Predicting mean profiles in compressible turbulent channel and pipe flows

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(Received 11 October 2022; accepted 10 March 2023; published 22 March 2023)

A modified iterative method is proposed to predict the mean profiles in compressible channel and pipe turbulence by solving a nonlinear system, which is constituted by the perfect gas assumption, the Sutherland's viscosity law, the variable-property scaling models for compressible wall turbulence, the incompressible velocity law of the wall and the relationship between the mean temperature and velocity. Systematic parametric studies display that the proposed iterative method is robust and does not depend on the initial guess of the undetermined parameters. The results also indicate that the present method works very well in both channel and pipe flows at various Reynolds and Mach numbers, and the relative errors of mean temperature profiles and velocity profiles, as compared with the available direct numerical simulation databases, in most cases are within 2% and 5%, respectively.

DOI: 10.1103/PhysRevFluids.8.034604

I. INTRODUCTION

Canonical compressible wall-bounded turbulent flows, including flat plate flows, channel flows and pipe flows, are the typical problems to study the flow physics of compressible turbulence due to its simple geometry. Compared with the incompressible wall-bounded turbulence, where the fluid properties are assumed constant and the classical logarithmic law of the mean velocity has been reported in the canonical flows, the fluid properties in compressible turbulence are no longer constant, but varying spatially and temporally due to coupling of the velocity and temperature fields through the compressible Navier-Stokes equations. The variations of flow properties will complicate the mean velocity profiles in compressible wall-bounded turbulence. Nevertheless, the well-known Morkovin's hypothesis, which was restated by Bradshaw [1] that "the direct effects of density fluctuations on turbulence are small if the root-mean-square density fluctuation is small compared with the absolute density," allows the community to handle the turbulent structures of boundary layers and wakes with Ma $\leqslant 5$, which are closely the same as those in the corresponding constant-density flows. The validity of the Morkovin's hypothesis has been verified in several canonical flows with varying Ma $\leqslant 5$ [2–4]. Nowadays, some studies further showed that for compressible turbulent boundary layers the Morkovin's hypothesis can hold even in the hypersonic range at Ma $= 12$ [5] and Ma $= 20$ [6] when neglecting thermochemical effects.

At the same time, accounting for mean property variations in compressible turbulent wall-bounded flows, i.e. transforming the mean profiles of a compressible state to a corresponding incompressible state, is "state-of-the-art" [7], and various transformation methods [7–11] had been proposed in the past since the pioneer work by Van Driest [12]. Following the unified expression summarized by Modesti and Pirozzoli [13], most of the existed transformations (except Griffin *et al.* [11]) can be expressed in terms of mapping functions f_M and g_M for wall distance y_I and mean

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velocity u_I, which are the equivalent incompressible distributions obtained from the transformation "M,"

$$y_I = \int_0^y f_M dy, \quad u_I = \int_0^{\overline{u}} g_M d\overline{u}. \tag{1}$$

Here, f_M and g_M are functions of $R = \overline{\rho}/\overline{\rho}_w$, the ratio of local density to its value at the wall, and $M = \overline{\mu}/\overline{\mu}_w$, the ratio of local dynamics viscosity to its value at the wall. Here and thereafter, $\overline{(.)}$ is the Reynolds averaging operator, y is the wall distance, u is the velocity, ρ is the density, $\overline{\rho}_w$ is the density at the wall, μ the dynamic viscosity and $\overline{\mu}_w$ the viscosity at the wall. In the above unified expression, the mapping functions of the so-called "van Driest" (VD) transformation [12] are

$$f_{VD} = 1, \quad g_{VD} = R^{1/2}, \tag{2}$$

and the mapping functions of the transformation proposed by Trettel and Larsson [7] (denoted as the TL transformation) are

$$f_{TL} = \frac{d}{dy}\left(\frac{yR^{1/2}}{M}\right), \quad g_{TL} = M\frac{d}{dy}\left(\frac{yR^{1/2}}{M}\right). \tag{3}$$

Previous investigations indicated that the "VD" transformation is appropriate for the heating wall as well as the adiabatic wall [14], whereas it works bad for cooling walls [14,15]. The TL transformation showed a good collapse for compressible channels and pipes [7,13,16]. For a compressible flat plate, there is an upward shift in the log-layer [17].

A recently total-stress-based transformation proposed by Griffin *et al.* [11] successfully collapsed the velocity distributions for fully developed channel flows, pipe flows, and boundary layers with or without heat transfer. In the total-stress-based transformation, distinct effects of the compressibility on the viscous stress and turbulent shear stress were considered, where it treated the viscous stress by accounting for the mean property variations with the semi-local nondimensionalization, whereas it treated the Reynolds shear stress to maintain the approximate equilibrium of turbulence production and dissipation. The transformed velocity of the simplified constant-stress-layer model is defined as follows:

$$U_t^+[y^*] = \int S_t^+ dy^*, \quad S_t^+ = \frac{S_{\mathrm{eq}}^+}{1 + S_{\mathrm{eq}}^+ - S_{TL}^+},$$
$$S_{\mathrm{eq}}^+[y^*] = \frac{1}{\mu^+}\frac{\partial \overline{u}^+}{\partial y^*}, \quad S_{TL}^+[y^*] = \mu^+\frac{\partial \overline{u}^+}{\partial y^+}, \tag{4}$$

where $y^* = y/l_{sl}$ is the semilocal wall-normal coordinate, $u_{sl} = \sqrt{\tau_w/\overline{\rho}[y]}$ and $l_{sl} = \overline{\nu}[y]/u_{sl}$ are, respectively, the semilocal velocity and length scales. The superscript $^+$ means a nondimensionalization by the classical wall units, i.e. the friction velocity $u_\tau = \sqrt{\tau_w/\overline{\rho}_w}$, the viscous length scale $\delta_v = \mu_w/(u_\tau \overline{\rho}_w)$ and $\overline{\rho}_w$. For examples, $y^+ = y/\delta_v$, and $\overline{u}^+ = \overline{u}/u_\tau$. In the following sections, the TL transformation as well as the total-stress-based transformation will be used.

While the above transformations were concentrated on a "forward" problem, i.e., collapsing the transformed velocity profiles of compressible wall-bounded turbulent flows at different Reynolds numbers, Mach numbers, and wall temperatures with the corresponding incompressible ones, the mean profiles of the compressible wall-bounded turbulence themselves, i.e., the profiles before the transformation, are of more importance and interest in practice, since they are essential for the aerospace design. Predicting these mean profiles as well as the related skin friction and heat flux at the wall at very low cost is in urgent need in the design process [18]. Thanks to the successful works of various transformations mentioned above and the well-studied incompressible wall-bounded turbulent scaling laws, an inverse problem, i.e., obtaining the mean profiles of compressible wall-bounded turbulence from the above-mentioned transformations and the known incompressible wall-bounded profiles, provides one possible approach. In the past few years, researchers have found that one can obtain the mean profiles (velocity, temperature, density, and viscosity) by solving the closed system formed by the perfect gas assumption, the viscosity law, the variable-property scaling models (transformations) for compressible wall turbulence, the incompressible velocity law of the wall, and the mean temperature-mean velocity relationship [18–21]. Huang *et al.* [19] proposed a general approach to get mean velocity and temperature profiles at desired Mach number and T_w/T_r (T_w and T_r are the wall temperature and recovery temperature) for compressible turbulent boundary layers based on a density-weighted transformation. They also proposed a self-consistent skin friction law from the proposed velocity profile, which was at least as good as those obtained by using the Van Driest II transformation. Zhang [20] also studied the quantitative predictions of the mean profiles in compressible turbulent boundary layers based on the generalized mean temperature-mean velocity relationship and a series of Mach number invariants and wall-temperature invariants in his Ph.D. thesis. He proposed three different iterative approaches to predict the mean profiles in adiabatic compressible turbulent boundary layers, and the first two could further predict the profiles of Reynolds stresses and temperature variance if the Mach number invariance property of the Reynolds stresses as well as the generalized Reynolds analogy theory between the streamwise velocity fluctuations and the temperature fluctuations were used. For the diabatic compressible boundary layers, he suggested combining the wall temperature invariance property and the predictions of adiabatic cases. In his thesis, he showed the prediction results and found that the proposed approaches can well predict the profiles for the adiabatic and quasi-adiabatic cases, while the predictions on the diabatic cases are rather poor. Recently, Kumar and Larsson [18] formed a modular method by extending the approach by Huang *et al.* [19] to estimate the velocity and temperature profiles in high-speed boundary layers. The recent improvements on the transformation as well as the temperature-velocity relationship were used and tested. The results showed that by using the more accurate relations the errors of the skin friction and wall heat flux could be reduced from 16% to 8% and from 13% to 11%, respectively, as compared to the Van Driest II method.

