hat{b}^+)^2\hat{b}^2 + 2\hat{b}^+\hat{b} + \tfrac{1}{4}\big]\big)|\Psi\rangle = 0, \quad |\Psi\rangle \in \mathbf{H}. \tag{16}$$

The eigenvalues $\mathscr{E}$ of the operator $\hat{H}$ and specific values of the quantity Γ depend on the domain V in which the motion of the vortex ring in question occurs. Before considering a specific example, we would like to say a few words about the possible values of quantized circulation $\Gamma = \Gamma_n$ in a turbulent flow. In our opinion:

(1) The conventional formula

$$\Gamma_n \equiv \oint_\gamma \mathbf{u}(\ell)d\boldsymbol{\ell} = \frac{n\hbar}{\mu} \quad n = 0, 1, 2, \dots \tag{17}$$

can be refined for a turbulence. Apparently, the set of values of the quantized quantity Γ_n is significantly wider here than the natural series. Of course, in the simplest cases, the formula Eq. (17) remains valid, possibly with some correcting terms (see author's work [14]). Note that the known experimental measurements of the magnitude of Γ were made for special single vortices and not in a turbulent flow. An overview of the results on this issue is given Ref. [2]. In Ref. [6], the formula Eq. (17) was confirmed by the results of numerical modeling in the framework of the Gross-Pitaevskii model. Let us note that these works predict the large peaks in the integer values and certain small peaks at the noninteger values, which was explained by errors. From the author's point of view, alternative models are quite appropriate for such a complex phenomenon as a quantum turbulence.

(2) The rule Eq. (17) is usually postulated. As has been repeatedly stated in the literature, such quantization rules are similar to the quantization rules in the old Bohr quantum theory. The author believes that quantum values Γ_n should be deduced from the general postulates of quantum theory and not postulated separately.

(3) As for the arguments that rule Eq. (17) is a consequence of the unambiguity of the wave function of certain quasiparticles (in a two-fluid model, which we are not considering here), we will make the following remark. We consider the fluid medium where the closed vortex loops with zero thickness are present. This medium can be considered as a realization of multiconnected space. Thus, any quasiparticle here can possess the fractional statistics. In this case, the wave function of a quasiparticle can receive a phase multiplier when moving along a closed path around a vortex filament. Therefore, the condition Eq. (17) may not hold even for a single vortex in general. Here we should mention Ref. [15], where the anyon superconductivity was investigated (as is well-known, this phenomenon is similar to superfluidity).

Let us consider the following example. We assume that the fluid moves in the domain V, which is determined by the following boundary conditions ([in this case, $\mathbf{H}_{pq} = L^2(V)$]:

$$x^2 + y^2 \leqslant R_0^2, \quad z \in [0, 2\pi R_1] \ (\mathrm{mod}\, 2\pi R_1), \quad R_1 = \mathrm{const}.$$

This domain models a round tube with a radius R_0 in the shape of a torus of radius $R_1 \gg R_0$.

The effects on the boundaries of domain V can be modeled by conditions on the wave function $\Psi(\boldsymbol{r}) \in \boldsymbol{H}_{pq}$ on the surface $x^2 + y^2 = R_0^2$. It is clear that $\Psi(\boldsymbol{r}) = \psi(x, y)\psi(z)$ here. Therefore, we can consider the following conditions:

$$c_1 \frac{\partial \psi(x, y)}{\partial \boldsymbol{n}}\Big|_{x,y\in S} + c_2\psi(x, y)\Big|_{x,y\in S} = 0,$$

where $\boldsymbol{n}$ is the normal vector for the circle $S : \{x, y : x^2 + y^2 = R_0^2\}$. Let us consider the simplest case when $c_1 = 0$ and $c_2 = \text{const}$. It is known that eigenvalues $-\lambda^2$ of the Laplace operator in this case will be values $\lambda_{n,k}^2 = (\zeta_k^{(n)}/R_0)^2$, where quantities $\zeta_k^{(n)}$, $k = 1, 2, \ldots$ stand for zeros of the Bessel function $J_n(\rho)$. Thus, the eigenvalues $-\lambda^2$ of the Laplace operator Δ in the domain V take the following values:

$$\lambda_{m,\ell,k}^2 = \left(\frac{m}{2R_1}\right)^2 + \left(\frac{\zeta_k^{(\ell)}}{R_0}\right)^2, \quad m, \ell = 0, 1, 2, \ldots \quad k = 1, 2, \ldots.$$

