Energy dissipation rate in the inertial sublayer of turbulent channel flow at large but finite Re_τ

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This paper presents a theory of the position dependence of the statistical average $\langle\varepsilon\rangle$ of the energy dissipation rate ε per unit mass in the inertial sublayer of turbulent channel flow. The theory gives $\langle\varepsilon\rangle y/u_\tau^3 \sim 1/\kappa_\varepsilon + C_p(y/h) + C_v(l_\tau/y)$ for small but finite ratios y/h and l_τ/y, at large but finite friction Reynolds number $\mathrm{Re}_\tau = h/l_\tau$, where y is the distance from the wall, h is the channel half-width, u_τ and l_τ are the friction velocity and length respectively, and κ_ε, C_p, and C_v are nondimensional constants. The theory agrees well with the data of a series of direct numerical simulations of turbulent channel flow with Re_τ up to approximately 8000. The data suggest $\kappa_\varepsilon \approx 0.44$, which is distinctively different from the widely accepted value (≈ 0.40 or so) for the von Kármán constant for the mean velocity in the log-law region of wall-bounded flows.

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I. INTRODUCTION

The statistical average $\langle\varepsilon\rangle$ of the local rate of dissipation ε of the kinetic energy per unit mass in turbulent flows plays key roles in theories and modeling of turbulence, where the brackets $\langle\ldots\rangle$ denote an appropriate average. In this paper, we consider the position dependence of $\langle\varepsilon\rangle$ in statistically stationary turbulent channel flow (TCF) of an incompressible fluid.

To date, studies on TCF suggest that certain basic flow characteristics in TCF obey simple laws in the limit of $\delta_h \equiv y/h \to 0$ and $\delta_\nu \equiv l_\tau/y \to 0$, where h is the channel half-width, y is the distance from the wall, ν is the kinematic viscosity, and l_τ is the friction length. For example, it has been suggested that the streamwise mean flow velocity U in statistically stationary TCF at sufficiently small δ_h and δ_ν fits well to

$$\frac{dU^+}{dy^+} = \frac{1}{\kappa y^+}, \tag{1}$$

where we have used that U is time independent and depends on the position only through y, κ is a nondimensional constant called the von Kármán constant, and the superscript "+" denotes nondimensionalization using the friction length l_τ and the friction velocity u_τ, where $l_\tau = \nu/u_\tau$.

It is natural to except that a similar scenario may be true for the local rate $\langle\varepsilon\rangle$ of dissipation, i.e., $\langle\varepsilon\rangle$ obeys a simple law. In fact, recent direct numerical simulations (DNSs) [1–5] suggest that $\langle\varepsilon\rangle$ at sufficiently small δ_h and δ_ν, fits well to

$$\langle \varepsilon^+ \rangle = \frac{1}{\kappa_\varepsilon y^+}, \tag{2}$$

where κ_ε is a nondimensional constant, and ε is dissipation due to the fluctuating part of the velocity fields. If one assumes (i) the mean flow velocity $U(y)$ obeys (1), (ii) the mean Reynolds shear stress is given by -1 in wall units, and (iii) the mean turbulence energy production is balanced by the dissipation rate (see, e.g., [6,7]), then one obtains not only (2) but also $\kappa = \kappa_\varepsilon$. However, a close inspection of the DNS data also suggests that although $\langle \varepsilon \rangle$ in DNS fits fairly well to (2), there are some differences between the data and the theory at least in the following two senses: First, the y dependence of $\langle \varepsilon^+ \rangle$ is not exactly in proportion to $1/y^+$, and second, the coefficient κ_ε does not exactly agree with κ in DNSs [3,5].

Such small but finite differences between the DNS and the theory are not surprising because of the simple fact that the friction Reynolds number

$$\mathrm{Re}_\tau \equiv \frac{u_\tau h}{\nu} = \frac{h}{l_\tau} = \left(\frac{h}{y}\right) \times \left(\frac{y}{l_\tau}\right) = \left(\frac{1}{\delta_h}\right)\left(\frac{1}{\delta_\nu}\right) \tag{3}$$

must be finite in any DNS, so that δ_h and δ_ν must be also finite, no matter how small they may be. A theory such as (2), even if correct, is supposed to represent the statistics only in the limit of $\delta_h \to 0$ and $\delta_\nu \to 0$. Hence, the existence alone of a small difference between the theory and DNS or experimental data at any finite Re_τ cannot invalidate the theory. At the same time, however, the smallness of difference alone is insufficient to validate the theory.

To understand $\langle \varepsilon \rangle$ in the limit of $\delta_h \to 0$ and $\delta_\nu \to 0$, as well as $\langle \varepsilon \rangle$ at small but finite δ_h and δ_ν, it is necessary to have a proper understanding of the influence of the finiteness, i.e., the small but finite δ_h and δ_ν at finite but large Re_τ.

In this paper, we propose a theory to promote the understanding of the influence of small but finite δ_h and δ_ν on ε in the layer of y such that $l_\tau \ll y \ll h$, i.e., $\delta_h \ll 1$, $\delta_\nu \ll 1$. The theory is based on the spirit of linear response theory (LRT) of turbulence [5,8–10], so that the influence of small but finite δ_h and δ_ν is regarded as a disturbance added to a certain state determined in the limit of $\delta_h \to 0$ and $\delta_\nu \to 0$. Also, we compare the conjectures from the theory with the data of a series of DNSs of TCF with Re_τ up to approximately 8000. It is shown that the conjectures agree well with the DNS results.

II. LINEAR RESPONSE THEORY OF $\langle \varepsilon \rangle$

We consider the TCF of an incompressible fluid that obeys the Naiver-Stokes (NS) equation and the incompressibility condition. In this study, $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}(\mathbf{x}, t)$ is the fluid velocity, $\widetilde{p} = \widetilde{p}(\mathbf{x}, t)$ is the pressure, and ρ is the fluid density. We assume that the TCF is under a constant mean pressure gradient $\alpha \equiv -(1/\rho)\partial\langle \widetilde{p} \rangle/\partial x$ (> 0) in the x direction and bounded by two planes placed at $y = 0$ and $y = 2h$, the mean flow $\mathbf{U}$ is unidirectional and corresponds to the form $\mathbf{U} = (U(y),\ 0,\ 0)$ in a Cartesian coordinate system $\mathbf{x} = (x_1,\ x_2,\ x_3) = (x,\ y,\ z)$, and the fluctuating field $\mathbf{u} \equiv \widetilde{\mathbf{u}} - \mathbf{U} = (u_1,\ u_2,\ u_3) = (u,\ v,\ w)$ is statistically homogeneous in the x (streamwise) and z (spanwise) directions.