Although a lot of research has studied the inverse problem in compressible turbulent boundary layers (external flows), the studies in internal flows (compressible channel and pipe flows) are still relatively rare. Griffin *et al.* [21] mentioned that the centerline temperature can be iteratively determined with the condition $d\overline{T}/dy = 0$ ($\overline{T}$ is the mean temperature) at the centerline for channel flows. However, neither the detailed process nor the predicted results were shown in their paper. In this paper, we proposed a modified iterative method for channel and pipe flows according to the approach by Huang *et al.* [19]. Different from the original iterative method, which is a straight extension of the approach by Huang *et al.* [19], we propose to replace the condition $d\overline{T}/dy = 0$ (or $d\overline{u}/dy = 0$) at the centerline by the semiempirical central mean temperature scaling recently proposed by Song *et al.* [22]. The proposed iterative method was validated against several direct numerical simulation (DNS) databases at various Reynolds and Mach numbers, and the results showed that the proposed method is stable and of good accuracy.

II. ITERATIVE METHOD FOR MEAN PROFILES IN CHANNEL AND PIPE FLOWS

Following the approach in compressible turbulent boundary layers [18–21], the nonlinear system constituted by the perfect gas assumption, the viscosity law, the variable-property scaling models (transformations) for compressible wall turbulence, the incompressible velocity law of the wall, and the mean temperature–mean velocity relationship will be used to estimate the mean profiles in compressible turbulent channel and pipe flows in this section. The above equations are modular and each of them can be easily replaced for a better one in the future. The scaling law and the iterative method used in this paper will be presented in detail in the following. All variables are nondimensionalized with the bulk-averaged density ρ_b, the bulk velocity u_b, the reference temperature T_{ref}, the reference viscosity μ_{ref} at T_{ref} and the reference length L, which is the channel half-height h for channel flows and the pipe radius R for pipe flows.

Firstly, we consider the boundary layer assumption that the pressure is invariant with wall-normal direction, i.e., $\overline{P}[y] \approx \overline{P}[y = 0]$. By considering the Morkovin's assumption and the perfect gas assumption, we obtain

$$\overline{\rho}[y] = \frac{\overline{\rho}_w \overline{T}_w}{\overline{T}[y]}. \tag{5}$$

Similarly, based on the Morkovin's assumption and the Sutherland's viscosity law, the mean viscosity can expressed as

$$\overline{\mu}[y] = (\overline{T}[y])^{3/2} \frac{1 + S/T_{\text{ref}}}{\overline{T}[y] + S/T_{\text{ref}}}, \tag{6}$$

where S is the Sutherland temperature.

As discussed earlier, multiple transformations have been proposed in the literature to produce an excellent collapse of the mean velocity profiles by taking into account the mean thermodynamic variables. In the present work, we mainly use the TL transformation proposed by Trettel and Larsson [7], i.e., Eq. (3). This is another equation that we need in the iterative process. With the transformation, the transformed velocity profile at $\text{Re} = \rho_b u_b L/\mu_{\text{ref}}$ is consistent with the incompressible velocity profile at friction Reynolds number $\text{Re}_\tau = u_\tau L/\nu$ (ν is the kinematic viscosity) if the transformed friction Reynolds number $\text{Re}_\tau^* = \text{Re}[(\overline{\rho}_c \overline{\rho}_w)^{1/2} u_\tau 1/\overline{\mu}_c]$ equals to Re_τ. That is, we need the incompressible velocity profile:

$$\overline{U}^+_{\text{incomp}} = f(y_I^+, \text{Re}_\tau^*). \tag{7}$$

The incompressible velocity profiles can be obtained by two means. One of them is from the open source DNS databases (see Ref. [23]), and the other is from the published analytical expressions [24,25]. The analytical mean velocity profile proposed by She *et al.* [24] will be mainly used in this work and it is

$$\begin{aligned}
U^+ &= \int_0^{y^+} S^+ dy^+, \quad S^+ = \left(-1 + \sqrt{1 + 4\tau^+ l_{uv}^{+2}}\right)/\left(2 l_{uv}^{+2}\right), \\
l_{uv}^+ &= l_0 \left(\frac{y^+}{y^+_{\text{sub}}}\right)^{3/2} \left(1 + \left(\frac{y^+}{y^+_{\text{sub}}}\right)^4\right)^{1/8} \left(1 + \left(\frac{y^+}{y^+_{\text{buf}}}\right)^4\right)^{-1/4} \frac{1 - r^m}{m(1-r)Z_c} \left(1 + \left(\frac{r_{\text{core}}}{r}\right)^2\right)^{1/4}, \\
l_0 &= \kappa \frac{y^{+2}_{\text{sub}}}{y^+_{\text{buf}}}, \, y^+_{\text{sub}} \approx 9.7, \, y^+_{\text{buf}} \approx 41, \, r_{\text{core}} \approx 0.27, \, Z_c = \left(1 + r^2_{\text{core}}\right)^{1/4}, \\
\kappa &\approx 0.45, \, r = (\text{Re}_\tau - y^+)/\text{Re}_\tau,
\end{aligned} \tag{8}$$

where $m = 4$ for channel and $m = 5$ for pipe, τ^+ is the total stress, and it is $\tau^+ = 1 - y^+/\text{Re}_\tau$ for channel and pipe flows. She *et al.* [24] have demonstrated in their paper that their analytical expression was quite accurate for both channel and pipe flows at $\text{Re}_\tau \gtrsim 940$. We also obtained the profiles for incompressible turbulent channel and pipe flows at various Re_τ ($180 \leqslant \text{Re}_\tau \leqslant 6000$) by using the above analytical expression, and compared the results with the recent DNS data for channel (Lee and Moser [26]) and pipe flows (Pirozzoli *et al.* [27]). The results showed (not shown here for brevity) that the relative errors across the wall-normal direction are within 4%, documenting the accuracy of the above analytical expression.

Finally, to close these equations, we should add the relationship between the mean temperature $\overline{T}[y]$ and the mean velocity $\overline{u}[y]$. Many algebraic relationships between the mean temperature and the mean velocity were proposed in the past [12,28–30]. In the present work, the latest formula by Zhang *et al.* [30] was chosen, and it is

$$\frac{\overline{T}}{\overline{T}_\delta} = \frac{\overline{T}_w}{\overline{T}_\delta} + \frac{\overline{T}_{rg} - \overline{T}_w}{\overline{T}_\delta} \frac{\overline{u}}{\overline{u}_\delta} + \frac{\overline{T}_\delta - \overline{T}_{rg}}{\overline{T}_\delta} \left(\frac{\overline{u}}{\overline{u}_\delta}\right)^2, \tag{9}$$

where $\overline{T}_{rg} = \overline{T}_\delta + r_g\overline{u}_\delta^2/(2C_p)$ is the generalized recovery temperature, $r_g = (\overline{T}_w - \overline{T}_\delta)/[\overline{u}_\delta^2/(2C_p)] - [2\mathrm{Pr}/\overline{u}_\delta]\cdot[\overline{q}_{yw}/\overline{\tau}_w]$ is the generalized recovery factor, $\overline{T}_\delta$ ($\overline{T}_c$) and $\overline{u}_\delta$ ($\overline{u}_c$) are the mean temperature and velocity at channel or pipe center, Pr is the Prandtl number, C_p is the specific heat at constant pressure, γ is the specific heat ratio, $\mathrm{Ma} = u_b/\sqrt{\gamma R T_{\mathrm{ref}}}$ is the Mach number (R is the gas constant), $\overline{q}_{yw}$ and $\overline{\tau}_w$ are the mean heat flux and skin friction at the wall, respectively. For fully developed turbulent channel and pipe flows, $\overline{q}_{yw} = -u_b|\overline{\tau}_w|$ holds due to the global energy conservation [31,32]. Therefore, Eq. (9) can be further rewritten as with the equation $C_p T_{\mathrm{ref}}/u_b^2 = 1/[\mathrm{Ma}^2(\gamma-1)]$,

$$\frac{\overline{T}}{\overline{T}_w} = 1 + \mathrm{Pr}\overline{u}_c(\gamma-1)\mathrm{Ma}^2\left(\frac{\overline{u}}{\overline{u}_c}\right) + \left[\frac{\overline{T}_c}{\overline{T}_w} - 1 - \mathrm{Pr}\overline{u}_c(\gamma-1)\mathrm{Ma}^2\right]\left(\frac{\overline{u}}{\overline{u}_c}\right)^2. \tag{10}$$