As a result, we find the following formulas for energy $\mathscr{E}$ and circulation Γ:

$$E_{n,m,\ell,k} = \frac{\hbar^2 \lambda_{m,\ell,k}^2}{2\mu_0} + \frac{\beta_2 \hbar}{t_0}\left(n + \frac{1}{2}\right), \tag{18}$$

$$\Gamma_{n,m,\ell,k} = \pm \frac{2\hbar R_0 \lambda_{m,\ell,k}}{\mu_0(2n+1)} = \frac{\pm 2\hbar}{\mu_0(2n+1)} \sqrt{\left(\frac{mR_0}{2R_1}\right)^2 + \left(\zeta_k^{(\ell)}\right)^2}, \tag{19}$$

where natural numbers $n, m, \ell = 0, 1, 2, \ldots$ and number $k = 1, 2, \ldots$. As is well-known, the asymptotic behavior of values $\zeta_k^{(\ell)}$ for large values ℓ and k will be $\zeta_k^{(\ell)} \simeq (3\pi/4) + (\pi/2)\ell + \pi k$. Therefore, any asymptotics $s \to \infty$, where number s is the number m, ℓ or k gives the formula Eq. (17) for circulation $\Gamma_{n,m,\ell,k}$ if other quantum numbers are fixed.

Let's establish the properties of the set $\{\Gamma_{n,m,\ell,k}\}$.

(1) The set $\{\Gamma_{n,m,\ell,k}\}$ has a fractal structure. First, we verify the property of self-similarity. It is more convenient to take the set

$$\{\Gamma^2\} = \left\{\Gamma_{n,m,\ell,k}^2 ;\ n, m, \ell = 0, 1, 2, \ldots, k = 1, 2, \ldots\right\}$$

for this purpose. Indeed,

$$\{\Gamma^2\} = \bigcup_{n,m} \hat{D}_n \hat{T}_m \{\Upsilon\},$$

where the set $\{\Upsilon\}$ is the set of points $2\pi\zeta_k^{(\ell)} R_1/R_0$ on the real axis, symbol $\hat{T}_m$ stands for translation $x \to x + \text{const} \times m^2$ and symbol $\hat{D}_n$ stands for dilatation $x \to x \cdot (\hbar/\mu_0(2n+1))^2$. To calculate the fractal dimension, let's write the set $\{\Gamma_{n,m,\ell,k}\}$ in the form

$$\{\Gamma_{n,m,\ell,k}\} = \bigcup_{m,\ell,k} \{X_{m,\ell,k}\},$$

where the sets $\{X_{m,\ell,k}\}$ are the sequences $X_n = \Gamma_{n,m,\ell,k}$, $n = 0, 1, 2, \ldots$, where numbers m, ℓ, k are fixed. These sequences have asymptotic behavior as const/n when $n \to \infty$. Then, the distance δ between neighboring elements X_n and X_{n+1} at $n \to \infty$ is equal to $\delta(n) = 1/n^2$. Applying the standard formula for calculating the fractal dimension $\mathscr{D}$, we find

$$\mathscr{D}_X = \lim_{n\to\infty} \frac{\ln n}{\ln(1/\delta(n))} = \frac{1}{2}.$$

This result is well-known for the fractal dimension of the natural series. Therefore, the set of the circulation values in our model demonstrates fractal properties. In our opinion, such a structure of the set $\{\Gamma_{n,m,\ell,k}\}$ is more suitable for describing a turbulent flow then the regular structure due to the formula Eq. (17).