Under the assumptions of statistical stationarity and homogeneity in the x and z directions of the TCF, the average of $\langle \varepsilon \rangle$ of the energy dissipation rate per unit mass due to the fluctuation part $\mathbf{u}$ of velocity is independent of time, and it depends on the position vector $\mathbf{x}$ only through y. Then it is natural to assume that in such a TCF $\langle \varepsilon \rangle$ is uniquely determined by the distance y from the wall, the mean pressure gradient represented by the parameter α , the channel half-width h, and the kinematic viscosity ν. Thus, we may write $\langle \varepsilon \rangle$ as

$$\langle \varepsilon \rangle = f(y,\ \alpha\ ,\ h,\ \nu), \tag{4}$$

where f is an appropriate function of only the four parameters y, α, h, and ν.

To consider $\langle \varepsilon \rangle$ at small but finite δ_h and δ_ν, it is convenient to introduce the change of variables from the set (y, α, h, ν) to $(y, u_\tau, \delta_h, \delta_\nu)$, where

$$u_\tau^2 \equiv \alpha h, \quad \delta_h \equiv \frac{y}{h} = \frac{y^+}{\mathrm{Re}_\tau}, \quad \delta_\nu \equiv \frac{\nu}{u_\tau y} = \frac{1}{y^+}, \tag{5}$$

and we have used (3). In terms of $(y, u_\tau, \delta_h, \delta_\nu)$, one may write (4) without loss of generality as

$$\langle \varepsilon \rangle = F(y, u_\tau, \delta_h, \delta_\nu), \tag{6}$$

where F is an appropriate function of only the four parameters $(y, u_\tau, \delta_h, \delta_\nu)$. We introduce here the following assumption:

Assumption 1. In the limit $\nu \to 0$ for given finite nonzero y, h, and α, the dissipation rate $\langle \varepsilon \rangle$ given by (4) tends to a finite nonzero value, say $\langle \varepsilon \rangle_0$, that is determined by only (y, h, α).

This assumption is similar to the well-known conjecture that in the limit $\nu \to 0$, the normalized energy dissipation rate $\langle \varepsilon \rangle$ in homogeneous and isotropic turbulence (HIT) tends to a nonzero finite constant, i.e., $\langle \varepsilon \rangle \ell / u'^3 \to D$ for given u' and ℓ, where u' and ℓ are the characteristic velocity and length scales of the energy-containing eddies, respectively, and D is a nondimensional nonzero constant independent of u' and ℓ.

Assumption 1 implies that in the limit $\delta_\nu \to 0$, the function F defined by (6) for any given finite y, u_τ, and δ_h tends to a nonzero finite constant $\langle \varepsilon \rangle_0$, where $\langle \varepsilon \rangle_0$ is uniquely determined by y, u_τ, and δ_h; therefore, we may write

$$\langle \varepsilon \rangle_0 = \langle \varepsilon \rangle_0(y, u_\tau, \delta_h). \tag{7}$$

Regarding the function $\langle \varepsilon \rangle_0(y, u_\tau, \delta_h)$ in (7), we assume the following:

Assumption 2. In the limit $\delta_h \to 0$, for given finite nonzero y and u_τ, the energy dissipation rate $\langle \varepsilon \rangle_e$ given by (7) tends to a finite nonzero value, say $\langle \varepsilon \rangle_e$, where $\langle \varepsilon \rangle_e$ is determined by only y and u_τ.

A simple dimensional consideration based on Assumption 2 yields that $\langle \varepsilon \rangle_e$ is given by

$$\langle \varepsilon \rangle_e = C_e \frac{u_\tau^3}{y}, \tag{8}$$

where C_e is a nondimensional constant, which plays a role similar to D noted above. For later convenience, we put $\kappa_\varepsilon = 1/C_e$ in the following:

If we write $\langle \varepsilon \rangle$ as

$$\langle \varepsilon \rangle = \langle \varepsilon \rangle_e + \Delta \langle \varepsilon \rangle + \cdots, \tag{9}$$

then Assumptions 1 and 2 imply that in the limit $\delta_h \to 0$ and $\delta_\nu \to 0$, we have $\Delta \langle \varepsilon \rangle \to 0$, where $\Delta \langle \varepsilon \rangle$ depends on (y, α, h, ν) only through δ_h and δ_ν, and represents the change of $\langle \varepsilon \rangle$ from $\langle \varepsilon \rangle_e$ due to the small but finite $\delta_h = y/h$ and $\delta_\nu = l_\tau / y$.

We consider here the dependence of $\Delta \langle \varepsilon \rangle$ on δ_h and δ_ν from the viewpoint of the linear response theory (LRT) for turbulence. Readers may refer to references such as [5,8–10], and references cited therein for some details on the idea of LRT applied to turbulent flows. For the convenience of readers, we start with a brief review of the idea in Sec. II A. The review is along the lines of the references noted above, in particular that of [5,8,9].