Equations (5), (6), (3), (7), and (10) constitute a nonlinear closed system that can be used to solve the mean profiles for turbulent channel and pipe flows. The above system requires several input parameters, including Re, Ma, T_{ref}, $\overline{T}_w$, Pr, and γ. In addition to the above input parameters, there are several parameters to be determined, including $\overline{T}_c/\overline{T}_w$, $\overline{u}_c$, u_τ and $\overline{\rho}_w$. The four undetermined parameters require four additional conditions for the iteration to proceed. Similar to the direct numerical simulation of the channel and pipe flows, the conservation of the mass and the momentum flux are two critical conditions. Besides these, $d\overline{T}/dy = 0$ and $d\overline{u}/dy = 0$ at the centerline are the other two restrictions, which are based on the symmetry of the temperature and velocity profiles. However, it should be noted that $d\overline{T}/dy = 0$ and $d\overline{u}/dy = 0$ at the centerline are redundant, since the condition $d\overline{T}/dy = 0$ at the centerline is nearly equivalent to $d\overline{u}/dy = 0$ at the centerline. If these two conditions were used together to perform the iteration, the results depended largely on the initial guess of the four undetermined parameters (not shown), i.e., $\overline{T}_c/\overline{T}_w$, $\overline{u}_c$, u_τ, and $\overline{\rho}_w$. Therefore, the above nonlinear system is not completely closed, and one of the above two conditions at the centerline should be replaced with proper constraint. Here, we propose to use the semiempirical central mean temperature scaling recently proposed by Song *et al.* [22], where the central mean temperature can be estimated from the central mean velocity as

$$\frac{\overline{T}_c}{\overline{T}_w} = 1 + 1.034\mathrm{Pr}\overline{u}_c\frac{\gamma-1}{2}\mathrm{Ma}^2. \tag{11}$$

It has shown by Song *et al.* [22] that most of the relative errors in turbulent channel flows with Re ranging from 3000 to 34 000 and Ma ranging from 0.5 to 4.0 are below 1.5% compared with direct numerical simulation data. It should be noted that the above relation can guarantee the equivalence of $d\overline{T}/dy = 0$ and $d\overline{u}/dy = 0$ at the centerline. Therefore, the number of undetermined parameters reduces to three, and we only need three conditions to close the above nonlinear system. Besides the conservation of mass and momentum flux, either $d\overline{T}/dy = 0$ or $d\overline{u}/dy = 0$ at the centerline could be chosen. With this semiempirical scaling of $\overline{T}_c$ and $d\overline{T}/dy = 0$ at the centerline, we propose the modified iterative method, and the corresponding flowchart is shown in Fig. 1. The one with $d\overline{u}/dy = 0$ at the centerline can be proposed similarly, and the results are very close to those with the constraint $d\overline{T}/dy = 0$ at the centerline according to our validations.

To test the above proposed iterative method, a case at Ma = 1.5, Re = 7667, $T_{\mathrm{ref}} = 288.15K$, $T_w = 1$ (Please note that T_w is already normalized by T_{ref}), Pr = 0.72 and $\gamma = 1.4$ is chosen, and the DNS data is available by Modesti and Pirozzoli [13]. Three sets of initial guesses of the undetermined parameters are chosen, including guess1: $\overline{u}_c = 1.30$, $u_\tau = 0.03$, $\overline{\rho}_w = 1.4355$, guess2: $\overline{u}_c = 1.137$, $u_\tau = 0.03$, $\overline{\rho}_w = 1.372$, and guess3: $\overline{u}_c = 1.137$, $u_\tau = 0.0480$, $\overline{\rho}_w = 1.367$. $\overline{T}_c/\overline{T}_w$ is calculated by using Eq. (11). It should be noted that guess3 is the same as those values from DNS. With a uniform grid $N_y = 4001$ in the wall-normal direction, a preset tolerance $\epsilon = 10^{-6}$ and the incompressible velocity profile from She *et al.* [24], the predicted mean velocity, temperature, viscosity, and density profiles are shown in Fig. 2. It is evident that the predicted mean velocity profiles, as well as the mean temperature, viscosity, and density profiles, with the three different initial guesses converge to the same results, which are very close to the reference DNS data. Table I

TABLE I. Influence of initial guesses of the undetermined parameters at Ma = 1.5, Re = 7667, $T_{\text{ref}} = 288.15K$, $T_w = 1$, Pr = 0.72 and $\gamma = 1.4$ by the proposed iterative method. The DNS results are from Modesti and Pirozzoli [13].

Term	$\overline{T}_c$	|Error|(%)	$\overline{u}_c$	|Error|(%)	u_τ	|Error|(%)	$\overline{\rho}_w$	|Error|(%)	$\overline{\mu}_c/\overline{\mu}_w$	|Error|(%)	$\overline{\rho}_c/\overline{\rho}_w$	|Error|(%)
DNS	1.372	–	1.137	–	0.0480	–	1.367	–	1.272	–	0.727	–
guess1	1.389	1.24	1.160	2.02	0.0481	0.21	1.368	0.07	1.277	0.39	0.720	0.96
guess2	1.389	1.24	1.160	2.02	0.0481	0.21	1.368	0.07	1.277	0.39	0.720	0.96
guess3	1.389	1.24	1.160	2.02	0.0481	0.21	1.368	0.07	1.277	0.39	0.720	0.96

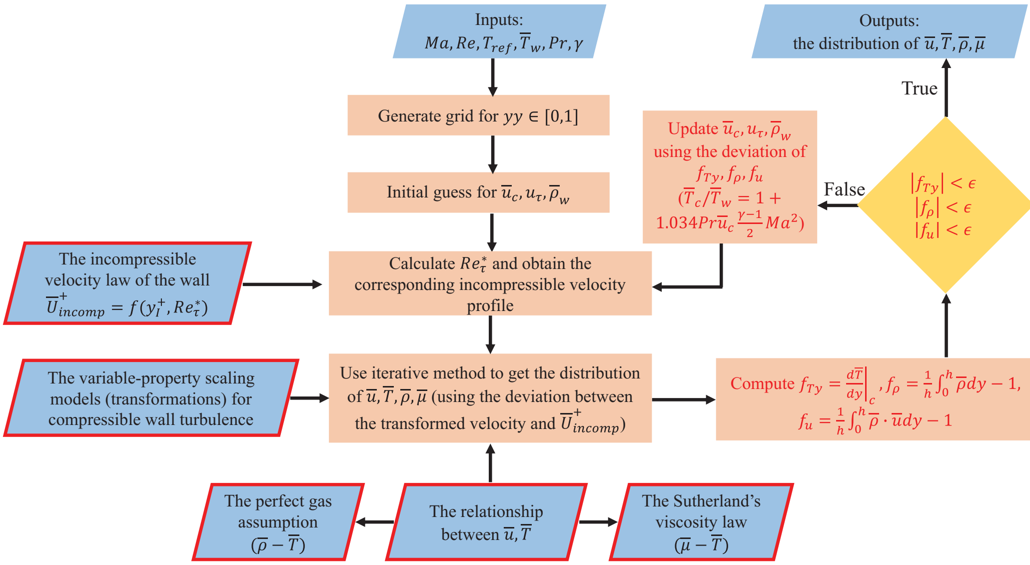

FIG. 1. Flowchart of the proposed iterative method with $d\overline{T}/dy = 0$ at the centerline. Boxes with solid red boundaries represent the required scaling law to choose.

lists the estimated results of $\overline{T}_c$, $\overline{u}_c$, u_τ, $\overline{\rho}_w$, $\overline{\mu}_c/\overline{\mu}_w$, and $\overline{\rho}_c/\overline{\rho}_w$, as well as the relative errors, with the three different initial guesses, and the results indicated that the relative errors reduced a lot for guess1 and guess2, as compared with those estimates by using the original iterative method (not shown for simplicity). Here, the relative error for a quantity f at a location y is defined as

$$Er_f[y] = \frac{f_{\rm DNS} - f_{\rm cal}}{f_{\rm DNS}} \times 100\%, \tag{12}$$

with $f_{\rm DNS}$ and $f_{\rm cal}$ being the DNS value and the predicted value at the location y, respectively. For guess1, the relative error of $\overline{T}_c$ can be reduced to approximate one order, i.e., from 9.33% to 1.24%. However, for guess3, the predicted results seem slightly worse than those obtained by the original method. Since the only difference between the original method and the proposed modified method is to use the semiempirical scaling for the central mean temperature, i.e., Eq. (11), we therefore attribute the weaker performance of the proposed modified method with the correct initialization of the undetermined parameters to the error of the semiempirical scaling. It should be emphasized that a uniform grid was used in the present study. This is just for convenience. We have also tried to use a stretched grid, and the same results were obtained with much fewer points. We believe that the stretched grid is preferable, especially for high Re cases.