(2) The set $\{\Gamma_{n,m,\ell,k}\}$ is bounded. Indeed, the following inequality takes place for the physical reasons:

$$\frac{\hbar^2 \lambda_{m,\ell,k}^2}{2\mu_0} < \mathscr{E}_{\max},$$

where the constant $\mathscr{E}_{\max}$ is some maximal energy in the considered flow. Consequently,

$$\Gamma < 2\pi R_0\sqrt{\frac{2\mathscr{E}_{\max}}{\mu_0}}. \tag{20}$$

The pure quantum states $|n; m, \ell, k\rangle \in \mathbf{H}_1$ that correspond to the values Eqs. (18) and (19) are written as follows:

$$|n; m, \ell, k\rangle = |m, \ell, k\rangle|n\rangle, \quad |m, \ell, k\rangle \in \mathbf{H}_{pq}, \quad |n\rangle \in \mathbf{H}_a, \tag{21}$$

where the notation $|m, \ell, k\rangle$ was used for the eigenvectors of Laplace operator.

Similarly, we can calculate the values of $\mathscr{E}$ and Γ in an arbitrary domain V. Indeed, let the numbers $-\lambda^2_{[s]}$ be the eigenvalues of the Laplace operator in domain V, where the notation $[s]$ means some multi-index (such as the complex index $\{m, \ell, k\}$ in example above). Corresponding formulas for the values of E and Γ will be similar to the formulas Eqs. (18) and (19). Having excluded the value $-\lambda^2_{[s]}$ from these formulas, we find a connection between energy and circulation:

$$\mathscr{E}_{[s],n} = \frac{\mu_0\Gamma^2_{[s],n}}{2\pi^2}\left(\frac{n+1/2}{R_0}\right)^2 + \beta_2\hbar\, v_0\left(\frac{n+1/2}{R_0}\right). \tag{22}$$

The radius R of considered vortices is also quantized. Indeed,

$$R^2 \to \hat{R}^2 = \frac{\hbar R_0^2}{t_0\mathscr{E}_0}\left(\hat{b}^+\hat{b} + \frac{1}{2}\right).$$

Therefore, we have following values for the radius $R = R_n$:

$$R_n = \sigma_{ph}R_0\sqrt{n+\frac{1}{2}}, \quad n = 0, 1, \ldots, . \tag{23}$$

The dimensionless constant $\sigma_{ph} = \sqrt{\hbar/\mu_0 v_0 R_0}$ was introduced here. This constant, in addition to the previously introduced dimensionless constant α_{ph}, naturally appears in the quantum version of the considered dynamical system. These constants depend on both specific fluid and domain V. For example, if the value μ_0 equals the ^{4}He mass, the value $v_0 \simeq 3.4\, m/c$ (the sound speed in the liquid helium) and the pipe radius $R_0 \simeq 0.03\, m$, we have values $\alpha_{ph} \simeq 2 \times 10^{-27}$ and $\sigma_{ph} \simeq 10^{-3}$ for these constants.

IV. DESCRIPTION OF THE MANY-VORTEX SYSTEMS

Here we describe the vortex loop as some pointlike particle with the internal degree of the freedom. This approach gives possibilities for studying many-vortex flows. First, we consider the fluid flow which contains N noninteracting vortices numbered by some multi-index $[n]$. For the sake of clarity, we will assume that the fluid is in the volume V, which was introduced in the previous section. We will also make the following assumptions:

(1) As is well-known, some space averaging is needed for the description of a turbulent flow in a concrete physical system. We assume that the number of vortices N is sufficiently large, so the space averaging volume δV contains a large number of the vortex loops with centers $\mathbf{q}_{[s]}$.

(2) The unequalities $R_n < l_a/2$ take place for every vortex of radius R_n where the value $l_a \simeq \sqrt[3]{V/N}$ is average distance between the vortex centers.

(3) Let the value l_1 be the distance in the fluid flow such that correlation for any parameters in points $\mathbf{q}_1$ and $\mathbf{q}_2$ is absent if $|\mathbf{q}_1 - \mathbf{q}_2| > l_1$. We suppose that the unequality $l_a \geqslant l_1$ takes place.