A. General

In general, the LRT is based on the assumption of the existence of a certain kind of equilibrium or basic state in a certain limit. Let $\langle B \rangle$ be the statistical average of observable B, and suppose that a disturbance, say X, is added to a system that is in an equilibrium state in the absence of X. Then in response to this disturbance X, $\langle B \rangle$ changes from $\langle B \rangle_e$ to

$$\langle B \rangle = \langle B \rangle_e + \Delta \langle B \rangle, \tag{10}$$

where $\langle B\rangle_e$ is the average at the equilibrium state, and $\Delta\langle B\rangle$ denotes the change of $\langle B\rangle$ owing to X. In the LRT, it is assumed that if X is small in an appropriate sense, then $\Delta\langle B\rangle$ can be approximated to be linear in X, i.e.,

$$\Delta\langle B\rangle \sim \Delta_1\langle B\rangle, \tag{11}$$

where $\Delta_1\langle B\rangle$ denotes the first-order term in X and "$\sim$" denotes the equality with ignoring terms higher order in X, i.e., $\Delta_1\langle B\rangle$ can be written as

$$\Delta_1\langle B\rangle = cX. \tag{12}$$

Here X is to be understood as a measure representing the disturbance in an appropriate sense, and c is a coefficient determined by the nature of the equilibrium state, independently of X. Since $\Delta_1\langle B\rangle$ given by (12) is linear in X, the use of the word "linear" in "linear response theory" is consistent with (11) and (12). But one may proceed to include higher order terms, if necessary; see e.g., [5,8,9].

If the disturbance consists of X_m $(m = 1, 2, 3, \ldots)$, (12) is to be understood as

$$\Delta_1\langle B\rangle = \Delta_1\langle B\rangle_1 + \Delta_1\langle B\rangle_2 + \cdots, \tag{13}$$

where

$$\Delta_1\langle B\rangle_m \equiv c_m X_m \tag{14}$$

is the effect by the "disturbance" X_m, and $c_1, c_2, \ldots$ are constants determined by the nature of the equilibrium state independently of $X_1, X_2, \ldots$.

In this study the influence of small but finite δ_h and δ_ν is regarded as a disturbance added to a certain state determined in the limit of $\delta_h \to 0$ and $\delta_\nu \to 0$.

B. Application to $\Delta\langle\varepsilon\rangle$

In order to apply the idea of the LRT for the estimate of $\Delta\langle\varepsilon\rangle$, it is instructive to recall that the NS equation gives

$$\langle uv\rangle = -u_\tau^2 + \alpha y + \nu\frac{d}{dy}U \tag{15}$$

for turbulence that is statistically stationary and homogeneous in the x and z directions (see, e.g., [7]). Equation (15) implies that in the layer where the second and third terms on the right-hand side are negligible, $\langle uv\rangle$ is almost constant ($\sim -u_\tau^2$) independent of y. We call this layer the inertial sublayer (ISL). A simple analysis suggests that the ISL may be identified as the layer satisfying $l_\tau \ll y \ll h$, i.e., $\delta_h \ll 1$, $\delta_\nu \ll 1$.

The α term is associated with the mean pressure gradient α and is linear in α, while the ν term is associated with the viscous force and is linear in ν. Equation (15) implies that (i) in the limit these terms normalized by u_τ^2 tend to 0, $\langle uv\rangle$ tends to the value, say $\langle uv\rangle_e$ which is independent of the terms, and (ii) the difference $\Delta\langle uv\rangle \equiv \langle uv\rangle - \langle uv\rangle_e$ may be understood as a perturbation by the two terms, in the ISL.

These observations suggest to us to assume that (i) there are two kinds of forces in an appropriate sense such that one is associated with the mean pressure gradient and is linear in α, and the other is associated with the viscous force and is linear in ν, (ii) in the limit they tend to zero under appropriate normalization, the turbulence statistics are at a state independent of the forces, and (iii) if they are small but finite, then their effects are treated as disturbance added to the state.

By applying the idea of LRT outlined in Sec. II A, Kaneda *et al.* [8] proposed a theory for the statistics $\langle B\rangle$ in the ISL, where $B = \partial Z/\partial y$, and $Z = uv$, vv, or ww. In the application, they exploited an analogy between the momentum flux $\langle uv\rangle(\sim -u_\tau^2)$ per unit mass in the ISL ($l_\tau \ll y \ll h$), and the energy flux $\Pi(k)(\sim \langle\varepsilon\rangle)$ across the wave number k in the inertial subrange (ISR) of HIT that may be identified as the range satisfying $\eta \ll 1/k \ll L$. Here, L and η are respectively the length scales characterizing the energy containing range and energy dissipation range. Like $\langle\varepsilon\rangle$ and the eddy-scale r or equivalently the wave number k play key roles in Kolmogorov's theory [11], u_τ and y play key roles in the proposed theory.

In the application of the LRT, it is assumed that in the ISR the statistics of B under consideration are dominated by dynamics local in physical space as well as in scale, and the following:

Assumption (I). In the limit of $\delta_h \to 0$ and $\delta_\nu \to 0$, the statistics is at a certain kind of equilibrium or basic state that is determined by only y and u_τ.

This implies that $\langle B \rangle \to \langle B \rangle_e$ in the limit, where $\langle B \rangle_e$ is a function of only y and u_τ.

As regards $\Delta_1 \langle B \rangle_1$ in (13), the theory suggests to write it as

$$\Delta_1 \langle B \rangle = \Delta_1 \langle B \rangle_p + \Delta_1 \langle B \rangle_\nu, \tag{16}$$

and to introduce the following Assumptions (II) and (III) (see [9]):

Assumption (II). $\Delta_1 \langle B \rangle_p$ and $\Delta_1 \langle B \rangle_\nu$ are linear in α and ν, respectively, i.e., they can be written in the form $\Delta_1 \langle B \rangle_p = c_p^B \alpha$ and $\Delta_1 \langle B \rangle_\nu = c_\nu^B \nu$ where the coefficients c_p^B and c_ν^B are independent of α and ν.

Assumption (III). c_p^B and c_ν^B are determined by only y and u_τ.

These assumptions are based on the assumption of locality in the sense that in the ISR the statistics of B under consideration are dominated by dynamics local in physical space as well as in scale. If the statistics under consideration could be affected significantly by nonlocal dynamics, then it is unlikely that one-point statistics such as $\partial \langle vv \rangle / \partial y$ are characterized by only the local position y and the local statistics, in particular the flux $\langle uv \rangle$ ($\sim -u_\tau^2$ in the ISL) near the position under consideration. Note that the flux $\langle uv \rangle$ may be far from $-u_\tau^2$ outside the ISL.