Here, we would like to mention that the proposed modified iterative method is also modular in the sense that it works with multiple incompressible velocity law of the wall, variable-property scaling models, equations-of-state, velocity-temperature relations, and viscosity-temperature relations, which are shown in solid red boundaries in Fig. 1. More accurate profiles can be predicted if more accurate formulas were used.

III. FURTHER VALIDATION OF THE PROPOSED MODIFIED METHOD IN CHANNEL FLOWS

A. Validations in various Reynolds and Mach numbers

In this subsection, we will focus on our proposed modified iterative method, which is robust and has no concern with the initial guess of the undetermined parameters. The proposed method

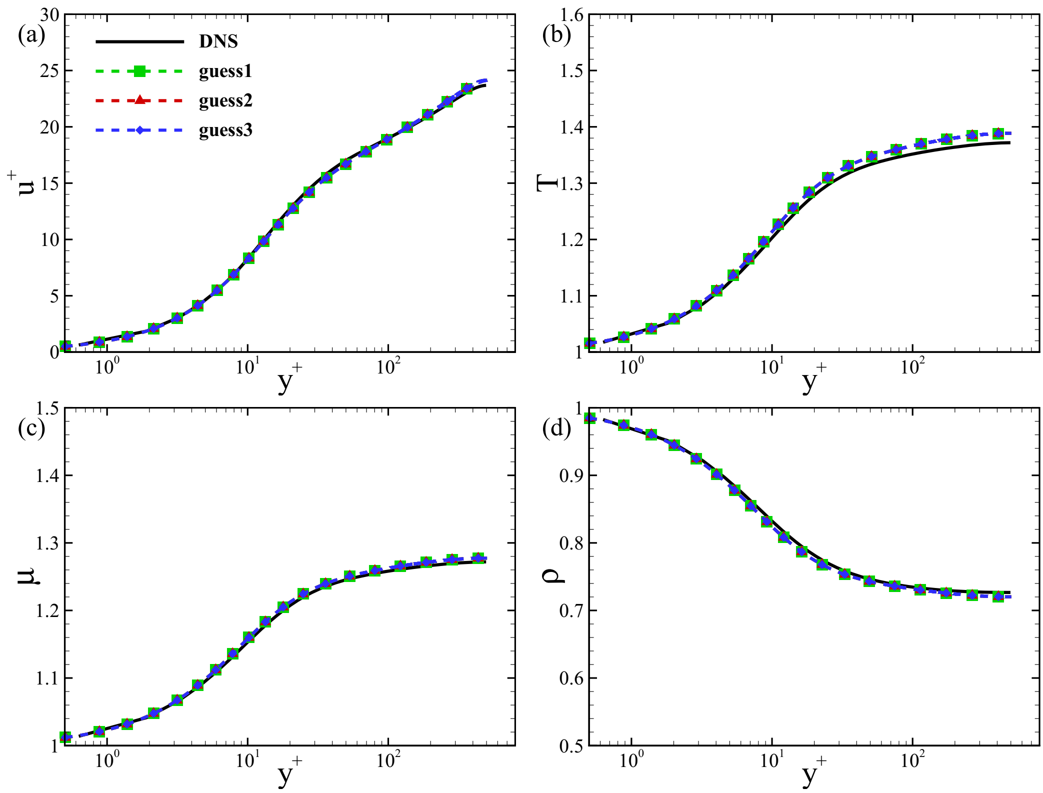

FIG. 2. (a) Mean velocity, (b) mean temperature, (c) mean viscosity, and (d) mean density profiles predicted by the proposed iterative method with three different sets of initial guesses in compressible turbulent channel flows. guess1: $\overline{u}_c = 1.30$, $u_\tau = 0.03$, $\overline{\rho}_w = 1.4355$; guess2: $\overline{u}_c = 1.137$, $u_\tau = 0.03$, $\overline{\rho}_w = 1.372$; and guess3: $\overline{u}_c = 1.137$, $u_\tau = 0.0480$, $\overline{\rho}_w = 1.367$. $\overline{T}_c/\overline{T}_w$ is calculated by using Eq. (11). The reference DNS data at Re = 7667 and Ma = 1.5 is from Modesti and Pirozzoli [13].

will be further validated by comparing with several available DNS databases of compressible turbulent channel flows [13,33,34], where Re ranges from 3000 to 34 000, and Ma varies in $\{0.5, 0.8, 1.5, 3.0\}$. Besides Re and Ma, other input parameters are fixed and the same as before, i.e., $T_{\text{ref}} = 288.15K$, $\overline{T}_w = 1$, $\text{Pr} = 0.72$, $\gamma = 1.4$.

Figure 3 shows the predicted mean velocity and temperature profiles, as well as the corresponding relative errors, across the channel at fixed Ma = 1.5 and various Re. The reference DNS data are from Yao and Hussain [33]. Hereafter, the cases will be denoted as "$XX - YY$", where Re = XX and Ma = YY. For the mean velocity profiles, it is seen from Fig. 3(a) that the proposed modified iterative method works pretty well, and the predicted mean velocity profiles match very well with the corresponding DNS data, except for the low-Reynolds-number case at Re = 3000, where some deviations can be observed in the range $15 < y^+ < 100$. Figure 3(b) further indicates that the relative errors of the predictions of mean velocity are pretty small for most of the cases across the channel, less than 4%, except that the case "3000-1.5" has a relative error up to 6.5%. For the mean temperature profiles as shown in Figs. 3(c) and 3(d), on the other hand, the predicted ones are in excellent agreement with the corresponding DNS data, and the relative errors are all within 2%. Nevertheless, due to the small error of the semiempirical scaling for the central mean temperature, the error of the predicted mean temperature will be inevitable near the channel center. From Fig. 3, we can conclude that at fixed Ma, the performance of the proposed modified iterative method at very low Re is not as good as that at higher Re, especially for the prediction of mean velocity profiles.

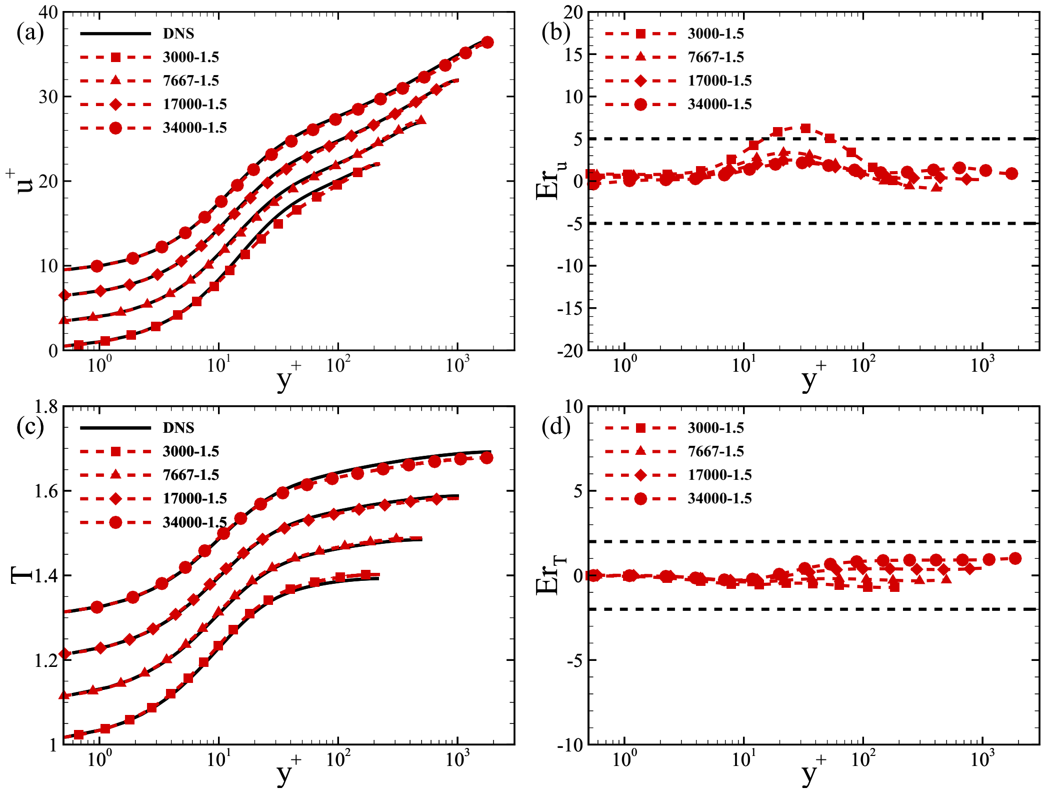

FIG. 3. Predicted mean profiles and the relative errors for channel flows at various Re and fixed Ma = 1.5: (a) velocity; (b) the relative error of velocity; (c) temperature, and (d) the relative error of temperature. The black lines represent the DNS results from Yao and Hussain [33]. The three profiles at higher Re in (a) and (c) are vertically shifted up step-by-step by 3 units and 0.1 units, respectively, for better visualizations.