As mentioned earlier, we suppose that coordinates $\mathbf{q}_{[k]}$ of our structured particle coincide with the center of the vortex ring. What is the quantized fluid velocity $\mathbf{u}_{[s]}$ in the point $\mathbf{q}_{[s]}$? This value can be calculated as follows:

$$\mathbf{u}_{[s]} = \mathbf{u}_v + \frac{\Gamma_{n,m,\ell,k}}{2\pi R_n}\mathbf{e}, \quad |\mathbf{e}| = 1,$$

where the velocity $\mathbf{u}_v = (\beta_1 R/t_0)\mathbf{e}$ of the vortex ring is defined in accordance with formula Eq. (3). Therefore, the formulas Eqs. (13), (19), and (23) that were deduced earlier give the following expression for the quantized fluid velocity in the point $\mathbf{q}_{[s]}$:

$$\mathbf{u}_{[s]} = \sigma_{ph} v_0 \left[-\beta_1 \sqrt{n + \frac{1}{2}} + \frac{1}{2(n+1/2)^{3/2}} \sqrt{\left(\frac{mR_0}{2\pi R_1}\right)^2 + \left(\zeta_k^{(\ell)}\right)^2} \right] \mathbf{e}, \tag{24}$$

where the numbers n, m, ℓ, k are random natural numbers and the vector $\mathbf{e}$ is a random unit vector. In accordance with the assumptions made, formula Eq. (24) models a quasirandom velocity distribution in a turbulent flow.

The suggested theory allows us to calculate the partition function $\mathscr{Z}$ for any concrete domain V. In the simplest case, when there is no interaction between vortices, the $\mathscr{Z}$ function can be written out explicitly. Taking into account the example from the previous section, we can write the following expression:

$$\mathscr{Z} = N \sum_{n,m,\ell,k} \exp\left(-\frac{\mathscr{E}_{n,m,\ell,k}}{k_B T}\right),$$

where value T is the temperature, the constant k_B is Boltzmann constant, and the energy levels $E_{n,m,\ell,k}$ have been determined earlier by the formula Eq. (18).

Of course, a system of noninteracting vortices is too unsatisfactory an approximation to describe a turbulent flow. To get closer to reality, we must consider the interaction of vortices. The proposed theory allows us to do this. Indeed, we can apply the standard formalism of the many-body systems theory here. Let us introduce the N-vortex space $\mathbf{H}_N$:

$$\mathbf{H}_N = \underbrace{\mathbf{H}_1 \otimes \cdots \otimes \mathbf{H}_1}_{N} \equiv \mathfrak{H}_{pq}^N \otimes \mathfrak{H}_b^N,$$

where

$$\mathfrak{H}_{pq}^N = \underbrace{\mathbf{H}_{pq} \otimes \cdots \otimes \mathbf{H}_{pq}}_{N}, \quad \mathfrak{H}_b^N = \underbrace{\mathbf{H}_b \otimes \cdots \otimes \mathbf{H}_b}_{N}.$$

In Dirac notation, any vector $|\Phi^N\rangle \in \mathbf{H}_N$ takes the form ($N \geqslant 1$)

$$|\Phi^N\rangle = \sum_{n_1,\dots,n_N} \int \cdots \int d\mathbf{p}_1 \dots \mathbf{p}_N f^N(\mathbf{p}_1, \dots, \mathbf{p}_N)\, \varphi^N_{n_1,\dots,n_N} |\mathbf{p}_1\rangle \dots |\mathbf{p}_N\rangle |n_1\rangle \dots |n_N\rangle, \tag{25}$$

where the vectors $|\mathbf{p}_j\rangle$ are corresponding eigenvectors of the operators $\hat{\mathbf{p}}_j$.