Assumptions (I)–(III) were shown by Kaneda *et al.* [8] to give estimates in good agreement with the results of DNS for the case $B = \partial Z / \partial y$, and $Z = uv$, vv, or ww. The discussions in [8] suggest that Assumptions (I)–(III) may work well provided that the statistics of B are dominated by local dynamics. (Note: The discussions on $\Delta \langle B \rangle_1$ in [8] are not exactly the same as a those in [9], but they are essentially the same in the sense that $\Delta \langle B \rangle_1$ in both of them consist of two terms that are respectively in proportion to α and ν, and the resulting expressions are the same. We have followed here the line of [9]).

It is natural to assume that the statistics of ε is dominated by local small-scale dynamics. This encourages us to try the application of Assumptions (I)–(III) to the case $B = \varepsilon$.

The application of Assumption (I) then gives

$$\langle \varepsilon \rangle_e = C_e' \frac{u_\tau^3}{y}, \tag{17}$$

where C_e' is a nondimensional constant. Equation (17) is consistent with (8), as could be expected, provided that $C_e' = C_e$.

The application of Assumptions (II) and (III) and a simple dimensional analysis give

$$\Delta_1 \langle \varepsilon \rangle_p = c_p^\varepsilon \alpha \ = c_p^\varepsilon \frac{u_\tau^2}{h} = C_p \frac{u_\tau^3}{y} \left(\frac{y}{h} \right), \tag{18}$$

$$\Delta_1 \langle \varepsilon \rangle_\nu = c_\nu^\varepsilon \nu \ = c_\nu^\varepsilon u_\tau l_\tau = C_\nu \frac{u_\tau^3}{y} \left(\frac{l_\tau}{y} \right), \tag{19}$$

where C_p and C_ν are nondimensional constants, and we have used $\alpha = u_\tau^2 / h$, and $\nu = u_\tau l_\tau$. Equation (16) for $B = \varepsilon$ and (18) and (19) give

$$\Delta \langle \varepsilon \rangle \sim \frac{u_\tau^3}{y} \left[C_p \left(\frac{y}{h} \right) + C_\nu \left(\frac{l_\tau}{y} \right) \right]. \tag{20}$$

This is equivalent to

$$\Delta \langle \varepsilon \rangle \sim \frac{u_\tau^3}{y} [C_p (\delta_h)^\xi + C_\nu (\delta_\nu)^\zeta], \quad (\xi = \zeta = 1) \tag{21}$$

TABLE I. DNS parameters. U_b is the bulk velocity, L_x and L_z are respectively the fundamental periodicity length in the x and z directions, Δx, Δy, and Δz are the grid widths in the x, y, and z directions, respectively, T is the simulation time interval after the initial transient period.

Ref.	Run	Re_τ	U_b^+	L_x/h	L_z/h	Δx^+	Δy^+	Δz^+	T^+/Re_τ
[5]	R500	500	18.1	16.0	6.4	16.0	0.4–5.3	8.3	13.1
[5]	R1000	1000	20.0	16.0	6.4	16.0	0.6–8.0	8.3	12.0
[5]	R2000	2000	21.7	16.0	6.4	16.0	0.6–8.0	8.3	10.0
[5]	R4000	3996	23.4	16.0	6.4	16.0	0.6–8.0	8.3	14.0
[5]	R8000	7987	25.0	16.0	6.4	18.5	0.6–8.0	8.9	7.5
[12,13]	R10000	10049	26.0	2π	π	15.3	0.4–13.0	7.6	19.8

and implies that $\Delta\langle\varepsilon\rangle$ normalized by u_τ^3/y is approximately linear in δ_h and δ_ν for small but finite δ_h and δ_ν.

A naive idea based on these considerations suggests us to introduce, as a first step approximation, the following assumption:

Assumption 3. For sufficiently small δ_h and δ_ν, the correction $\Delta\langle\varepsilon\rangle$ in (9) due to small but finite δ_h and δ_ν can be approximated by (21).

Equations (8) and (9) and Assumption 3 give

$$\langle\varepsilon\rangle \sim \frac{u_\tau^3}{y}\left[C_e + C_p\left(\frac{y}{h}\right) + C_\nu\left(\frac{l_\tau}{y}\right)\right]. \tag{22}$$

In terms of wall units, (22) is equivalent to

$$y^+\langle\varepsilon^+\rangle \sim C_e + C_p\frac{y^+}{Re_\tau} + C_\nu\frac{1}{y^+}, \quad (C_e = 1/\kappa_\varepsilon). \tag{23}$$

As is clear from the above derivation, the estimates (22) and (23) are derived on the basis of Assumptions $1-3$. Assumption 3 is based on the assumption that $\Delta_1\langle B\rangle_p$ and $\Delta_1\langle B\rangle_\nu$ in (16) for $B=\varepsilon$ are linear in α and ν, respectively [see Assumption (II)], and the coefficients c_p^ε and c_ν^ε are determined by only y and u_τ [see Assumption (III)]. Behind Assumptions (II) and (III) is the idea of "locality" in the sense discussed after (13). In our view, Assumptions (I)–(III) are questionable if the nonlocal dynamics could significantly affect the statistics $\langle B\rangle$. Note also that δ_h and δ_ν are assumed to be small in the derivation of (21). This implies that the applicability of (21) is questionable unless δ_h and δ_ν are sufficiently small. The validity of Assumptions $1-3$ is not trivial. Hence, it is desirable to test by experiments and/or DNS the conjectures (22) or (23) that are derived from the assumptions.

III. DATA RESOURCES AND DETERMINATION OF THE CONSTANTS

In this study, we used the data from a series of DNSs of TCF [5] to both estimate the constants in (23) and verify the theoretical conjectures presented above. Some key parameters of the DNSs are listed in Table I. The parameters of the DNS run by Hoyas *et al.* [12] and by Oberlack *et al.* [13] are also included in the list, where the run is named R10000.