Now we turn our attention to the low Reynolds number cases, and three DNS data at $\mathrm{Re} = 3000, \mathrm{Ma} = 0.8, 1.5$ from Yao and Hussain [33], and $\mathrm{Re} = 4880, \mathrm{Ma} = 3.0$ from Modesti and Pirozzoli [13] will be used as reference. Figure 4 shows the predicted mean velocity and temperature profiles, as well as the relative errors, across the channel for the three cases. Again, more obvious deviations exist for the velocity predictions at lower Reynolds numbers with various Ma. Nevertheless, the predictions of mean velocity at $\mathrm{Re} = 3000, \mathrm{Ma} = 0.8$ are better than that at $\mathrm{Re} = 3000, \mathrm{Ma} = 1.5$, and we may conclude that the max relative error of the velocity profiles increases with Ma. For the deviations in the predicted mean velocity at the low Reynolds cases, we believe that the reason is twofold. On the one hand, there are some errors for the TL transformation at low Reynolds numbers, which can be depicted in Fig. 5 where the TL transformed mean velocity profiles at $\mathrm{Re} = 3000, \mathrm{Ma} = 1.5$ and $\mathrm{Re} = 4880, \mathrm{Ma} = 3.0$ deviate from the incompressible DNS profiles at $\mathrm{Re}_\tau = 140$ for $y^+ > 10$ (see also Fig. 8(a) in Modesti and Pirozzoli [13]). On the other hand, the analytical expression proposed by She *et al.* [24] at $\mathrm{Re}_\tau = 140$ further deviates from the incompressible DNS results at the same friction Reynolds number, resulting in more obvious deviation between the TL transformed velocity and the analytical profile at the same friction Reynolds number, as also shown in Fig. 5. If more accurate incompressible mean velocity profile or/and more accurate velocity transformation method were used, better predictions on the mean velocity profiles can be arrived (see Fig. 7).

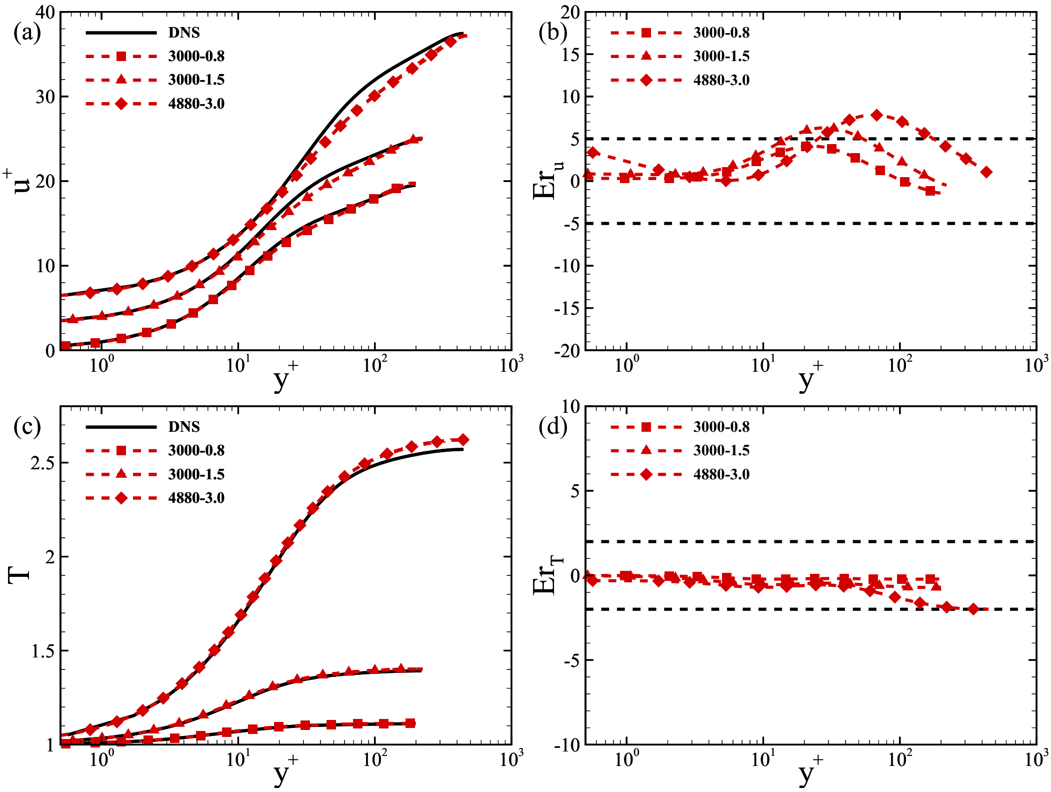

FIG. 4. Predicted mean profiles and the relative error for channel flows at different Ma. (a) velocity; (b) the relative error of velocity; (c) temperature, and (d) the relative error of temperature. The black lines represent the DNS results at Re = 4880, Ma = 3.0 from Modesti and Pirozzoli [13], Re = 3000, Ma = 1.5 and Re = 3000, Ma = 0.8 from Yao and Hussain [33]. The two profiles at higher Ma in (a) are vertically shifted up by 3 and 6 units for better visualizations.

Figure 6 shows the relative errors of the mean velocity, temperature, density and viscosity profiles for the 14 different DNS cases, where Re ranges from 3000 to 34 000, and Ma varies in {0.5, 0.8, 1.5, 3.0}. It is found that the relative errors of predicted mean temperature, density, and viscosity profiles are within 2%, while those for the predicted mean velocity profiles in most cases are less than 5%, except for those cases with lower transformed friction Reynolds number Re_τ^*. We have also validated the present modified iterative method at Ma = 3.0, 4.0 and $7500 \leqslant Re \leqslant 24000$, and the predicted central mean velocity and temperature are listed in Table II. The corresponding

TABLE II. $\overline{T}_c$ and $\overline{u}_c$ from the available DNS data with various Re and Ma, where the subscript *cal* represents the results from the proposed modified iteration method. The definitions of $\overline{T}_{c,\mathrm{error}}$ and $\overline{u}_{c,\mathrm{error}}$ are $(\overline{T}_c - \overline{T}_{c,\mathrm{cal}})/\overline{T}_c \times 100\%$ and $(\overline{u}_c - \overline{u}_{c,\mathrm{cal}})/\overline{u}_c \times 100\%$, respectively.

Type	Case	Re	Ma	$\overline{T}_c$	$\overline{u}_c$	$\overline{T}_{c,\mathrm{cal}}$	$\overline{u}_{c,\mathrm{cal}}$	$\overline{T}_{c,\mathrm{error}}(\%)$	$\overline{u}_{c,\mathrm{error}}(\%)$
Channel	Trettel and Larsson [7], Trettel [35]	7500	3.0	2.487	1.156	2.542	1.184	−2.21	−2.42
		15000	3.0	2.486	1.156	2.507	1.157	−0.84	−0.09
		24000	3.0	2.491	1.157	2.491	1.144	0.00	1.12
		10000	4.0	3.637	1.144	3.748	1.186	−3.05	−3.67

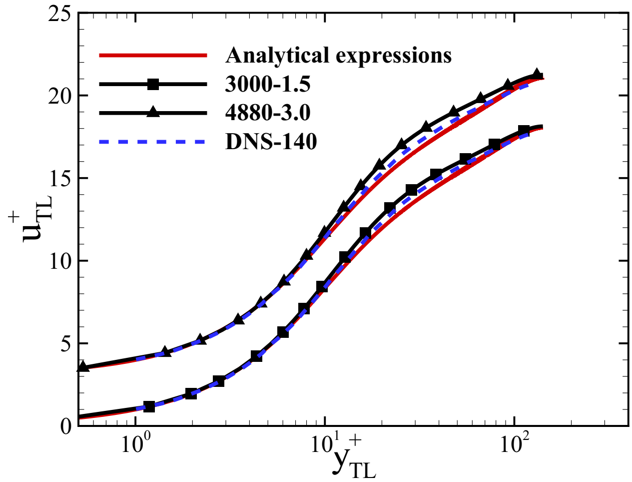

FIG. 5. Mean velocity profiles in channel flows. The black solid lines with symbols represent TL-transformed mean velocity profiles based on the DNS velocity profiles from Modesti and Pirozzoli [13], where Re_τ^* equals to 141, 142 for case "3000-1.5" and "4880-3.0", respectively. The red solid lines represent the corresponding incompressible profile obtained using the analytical expressions proposed by She *et al.* [24] at the same Re_τ^*. The blue dashed lines represent the incompressible DNS results at $Re_\tau = 140$, which were obtained from the figure in Modesti and Pirozzoli [13]. The profiles are shifted up for clarity.