The Fock space

$$\mathfrak{H} = \bigoplus_{N=0}^{\infty} \mathbf{H}_N = \mathfrak{H}_{pq} \otimes \mathfrak{H}_{\text{int}}, \quad \mathbf{H}_0 = |0_{pq}\rangle \otimes |0_b\rangle = C,$$

is defined in a standard way. Here we have introduced the notation

$$\mathfrak{H}_{pq} = \bigoplus_{N=0}^{\infty} \mathfrak{H}_{pq}^N, \quad \mathfrak{H}_{\text{int}} = \bigoplus_{N=0}^{\infty} \mathfrak{H}_b^N.$$

The creation and annihilation operators $\hat{a}^+_{pq}(\mathbf{p})$, $\hat{a}_{pq}(\mathbf{p})$ and $\hat{a}^+_{\text{int}}(n)$, $\hat{a}_{\text{int}}(n)$ act in the space $\mathfrak{H}$ as $\hat{a}^+_{pq}(\mathbf{p}) \otimes I_{\text{int}}$ and so on. They are defined in a standard way. For example, let us suppose that vectors $\Phi_b \in \mathfrak{H}_{\text{int}}$ take the form (vector in the form of a string):

$$\Phi_b = \left(\varphi^0, \varphi^1_{n_1}, \dots, \varphi^N_{n_1,\dots,n_N}, \dots\right).$$

Then the definition of operators $\hat{a}_{\rm int}(n)$ and $\hat{a}^+_{\rm int}(n)$ will be as follows ($\varphi^0 = 1$):

$$(\hat{a}_{\rm int}(n)\varphi^N)_{n_1,\dots,n_{N-1}} = \sqrt{N}\varphi^N_{n_1,\dots,n_{N-1},n}, \quad (\hat{a}_{\rm int}(n)\varphi^0) = 0,$$

$$(\hat{a}^+_{\rm int}(n)\varphi^N)_{n_1,\dots,n_{N+1}} = \frac{1}{\sqrt{N+1}}\sum_{j=1}^{N+1}\delta_{nn_j}\,\varphi^N_{n_1,\dots,\not{n}_j,\dots n_{N+1}},$$

where the symbol $\not{n}_j$ as well as the symbol n_0 both mean the absence of the corresponding number. In addition, the symbol n without any subscript is not a summation index in the formula Eq. (25). As usual, the constructions $(\dots\otimes I_a)$ and $(I_{pq}\otimes\dots)$ will not be explicitly written out. We will also consider the operators

$$\hat{\mathfrak{a}}^+(\mathbf{p};n) = \hat{a}^+_{pq}(\mathbf{p})\otimes\hat{a}^+_{\rm int}(n), \quad \hat{\mathfrak{a}}(\mathbf{p};n) = \hat{a}_{pq}(\mathbf{p})\otimes\hat{a}_{\rm int}(n)$$

which act in the space $\mathfrak{H}$.

Thus, the suggested method allows us to describe the processes of creation and annihilation of closed vortex rings of a variable radius. Indeed, let us consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{U}, \tag{26}$$

where

$$\hat{H}_0 = \frac{1}{2\mu_0}\int \mathbf{p}^2\hat{a}^+_{pq}(\mathbf{p})\hat{a}_{pq}(\mathbf{p}) + \frac{\beta_2\hbar}{t_0}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\hat{a}^+_{\rm int}(n)\hat{a}_{\rm int}(n). \tag{27}$$

This operator has continuous spectrum,

$$\hat{H}_0\Phi^N_{\mathscr{E}} = \mathscr{E}\Phi^N_{\mathscr{E}},$$

where the eigenvalues $\mathscr{E}$ are positive numbers. Let's find the eigenvectors $\Phi^N_{\mathscr{E}}\in\mathbf{H}_N$ of the operator $\hat{H}_0$. Introducing the designation

$$\Phi^N = \big(0,0,\dots,f^N(\mathbf{p}_1,\dots,\mathbf{p}_N)\varphi^N_{n_1,\dots,n_N},0,\dots\big)$$

and performing the direct calculations, we find for the vector Φ^N:

$$\hat{H}_0\Phi^N \equiv \mathscr{E}(\mathbf{p}_1,\dots\mathbf{p}_N;n_1,\dots n_N)\Phi^N, \tag{28}$$

where

$$\mathscr{E}(\mathbf{p}_1,\dots\mathbf{p}_N;n_1,\dots n_N) = \frac{1}{2\mu_0}\sum_{j=1}^{N}\mathbf{p}_j^2 + \frac{\beta_2\hbar}{t_0}\sum_{j=1}^{N}\left(n_j+\frac{1}{2}\right). \tag{29}$$