In the comparison, we use the so-called pseudodissipation $\langle\varepsilon_g\rangle \equiv \nu\langle g_{ij}^2\rangle$, rather than the true dissipation $\langle\varepsilon\rangle \equiv \nu\langle s_{ij}^2\rangle$ and omit the subscript g, so ε_g is written as ε for the sake of simplicity, where $g_{ij} \equiv \partial u_i/\partial x_j$, $s_{ij} \equiv (g_{ij}+g_{ji})/2$, i, $j = 1, 2, 3$ and the summation convention is used for repeated indices. Pseudodissipation is commonly used in studies of TCF, and is known not to differ much from the true dissipation in the inertial sublayer [5,14–17].

To apply the theory (23) to the estimate of $\langle\varepsilon\rangle$, we need to know the constants C_e, C_p, and C_ν. However, unfortunately, no theoretical derivation of the constants from the first principle, i.e., the

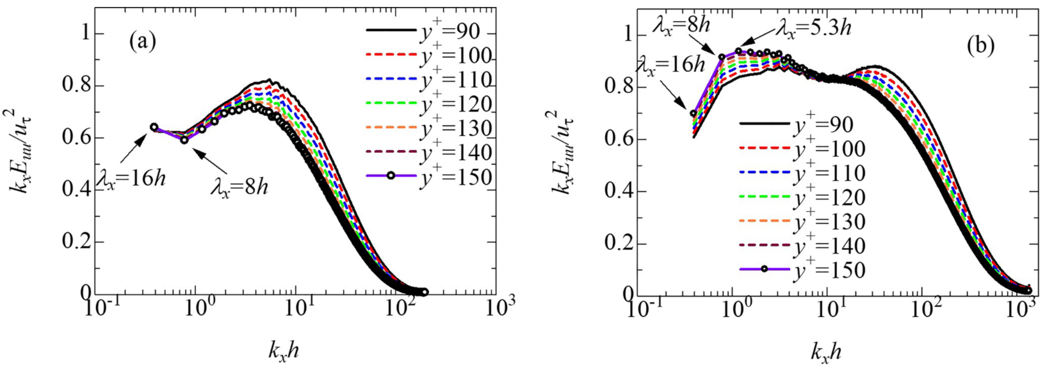

FIG. 1. The streamwise premultiplied velocity-correlation spectrum profiles, $k_x(= 2\pi/\lambda_x, \lambda_x)$ is the wavelength is streamwise wavelength, E_{uu} is the streamwise energy spectrum for u, (a) R1000 and (b) R8000.

NS equation, is known at present. Therefore, we evaluate them by applying a least fitting method to the DNS data. Since the constants C_e, C_p, and C_ν are supposed to be determined by the state at $\delta_h, \delta_\nu \to 0$, it is reasonable to use the data of DNS at a Re_τ value that is as large as possible. The friction Reynolds number is $\mathrm{Re}_\tau \approx 10\,049$ in R10000. To the best of our knowledge, $\mathrm{Re}_\tau \approx 10\,049$ is the highest Re_τ so far achieved in DNS of TCF. However, as seen in Table I, the fundamental periodicity length L_x in the streamwise direction in R10000 is $2\pi h$, while $L_x = 16h$ in R8000. In order to avoid artificial effects on the statistics of $\langle\varepsilon\rangle$ due to the use of periodic boundary conditions, it is presumably necessary in general that the length L_x is sufficiently large in an appropriate sense. In this respect, it is worthwhile to recall that there is a second peak in the streamwise premultiplied velocity-correlation spectrum at the wavelength $\lambda_x \approx 6h$ (see, e.g., [18–21]; see also Fig. 1(b), where a peak is in fact observed at $\lambda_x \approx 6h$ in the premultiplied spectrum of R8000). This suggests that it is desirable that the length L_x is sufficiently large as compared to $\lambda_x \approx 6h$. In view of these observations, we use in this study the data of R8000 ($L_x = 16h$) rather than those of R10000 ($L_x = 2\pi h$) for evaluating of the constants C_e, C_p, and C_ν.

The fitting based on the data of R8000 in the range $y^+ \in [100,\ 1600]$ yields

$$C_e = 2.28,\ \ C_p = -1.19,\ \ C_\nu = 9.42. \tag{24}$$

(Details of the selection of the fitting range are shown in the Appendix.) It is natural to assume that $\langle\varepsilon\rangle$ in the inertial sublayer is larger for larger ν, for given y, u_τ, and h. This implies $C_\nu > 0$. Similarly, it is natural to assume that $\langle\varepsilon\rangle$ in the sublayer is smaller for larger h, for a given y, u_τ, and ν. This implies $C_p < 0$. The values in (24) are consistent with these conjectures. In the following discussion, we assume $C_\nu > 0$ and $C_p < 0$.

IV. COMPARISON WITH DIRECT NUMERICAL SIMULATION

In the following, we examine the theoretical conjectures presented in Sec. II by comparing them with the data of a series of DNSs of TCF listed in Table I. Figure 2(a) shows the value of $y^+\langle\varepsilon^+\rangle$ as a function of y^+ in the range $y^+ \in [100,\ 0.2\mathrm{Re}_\tau]$. The theoretical conjecture (23) agrees fairly well with the DNS results for $\mathrm{Re}_\tau \geqslant 4000$. The difference at smaller Re_τ is not surprising in view of the fact that the theory is for large Re_τ.

Figure 2(b) shows $y^+\langle\varepsilon^+\rangle - [C_p y^+/\mathrm{Re}_\tau + C_\nu/y^+]$. Here, the curves overlap well to a constant in an appropriate range of y^+ and Re_τ, as could be expected from (23). This supports (23) and suggests that the coefficient C_e is close to $2.28 \approx 1/0.44$.