DNS values from Trettel and Larsson [7] and Trettel [35] are also listed. It is again seen that the accuracy of the present method is satisfactory, and that the relative errors for the center mean velocity and temperature are all within 4%. It should be clarified that Eq. (11) only relates the predicted center mean velocity and temperature, and it has nothing to do with their relative errors. As listed in Table II, the relative errors of the predicted center mean velocity and temperature are uncorrelated. For some cases, see $Re = 15000, Ma = 3.0$, the relative error of predicted center mean velocity is smaller, whereas that of the predicted center mean temperature is smaller (see other three cases). From the above results, we conclude that the proposed modified iterative method can be used to predict the mean profiles in the compressible turbulent channel flows, especially for those cases with higher Re.

B. Parametric studies

In this subsection, we will discuss the influences of the incompressible velocity profile and the variable-property scaling models (transformations). In the following discussions, the cases $Re = 7667, Ma = 1.5$ and $Re = 4880, Ma = 3.0$ will be used to study the influence of the incompressible velocity profile, and the cases $Re = 7667, Ma = 1.5$ and $Re = 34000, Ma = 1.5$ will be adopted to investigate the influence of the variable-property scaling models at high Re. We want to emphasize that $Ny = 4001$ and $\epsilon = 10^{-6}$ used in this paper are accurate enough for these cases, and related validations are not shown here for brevity.

In the past decades, many different analytical expressions were proposed for the incompressible mean velocity profiles with different accuracy. Nevertheless, the DNS profiles should be the most accurate ones. In the last section, we have shown that the analytical expression by She *et al.* [24] may induce the error of the predictions on mean velocity at lower Reynolds numbers. Here, we will compare the results with different incompressible velocity profiles to further verify our conjecture. Figure 7 displays the predicted mean velocity and temperature profiles, as well as the relative errors, at $Re = 7667, Ma = 1.5$ and $Re = 4880, Ma = 3.0$ by using different incompressible velocity profiles, including the analytical results by She *et al.* [24] and the incompressible velocity profiles

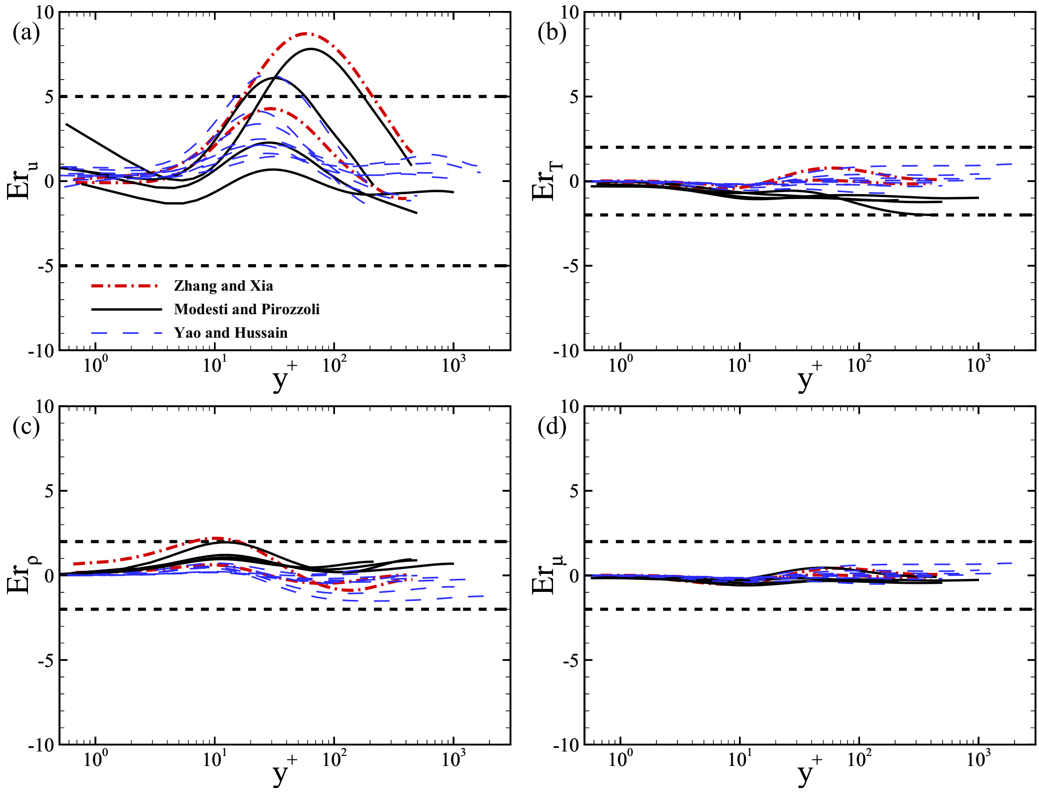

FIG. 6. Relative errors [see the definition in Eq. (12)] of (a) mean velocity; (b) mean temperature; (c) the mean density, and (d) mean viscosity. The 14 DNS results are from Zhang and Xia [34] (two cases), Modesti and Pirozzoli [13] (four cases), Yao and Hussain [33] (eight cases), and the mean viscosity in Yao and Hussain are calculated from mean temperature by using Eq. (6).

obtained by mapping the DNS data (denoted as ”DNS-Mapping”). Here, since the incompressible DNS mean velocity profile at $Re_\tau = 142$ and 333 are not available, the DNS mean velocity profiles at $Re_\tau = 150$ [36] and 395 [37] are used to do the mapping for the cases $Re = 4880$, $Ma = 3.0$ and $Re = 7667$, $Ma = 1.5$, respectively. For the higher Reynolds number case at $Re = 7667$, $Ma = 1.5$, it is seen that both incompressible velocity profiles can arrive at almost the same mean velocity and temperature profiles. The relative errors of velocity and temperature are all within 5% and 2%, respectively, across the channel. For the lower Reynolds number case at $Re = 4880$, $Ma = 3.0$, slight deviations can be observed for the mean temperature profiles in the center region, while more obvious differences exist for the mean velocity predictions in the range $15 < y^+ < 200$. The max relative error of velocity exceeds 5% with the “Analytical expression” and it is less than 5% with the “DNS-Mapping”. For temperature, the relative errors are both within 2%. Compared with the analytical expression by She *et al.* [24], the DNS profile provides a better prediction of the mean velocity, although the predicted mean velocity still deviates from the reference DNS result, which is due to the error of the TL transformation. Combing the results in Figs. 7 and 5, we arrive at the conclusion that at higher Reynolds number cases, the analytical expression by She *et al.* [24] is accurate enough to get good predictions on the mean velocity profiles, whereas at lower Reynolds number cases, the analytical expression by She *et al.* [24] works relative poorly, and the DNS profiles can be used to improve the predictions on mean profiles, especially on the mean velocity profiles.