Therefore,

$$\big|\Phi^N_{\mathscr{E}}\big\rangle = \sum_{n_1,\dots,n_N}\int\cdots\int d\mathbf{p}_1\dots\mathbf{p}_N f^N_{\mathscr{E}}(\mathbf{p}_1,\dots,\mathbf{p}_N;n_1,\dots n_N)$$
$$\times\,\varphi^N_{n_1,\dots,n_N}|\mathbf{p}_1\rangle\dots|\mathbf{p}_N\rangle|n_1\rangle\dots|n_N\rangle, \tag{30}$$

where the function $f^N_{\mathscr{E}}$ is proportional to the Dirac δ function:

$$f^N_{\mathscr{E}}(\mathbf{p}_1,\dots,\mathbf{p}_N;n_1,\dots n_N) = \text{const}\times\delta(\mathscr{E}-\mathscr{E}(\mathbf{p}_1,\dots\mathbf{p}_N;n_1,\dots n_N)).$$

Thus, the vector $|\Phi^N_{\mathscr{E}}\rangle$ is some entangled state of states of N vortexes with momenta $\mathbf{p}_1,\dots,\mathbf{p}_N$ and radii R_i, $i=1,\dots,N$ which are defined by the numbers n_i in accordance with the formula Eq. (23).

Let us discuss the term $\hat{U}$ that is responsible for the interaction in the formula Eq. (26). In general, operator $\hat{U}$ has the following form:

$$\hat{U} = \sum_{m,n=1}^{\infty} \varepsilon_{m,n} \hat{U}_{m\leftrightarrow n}, \tag{31}$$

where the sequence $\varepsilon_{m,n}$ is some finite decreasing sequence. The constants $\varepsilon_{m,n}$ are the coupling constants which define the intensity of the vortex interaction. Operators $\hat{U}_{m\leftrightarrow n}$ define the reconnection of the considered vortex filaments in the flow. Each such operator describes the transformation of m vortex rings into n rings and vice versa, $n \to m$. To fix the exact form of operators $\hat{U}_{m\leftrightarrow n}$, additional assumptions are needed.

For example, let us consider the operator $\hat{U}_{2\leftrightarrow 2}$ in case of a paired interaction between our structured particles. Thus, any two particles located in the points with coordinates $\mathbf{q}_1$ and $\mathbf{q}_2$ interact by means of potential $\hat{\mathscr{V}}(\mathbf{q}_1 - \mathbf{q}_2)$. In this case,

$$\hat{U}_{2\leftrightarrow 2} = \sum_{n_1,n_2,n_1',n_2'} \hat{U}_{2\leftrightarrow 2}(n_1, n_2, n_1', n_2'),$$

where (see Ref. [16], for instance)

$$\hat{U}_{2\leftrightarrow 2}(n_1, n_2, n_1', n_2') = \int \cdots \int d\boldsymbol{q}_1 d\boldsymbol{q}_2 \hat{a}^+(\boldsymbol{q}_1; n_1)\hat{a}^+(\boldsymbol{q}_2; n_2)\mathscr{V}(\boldsymbol{q}_1 - \boldsymbol{q}_2)\hat{a}(\boldsymbol{q}_2; n_2')\hat{a}(\boldsymbol{q}_1; n_1'). \tag{32}$$

The notations

$$\hat{a}(\mathbf{q}; n) = \int \hat{a}(\mathbf{p}; n) e^{i\mathbf{qp}} d\mathbf{p}, \quad \hat{a}^+(\mathbf{q}; n) = \int \hat{a}^+(\mathbf{p}; n) e^{-i\mathbf{qp}} d\mathbf{p}$$

were introduced here. Note that the conjugation rules

$$\hat{U}^+_{2\leftrightarrow 2}(n_1, n_2, n_1', n_2') = \hat{U}_{2\leftrightarrow 2}(n_1', n_2', n_1, n_2)$$

are fulfilled so the operator $\hat{U}_{2\leftrightarrow 2}$ is self-adjoint operator.