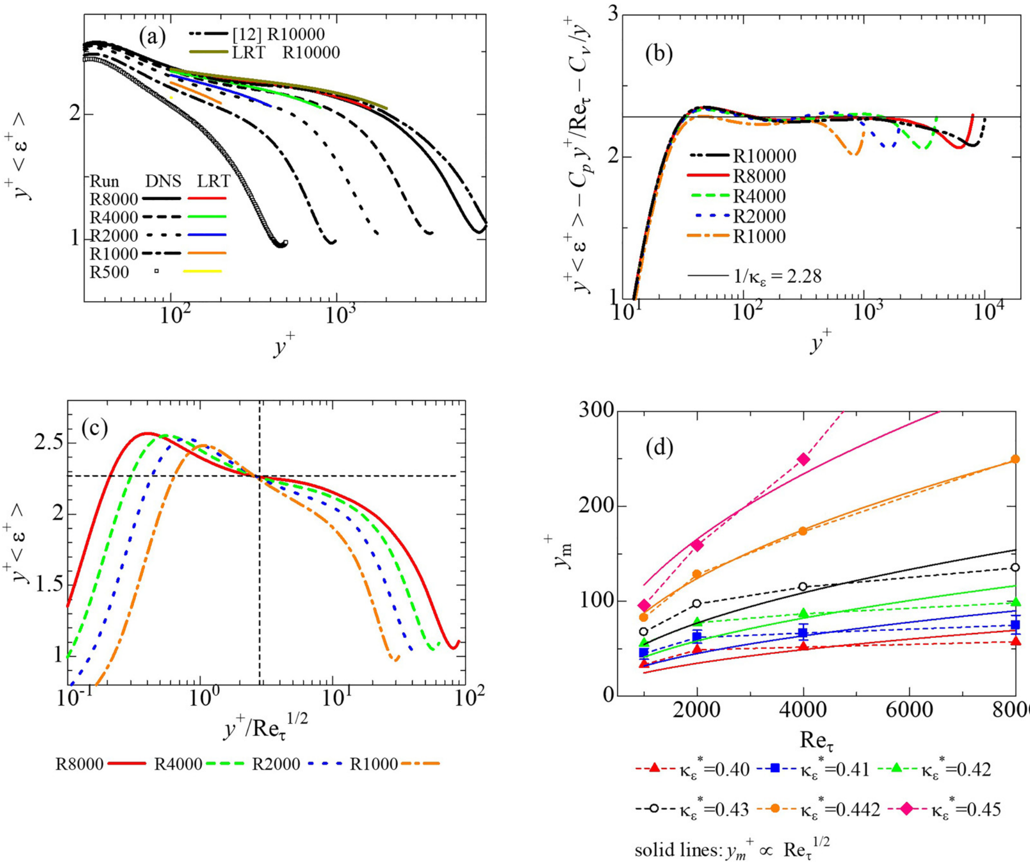

FIG. 2. Comparison between the theoretical conjectures of LRT and DNS data, (a) $y^+\langle\varepsilon^+\rangle$ vs y^+, where color solid lines show the value by (23) with the coefficients listed (24) for R8000. (b) $y^+\langle\varepsilon^+\rangle-[C_p y^+/\mathrm{Re}_\tau + C_v/y^+]$ vs y^+, (c) DNS data by R1000, R2000, R4000, and R8000 for $y^+\langle\varepsilon^+\rangle$ vs $y^+/\mathrm{Re}_\tau^{1/2}$. The horizontal and vertical dashed lines show respectively $y^+\langle\varepsilon^+\rangle = C_\mathrm{e}(= 2.28)$, and $y^+/\mathrm{Re}_\tau^{1/2} = (-C_v/C_p)^{1/2}(= 2.81)$. (d) Symbols: y_m^+ satisfying $y_m^+\langle\varepsilon^+\rangle|_{y^+=y_m^+} = 1/\kappa_\varepsilon^*$ for $\kappa_\varepsilon^* = 0.40, 0.41, 0.42, 0.43, 0.442, 0.45$ for R1000, R2000, R4000, and R8000. Solid lines show the values $y_m^+ = G\mathrm{Re}_\tau^{1/2}$, where G is a constant chosen by a least square fitting.

Recently, Abe and Antonia [3] proposed a theory of $\langle\varepsilon\rangle$ on the basis of matched asymptotic expansions, as follows

$$y^+\langle\varepsilon^+\rangle = \frac{1}{\kappa_\varepsilon} - d\,\frac{y^+}{\mathrm{Re}_\tau}, \tag{25}$$

where κ_ε and d are constant. They obtained $1/\kappa_\varepsilon = 2.45$ and $d = 1.7$ to fit the DNS data up to $\mathrm{Re}_\tau = 5200$. Figure 3(a) shows the value of $y^+\langle\varepsilon^+\rangle$ as a function of y^+ in the range $y^+ \in [30\text{-}0.2\,\mathrm{Re}_\tau]$ [3], where the colored solid lines show the value by (25) with $1/\kappa_\varepsilon = 2.45$ and $d = 1.7$. Equation (25) implies that the curves of $[y^+\langle\varepsilon^+\rangle + dy^+/\mathrm{Re}_\tau]$ vs y^+ would overlap well to a constant ($\sim 1/\kappa_\varepsilon = 2.45$) at small enough y^+/Re_τ However, Fig. 3(b) suggests that the overlap of the curves to the constant 2.45 is not as good as that shown in Fig. 2(b). Figure 3(b) suggests that $[y^+\langle\varepsilon^+\rangle + dy^+/\mathrm{Re}_\tau]$ approaches to a constant in an appropriate range of y^+ as $\mathrm{Re}_\tau \to \infty$, but the constant is different from the fitting value of 2.45.

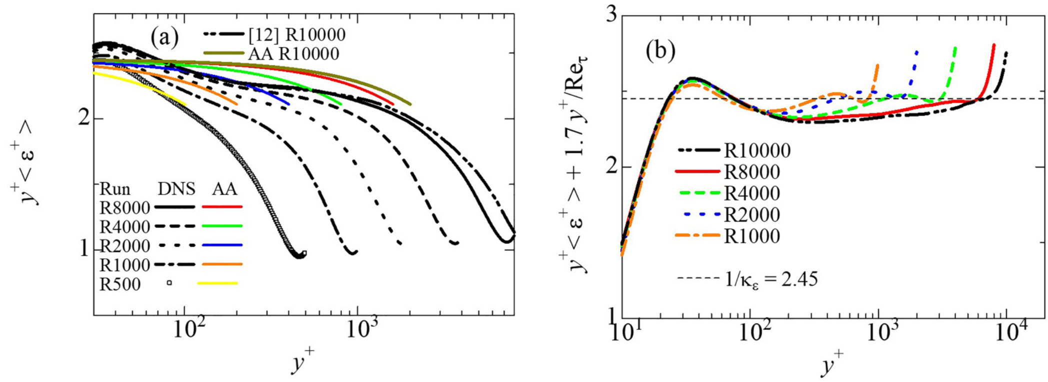

FIG. 3. Comparison between the theoretical conjectures of Abe and Antonia (AA) [3] and DNS data, (a) $y^+\langle\varepsilon^+\rangle$ vs y^+, (b) $[y^+\langle\varepsilon^+\rangle + 1.7\,y^+/\mathrm{Re}_\tau]$ vs y^+.