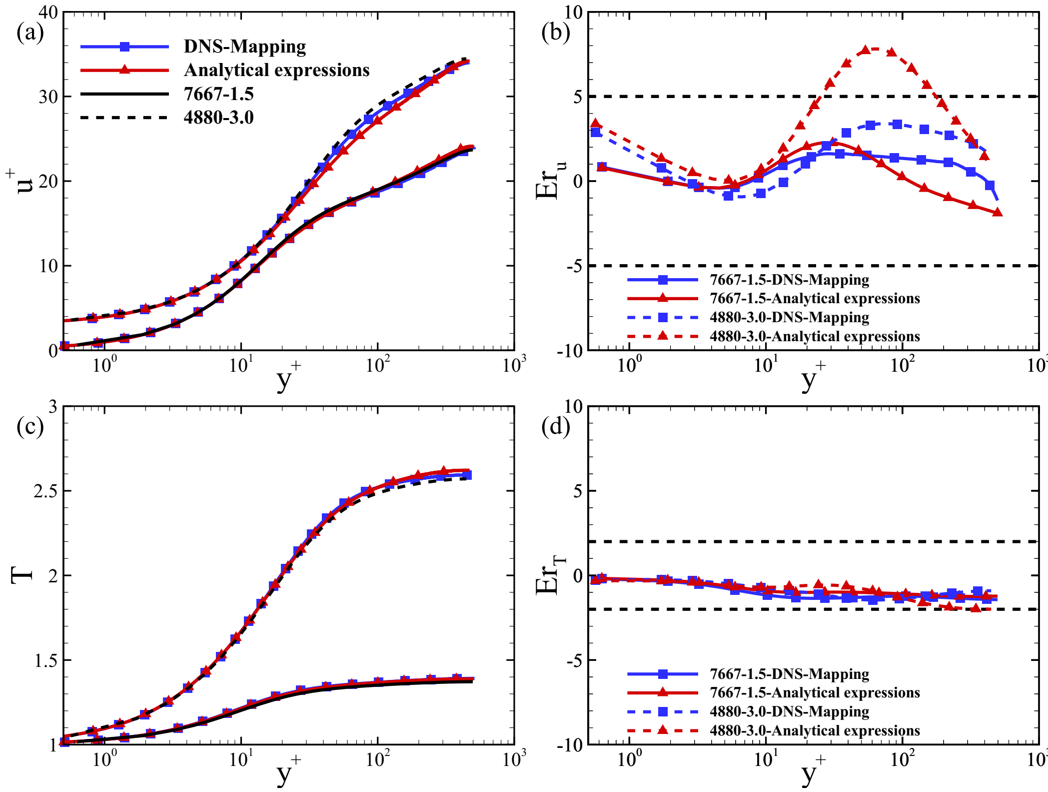

FIG. 7. (a), (b) Mean velocity profiles and the relative errors and (c), (d) mean temperature profiles and the relative errors of compressible channel flows with different incompressible mean velocity profiles. The black solid and dashed lines represent the DNS results from Modesti and Pirozzoli [13] at Re = 7667, Ma = 1.5 and Re = 4880, Ma = 3.0, respectively. The blue lines with square symbols denote the predicted mean profiles by using the incompressible DNS mean velocity profiles at $Re_\tau = 150$ (for Re = 4880, Ma = 3.0) [36] and 395 (for Re = 7667, Ma = 1.5) [37]. The red lines with triangle symbols denote the predicted results using the analytical expressions by She *et al.* [24]. The profiles in (a, c) are vertically shifted for better visualizations.

Figure 8 shows the predicted mean velocity and temperature profiles, as well as the relative errors, by using the recently proposed total-stress-based velocity transformation by Griffin *et al.* [11] and the TL transformation at Re = 7667, Ma = 1.5 and Re = 34000, Ma = 1.5, where the DNS results are available from Yao and Hussain [33]. It is seen that the differences of the predicted mean profiles between these two transformations are relatively small, as shown in Figs. 8(b) and 8(d) that the relative errors of velocity and temperature are all within 5% and 2%, respectively, across the channel, indicating the good agreement with the reference DNS results. This means that as long as the transformation works well, the proposed modified iterative method can obtain excellent results.

IV. VALIDATIONS IN PIPE FLOWS

In this section, the accuracy of the proposed modified iterative method is also validated in compressible pipe flows. Four DNS results with different Re and Ma from Modesti and Pirozzoli [16] are used as references. Figure 9 shows the predicted mean velocity and temperature profiles as well as the relative errors from the four cases, and Fig. 10 displays the relative errors of the predicted mean density and viscosity profiles. The results are quite similar to those in channel flows, verifying the usability of the proposed modified iterative method in compressible pipe flows. That

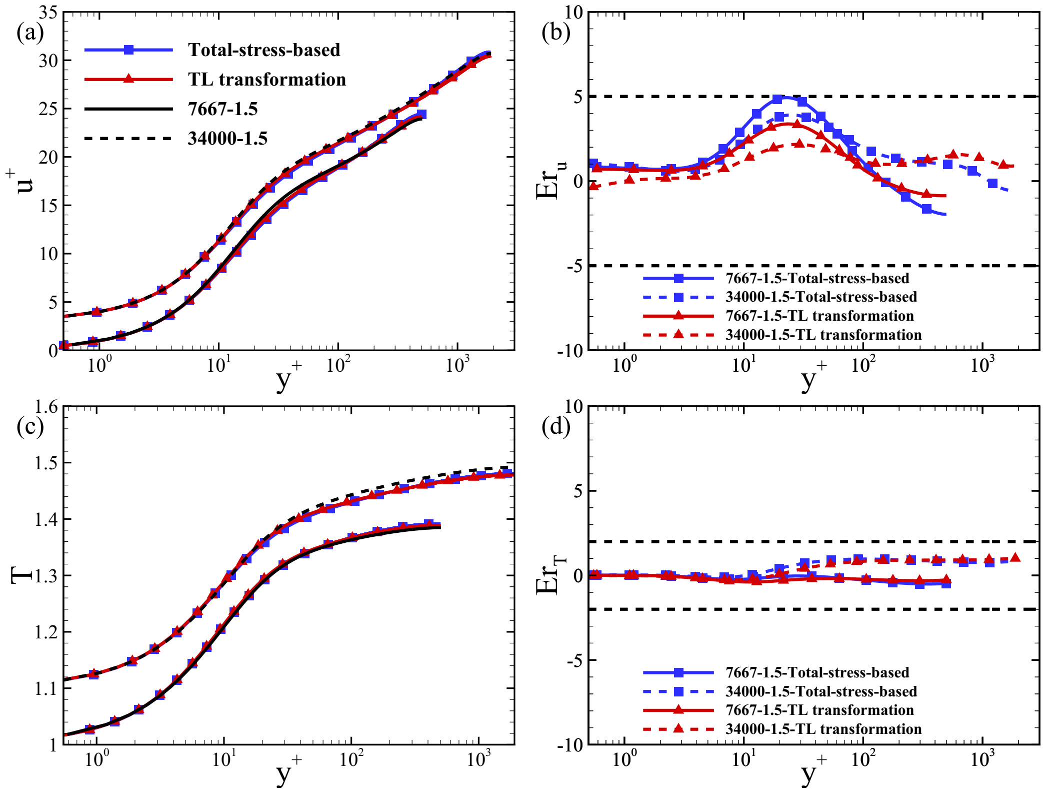

FIG. 8. (a), (b) Mean velocity profiles and the relative errors and (c), (d) mean temperature profiles and the relative errors of compressible channel flows with different variable-property models. The black solid and dashed lines represent the reference DNS results from Yao and Hussain [33] at Re = 7667, Ma = 1.5 and Re = 34000, Ma = 1.5, respectively. The blue lines with square symbols means the "Total-stress-based" transformation proposed by Griffin *et al.* [11], while those red lines with triangle symbols mean the "TL transformation" proposed by Trettel and Larsson [7]. The profiles in (a), (c) are vertically shifted for better visualizations.

is, the results are quite good for higher Reynolds number cases for the mean profiles of velocity, temperature, density and viscosity, where the relative errors for the predicted mean velocity are within 5%, and they are within 2% for the predicted mean temperature, density and viscosity. However, if the transformed friction Reynolds number was low, the predictions may be not as good as those at higher Reynolds number cases, especially for the mean velocity predictions, where the largest relative errors could be up to 9.8%. From the comparison among the incompressible DNS data at $\mathrm{Re}_\tau = 140$, the analytical profile from She *et al.* [24] at $\mathrm{Re}_\tau = 140$, and the TL transformed velocity profiles at Re = 3000, Ma = 1.5 and Re = 5150, Ma = 3.0 (not shown here), the same results can be made to explain the poorer performance of the proposed modified iterative method at lower Reynolds numbers, i.e., the errors of the analytical expression of She *et al.* [24] at low Re and the error of the TL transformation at low Re.