Let us consider the δ interaction between the particles:

$$\mathscr{V}(\mathbf{q}_1 - \mathbf{q}_2) = \delta(\mathbf{q}_1 - \mathbf{q}_2).$$

In this case, and taking into account the conservation laws for the momentum and the energy, we can write the following expression for the function $\hat{U}_{2\leftrightarrow 2}(n_1, n_2, n_1', n_2')$:

$$\hat{U}_{2\leftrightarrow 2}(n_1, n_2, n_1', n_2') = \delta_{n_1+n_2, n_1'+n_2'} \int \cdots \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2'$$
$$\times\, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\hat{a}^+(\mathbf{p}_1; n_1)\hat{a}^+(\mathbf{p}_2; n_2)\hat{a}(\mathbf{p}_1'; n_1')\hat{a}(\mathbf{p}_2'; n_2'),$$

As it seems, some function $\mathscr{U}_{n_1,n_2}(\mathbf{p}_1, \mathbf{p}_2)$ should be added in the integrand expression for the reasons of convergence of the integrals. For example, $\mathscr{U} = 1$ for $\mathscr{E}(\mathbf{p}_1, \mathbf{p}_2; n_1, n_2) \leqslant \mathscr{E}_{\max}$ and $\mathscr{U} = 0$ in the opposite case. The function $\mathscr{E}(\mathbf{p}_1, \mathbf{p}_2; n_1, n_2)$ is defined by the formula Eq. (29). Of course, other methods of the ultraviolet cutoff procedure are also possible.

Taking into account the vortex nature of our structured particles, it also makes sense to consider nonlocal interactions of a general kind. This can be done, for example, by replacing in the formula Eq. (32)

$$\hat{a}^+(\boldsymbol{q}_1; n_1)\hat{a}^+(\boldsymbol{q}_2; n_2)\mathscr{V}(\boldsymbol{q}_1 - \boldsymbol{q}_2)\hat{a}(\boldsymbol{q}_2; n_2')\hat{a}(\boldsymbol{q}_1; n_1')$$
$$\longrightarrow \int \cdots \int d\boldsymbol{q}_1' d\boldsymbol{q}_2' \hat{a}^+(\boldsymbol{q}_1'; n_1)\hat{a}^+(\boldsymbol{q}_2'; n_2)\mathscr{F}(\boldsymbol{q}_1 - \boldsymbol{q}_1', \boldsymbol{q}_2 - \boldsymbol{q}_2')\hat{a}(\boldsymbol{q}_2'; n_2')\hat{a}(\boldsymbol{q}_1'; n_1'),$$

where the function $\mathscr{F}(\cdot, \cdot)$ is certain form-factor.

Any summands $\hat{U}_{m\leftrightarrow n}$ in the sum Eq. (31) can be constructed similarly. Definitely, the description of a turbulent flow in the framework of the suggested method requires a large number of terms in the sum Eq. (31).

In this paper, we consider the simplest case when interacting rings retain their shape. Of course, the general case of interaction must take into account the change in the shape of the rings. The shape change can be taken into account by replacing the H_b space to the Fock space H_F, which is introduced in the work [8]. The author hopes to return to this issue in subsequent works.

Finally, we have the following general expression for the partition function of the quantum turbulent flow in our model:

$$\mathscr{Z} = \mathrm{Tr}\exp\left(-\frac{\hat{H}}{k_B T}\right). \tag{33}$$

V. CONCLUDING REMARKS

In this paper, we have proposed the basics of the approach to the description of a quantum turbulent flow as a system of interacting vortices. Specific calculations of any thermodynamic quantities with a help of the formula Eq. (33) involve the refinement of the model. So, we have to concretize the formula Eq. (31) in some way. In our opinion, the values $\varepsilon_{m,n}$ together with the vortex concentration V/N define the living time of the single vortex. Consequently, we can implement Prandtl's hypothesis about the length of the mixing path [17] in a turbulent flow within the framework of our theory. The complexity of describing such a phenomenon as quantum turbulence will require additional assumptions and subsequent investigations. The author hopes to return to this issue in the future.

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