Equation (23) gives

$$y^+\langle\varepsilon^+\rangle - C_e \sim \frac{C_p}{y^+}\left(\frac{y^+}{\sqrt{\mathrm{Re}_\tau}} - z_e\right)\left(\frac{y^+}{\sqrt{\mathrm{Re}_\tau}} + z_e\right),\ z_e \equiv \sqrt{\frac{C_\nu}{-C_p}}. \tag{26}$$

This implies that if $y^+\langle\varepsilon^+\rangle$ is plotted in the $(y^+/\sqrt{\mathrm{Re}_\tau},\ y^+\langle\varepsilon^+\rangle)$ plane as a function of $y^+/\sqrt{\mathrm{Re}_\tau}$ then although the curves are generally different for different values of Re_τ, all of them pass through the common point $(z_e,\ C_e)$ in the plane, irrespective of Re_τ. The plots in Fig. 2(c) are consistent with this conjecture in the sense that the curves overlap fairly well at a point. The ordinate value of the point provides an estimate C_e, that is free, i.e., independent from the least squares fitting method used to derive the values in (24). As seen in Fig. 2(c), the point is not far from (2.28, 2.81), which is obtained by substituting the values of C_e, C_p, and C_ν in (24) into $(C_e,\ \sqrt{C_\nu/-C_p})$.

Figures 2(b) and 2(c) show that two independent methods, i.e., one using least squares fitting and the other using plots of the DNS data in the $(y^+/\sqrt{\mathrm{Re}_\tau},\ y^+\langle\varepsilon^+\rangle)$ plane, give a similar value $\kappa_\varepsilon \approx 0.44$. This value is different from the widely accepted value (≈ 0.40 or so) for the von Kármán constant κ from the arguments based on experiments [22–26] or theory [27,28].

The difference of κ_ε from 0.40 or so can also be seen in Fig. 2(d), which shows the DNS values of y_m^+ such that $y^+\langle\varepsilon^+\rangle = 1/\kappa_\varepsilon^*$ at $y^+ = y_m^+$, for given $\kappa_\varepsilon^* = 0.40$, 0.41, 0.42, 0.43, 0.442, and 0.45 and R1000, R2000, R4000, and R8000. According to (26), if $1/\kappa_\varepsilon^* = C_e$, i.e., if $\kappa_\varepsilon = \kappa_\varepsilon^*$, then y_m^+ is given by $y_m^+ = z_e\sqrt{\mathrm{Re}_\tau} \propto \sqrt{\mathrm{Re}_\tau}$. This implies that if $\kappa_\varepsilon = \kappa_\varepsilon^*$, then the curves $y_m^+ = G\sqrt{\mathrm{Re}_\tau}$ for the given κ_ε^* fits well to the DNS data of y_m^+, provided that the constant G is chosen appropriately. The solid lines in Fig. 2(d) show the curves given by $y_m^+ = G\sqrt{\mathrm{Re}_\tau}$ for fixed κ_ε^*, in which G is determined by least squares fitting of $y_m^+ = G\sqrt{\mathrm{Re}_\tau}$ to the DNS data for fixed values of κ_ε^*. Among the six curves, the agreement between the DNS data and the curves is best for $\kappa_\varepsilon^* = 0.442$ (orange line). In addition to Figs. 2(b) and 2(c), this provides another support for the conjecture that κ_ε is not far from 0.442, and is distinctively different from 0.40 or so. The constant G thus obtained for $\kappa_\varepsilon^* = 0.442$ is ≈ 2.77, which is very close to $z_e \approx 2.81$ as could be expected.

V. DISCUSSION AND CONCLUSION

The relation (26) was derived from (23), i.e., (21) with $\xi = \zeta = 1$. In this regard, it may be of interest to note that a recent study of the influence of mean shear on velocity-gradient moments in turbulent shear flow suggests that the effects of small but finite viscosity ν on the moments are fairly well approximated by terms proportional to $\nu^{1/2}$ under appropriate normalization [5]. From this, one might assume $\xi = 1$ and $\zeta = 1/2$, instead of $\xi = \zeta = 1$ in (21). However, $\xi = 1$ and $\zeta = 1/2$

would give $y_m^+ \propto \mathrm{Re}_\tau^{1/3}$, while the DNS results in Fig. 2(d) look to be in favor of $y_m^+ \propto \mathrm{Re}_\tau^{1/2}$, rather than $y_m^+ \propto \mathrm{Re}_\tau^{1/3}$. This implies that the DNSs may favor the simple scalings $\xi = 1$ and $\xi = 1$ rather than $\xi = 1$ and $\zeta = 1/2$.

As noted in the Introduction, and as is well known, the simple relation $\kappa = \kappa_\varepsilon$ is derived by assuming the local equilibrium, i.e., the balance between turbulent production and energy dissipation rate $\langle \varepsilon \rangle$, and assuming the log law (1) for the mean velocity and $\langle uv \rangle \sim -u_\tau^2$. The difference between κ and κ_ε implies that at least one of these assumptions is questionable. The theory presented in this study is free from the assumption of local equilibrium [6,7].

The present theory is unique in the sense that it is for $\langle \varepsilon \rangle$ not only in the limit of infinite Re_τ but also at high but finite Re_τ, and is fully consistent with the existing data of the world's largest DNS. On the basis of these DNS data, the present theory suggests that κ_ε given by (23) is approximately 0.44.