V. FURTHER REMARKS

From the above discussions, it is easy to see that the present modified iterative method can well predict the mean profiles, including the velocity, temperature, density, and viscosity, in compressible turbulent channel and pipe flows, especially at higher Reynolds numbers. For the compressible

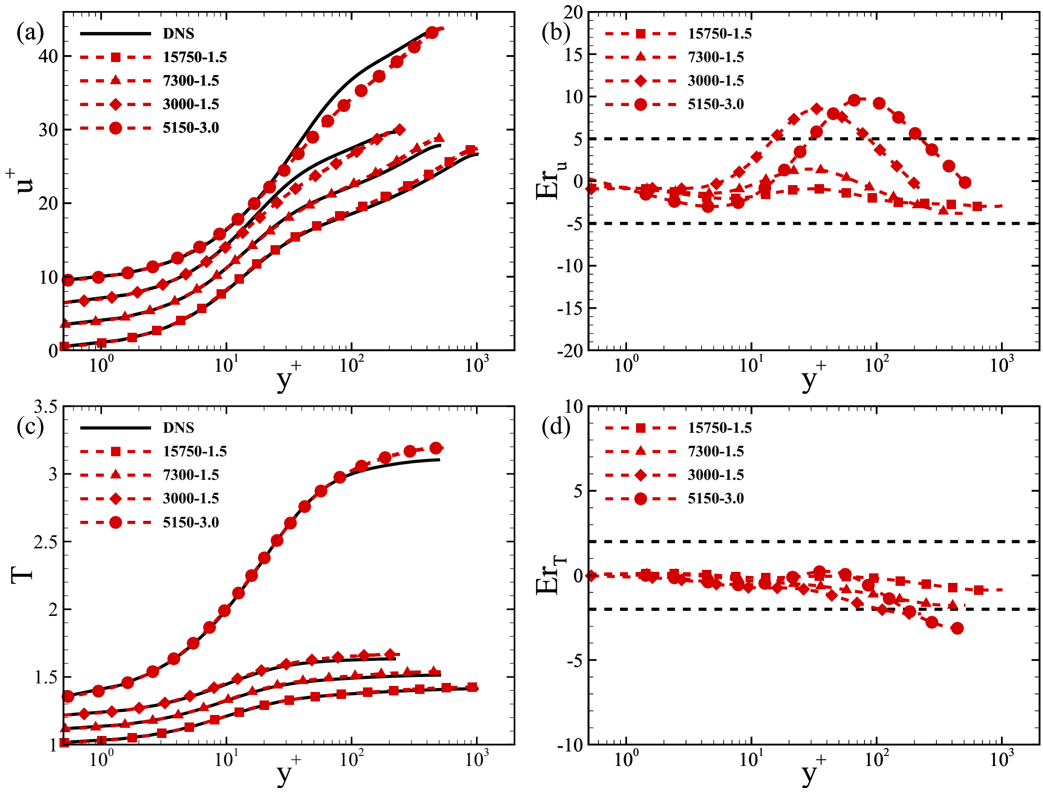

FIG. 9. Predicted mean profiles and the relative error for pipe flows. (a) velocity; (b) the relative error of velocity; (c) temperature, and (d) the relative error of temperature. The black lines represent the DNS results from Modesti and Pirozzoli [16]. The profiles in (a), (c) are shifted up for better visualization.

turbulent channel and pipe flows at low Reynolds numbers, although our method performs relatively poorer, DNS can provide more accurate profiles at an affordable cost. For the cases at extremely high Re, such as Re $O(10^6)$, which are more relevant to the practical applications, DNS is impossible in

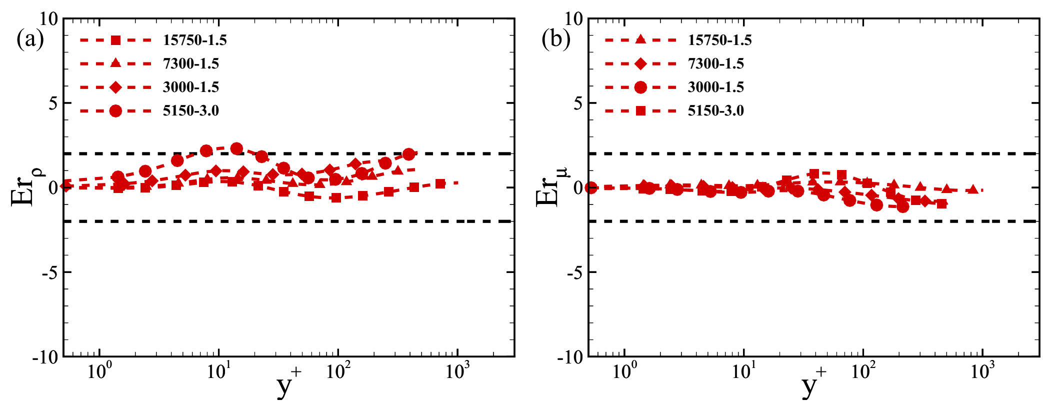

FIG. 10. Relative errors [see the definition in Eq. (12)] of (a) mean density and (b) mean viscosity for pipe flows. The reference DNS results are from Modesti and Pirozzoli [16].

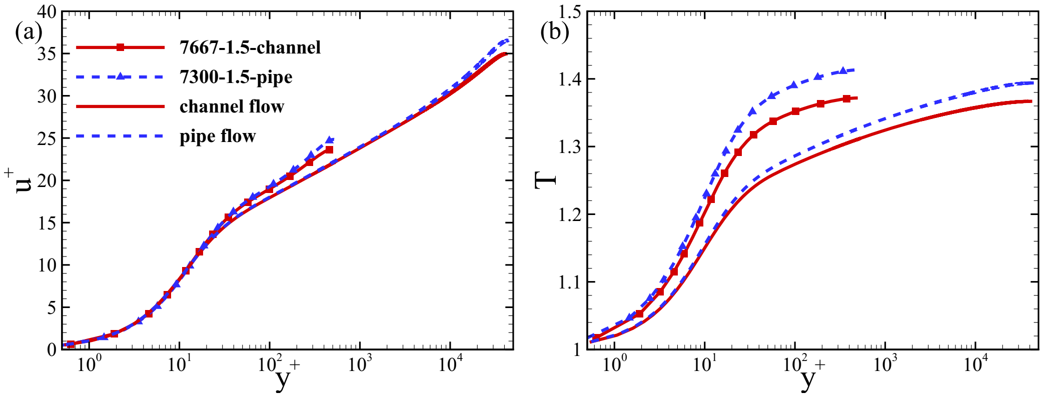

FIG. 11. (a) Mean velocity and (b) mean temperature profiles predicted by the proposed modified iterative method for channel and pipe flow at Re = 10^6, Ma = 1.5. The DNS results at Re = 7667, Ma = 1.5 for channel and Re = 7300, Ma = 1.5 for pipe flows from Modesti and Pirozzoli [13,16] are also shown for comparison.

the near future, and the present iterative method can provide the mean profiles. Figure 11 shows the predicted mean velocity and temperature profiles for channel and pipe flows at Re = 10^6, Ma = 1.5. The DNS results at two similar Re from Modesti and Pirozzoli [13,16], including Re = 7667, Ma = 1.5 for channel and Re = 7300, Ma = 1.5 for pipe, are also shown for comparison. There are two interesting results. One is that for the mean velocity profiles at the same Re and Ma, the main difference between the channel and pipe flows is mainly around the centerline. The other one is that at the same Re and Ma, the mean temperatures in pipe flows are generally higher than those in channel flows. However, it should be noted that the result related to the mean temperature conflicts with those results reported by Ghosh *et al.* [32] in their DNS data at the same low Re_τ. Our predicted mean profiles can provide useful data for compressible turbulent channel and pipe flows at higher Re. Furthermore, the predicted mean profiles can also help to guide the grid resolution and time step settings for DNS, as pointed out by Griffin *et al.* [21].

VI. CONCLUSION

In this paper, by combining the perfect gas assumption, the Sutherland's viscosity law, the variable-property scaling models for compressible wall turbulence, the incompressible velocity law of the wall, the relationship between the temperature and the velocity, and the Morkovin's assumption, a nonlinear closed system can be constituted to solve the mean profiles by iterative method in compressible turbulent channel and pipe flows. Based on the equivalence of $d\overline{T}/dy = 0$ and $d\overline{u}/dy = 0$ at the centerline, a modified iterative method, which is modular in the sense that it works with multiple incompressible velocity law of the wall, variable-property scaling models, equations-of-state, velocity-temperature relations, and viscosity-temperature relations, is proposed with the help of the semiempirical scaling for the central mean temperature in compressible turbulent channel and pipe flows with symmetric isothermal walls. The results show that the proposed modified iterative method, which is robust and has no concern with the initial guess of the undetermined parameters, can predict the mean velocity, temperature, density, and viscosity profiles very well at various Re and Ma. Comparing with available DNS databases, the relative errors of mean temperature profiles and velocity profiles in most cases are within 2% and 5%, respectively, in both channel and pipe flows. The present method can predict the mean profiles at higher Re and Ma, which can be further validated in the future when the DNS data or experimental measurements become available.

Since the mean profiles of velocity, temperature, density and viscosity can be predicted by using the present iterative method, the Reynolds stresses may also be predicted based on the collapse behaviors in semilocal coordinates [33]. In the future, we are going to work on this, and we hope that our method can provide the mean profiles, the Reynolds stresses profiles and the heat fluxes profiles at higher Re (and/or Ma) cases, which can be used to develop machine learning assisted Reynolds-averaged Navier-Stokes (RANS) models for compressible wall-bounded turbulence.

The code can be obtained in Ref. [38].

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC Grants No. 92152101, No. 91852205, and No. 92152301).

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