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APPENDIX

Let the fitting range of y be $[y_{\min},\ y_{\max}]$. In order to set the range appropriately, we need take into account the following:

(a) Equation (23) is assumed to be applicable only for the layer satisfying $l_\tau/y \ll 1$ and $y/h \ll 1$, not for the entire range of y. This implies that D_1 and D_2 must be "sufficiently large," where D_1 and D_2 are defined as $D_1 = y_{\min}/l_\tau = y^+_{\min}$ and $D_2 = h/y_{\max} = \mathrm{Re}_\tau/y^+_{\max}$.

(b) There exist the so-called inner-scaling layer and buffer layer near the boundary wall. The former is generally located at $0 < y^+ < L^+$, where L^+ is approximately 15 (e.g., the wall-normal height of the peak value of the streamwise turbulent intensity).

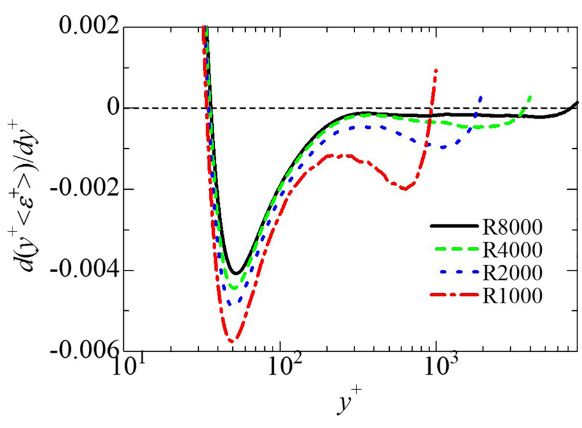

FIG. 4. $d(y^+\langle\varepsilon^+\rangle)/dy^+$ vs y^+.

TABLE II. The effect of the fitting range on constant parameters.

Run	D_2	D_3	D_4	L^+	$y^+_{\rm min}$	$y^+_{\rm max}$	C_e	C_p	C_ν
R8000	5.0	5.3	20.0	15	80	1600	2.27	−1.14	11.6
R8000	5.0	6.7	16.0	15	100	1600	2.28	−1.19	9.42
R8000	5.0	8.0	13.3	15	120	1600	2.29	−1.24	7.16

The DNS data concerning (15) (see, e.g. [21]) suggest that unless $y \gg L$, the effect of the viscous term is not negligible. This suggests that D_3 need be sufficiently large for (23) to apply, where $D_3 = y/L = y^+/L^+$.

(c) To compare the y dependence of $\langle\varepsilon\rangle$ by DNS and that by (23), D_4 needs to be sufficiently large, where $D_4 = y_{\rm max}/y_{\rm min}$. However, it is to be recalled that we have a trivial constraint

$$\mathrm{Re}_\tau = \frac{h}{l_\tau} = \underbrace{\frac{h}{y_{\rm max}}}_{D_2} \times \underbrace{\frac{y_{\rm max}}{y_{\rm min}}}_{D_4} \times \underbrace{\frac{y_{\rm min}}{L}}_{D_3} \times \underbrace{\frac{L}{l_\tau}}_{L^+} = D_2 D_4 D_3 L^+. \tag{27}$$

This constraint implies that there is a tradeoff between the constants D_2, D_3, and D_4; i.e., under any given finite Re_τ, setting a large value of any of the constants is possible only by sacrificing the largeness of at least one of the other constants.

It is not trivial how sufficiently large the constants D_2, D_3, and D_4 must be for the conditions noted in (a), (b), and (c) to be satisfied. However, it is observed in Fig. 2(a) that in the curve of $y^+\langle\varepsilon^+\rangle$ vs y^+ in R8000, there is no inflection point in the y range approximately [100, 1600]; in other words, the y dependence is not so complicated in the range. In practice, we therefore set $[y^+_{\rm min}, y^+_{\rm max}] = [100, 1600]$ for the fitting. This results in (24).

The choice $y^+_{\rm max} = 1600$ implies $y^+_{\rm max} = \mathrm{Re}_\tau/5$ ($D_2 = 5$), for $\mathrm{Re}_\tau = 8000$. This choice of $y^+_{\rm max} = \mathrm{Re}_\tau/5$ is consistent with the understanding that the so-called outer layer is located approximately $y/h > 0.2$, i.e., $y^+ > \mathrm{Re}_\tau/5$. The latter suggests that $y^+_{\rm max}$ needs to be smaller than $\mathrm{Re}_\tau/5$.

The choice of $y^+_{\rm min} = 100$ is consistent with the observation that $d(y^+\langle\varepsilon^+\rangle)/dy^+$ has a sharp peak at $y^+ \approx 50$, as seen in Fig. 4, which suggests that the statistics of $\langle\varepsilon^+\rangle$ at $y^+ \approx 50$ are dominated by factors different from those in the range $y^+ > 50$ or so. This suggests that $y^+_{\rm min}$ needs to be substantially larger than 50, i.e., $D_3 > 3.3$, where $L^+ \approx 15$.

To get some idea on the sensitivity of the estimates of the constants C_e, C_p, and C_ν to the choice of the fitting range, we estimated the constants by selecting some values of the range other than that of $[y^+_{\rm min}, y^+_{\rm max}] = [100, 1600]$ noted above. The results of least-squares fitting of (23) to the DNS data are shown in Table II. It is seen that C_p and C_ν are somewhat sensitive to the choice of the fitting range, but C_e is not so sensitive to that choice. The constants C_e, C_p, and C_ν by least-squares fitting of (23) in the range of $[y^+_{\rm min}, y^+_{\rm max}] = [100, 2000]$ for R10000 is 2.27, −1.23, and 9.49, respectively. Note that (27) implies that $\mathrm{Re}_\tau > 8040$, if $L^+ = 15$, $D_2 = 5$, $D_3 = 6.7$, and $D_4 > 16$. Therefore, in addition to R8000, the DNS data of at least $\mathrm{Re}_\tau \approx 16\,000$ with large computational domains would be required to discuss the confidence interval for the constants.

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