Dense, steady, fully developed fluid-particle flows over inclined, erodible beds

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We phrase boundary-value problems for a dense, steady, fully developed, gravitational flow of identical inelastic spheres and water over an inclined erodible bed in the presence of sidewalls. The mass density and the size of the spheres are that of the particles employed in laboratory experiments by Armanini *et al.*, [J. Fluid Mech. **532**, 269 (2005)]. We obtain numerical solutions for the profiles of the solid volume fraction, the strength of the particle velocity fluctuations, and the mean velocities of the particles and fluid. We compare these with those measured in the experiments.

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I. INTRODUCTION

Flows of mixtures of particles and fluid down inclines occur often in Nature and in industrial applications. Consequently, it is important to be able to describe and predict them. Because such flows take place under the influence of gravity, they typically involve particle volume fractions on the order of 0.50. In this sense, they are dense.

One possible description of such flows is a nonlocal extension [1–3] of the $\mu - I$ rheology [4] to describe particle interactions that are dominated by viscosity [5]. Another is an extension of the kinetic theory that takes into account, in a phenomenological way, the existence of velocity correlations [6–8]. In fact, the two approaches have much in common [9,10]. Here, we employ the latter to phrase boundary-value problems for a dense, steady, fully developed gravitational flow of identical inelastic spheres over an inclined erodible bed in the presence of sidewalls.

In earlier work, Jenkins and Hanes [11] introduced kinetic theory to describe nearly elastic particle interactions in a steady, turbulent, shearing flow of particles and fluid above a horizontal particle bed and employed an eddy viscosity to model the turbulence. The loss of energy in a collision between particles was taken to be insensitive to the presence of the fluid. In a similar study of collisional fluid-particle shearing flows, Berzi [12] improved upon this by incorporating both the viscous resistance of the fluid in the collisions, in a way suggested by Joseph *et al.* [13], and the influence of particle correlations on the transport coefficients of Garzo and Dufty [14] for very inelastic collisions, as suggested by Jenkins and Berzi [15], but he retained the eddy-viscosity model of turbulence.

Berzi [12] obtained approximate solutions to the resulting equations for three regions of decreasing concentration in steady, uniform flows above a horizontal particle bed, determined from these expressions the average particle flux and the thickness of the regions, and established that these compare favorably with measurements over a range of particles and flows. Berzi and Fraccarollo

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[16] extend the work of Berzi [12] to steady, uniform flows over inclined beds. Again, approximate analytic solutions provide expressions for volume flow rates and the thickness of the flow regions that are in good agreement with existing measurements. Berzi and Fraccarollo [17] reduced the number of regions and introduced turbulent suspension into the region of smaller particle concentration. This improved the agreement of the average quantities with experiments. Jenkins and Larcher [18] also employed the extension of the kinetic theory for dense, fluid-particle flows and used approximate integrations of the momentum and energy balances for the particles to obtain analytic expressions for the depth of the flow and the volume flow rate of the fully saturated mixture as functions of inclination and depth-averaged concentration.

This work differs from that of Berzi and Fraccarollo [16,17] in several important respects:

(i) As do Jenkins and Larcher [18], we focus on flows that involve a single dense, collisional region in which the fluid and particle phase have the same height above an inclined particle bed. Such flows are observed in the experiments of Armanini *et al.* [19]. Consideration of a single region in which the volume fraction exceeds 0.49 permits significant simplification of the transport coefficients of kinetic theory: there is a common dependence on the concentration that permits it to be eliminated from in favor of the granular temperature and the particle pressure. Then, for dense, dry flows, analytic solutions for the profiles of concentration, mean velocity, and velocity fluctuations may be obtained [20];

(ii) Instead of the eddy-viscosity model for the turbulence employed by Berzi and Fraccarollo [16,17], we take the viscosity of the fluid to be due to velocity fluctuations of the fluid induced by those of the particles [21]. This is consistent with the observation by Bagnold [22] that turbulent velocity fluctuations of the fluid are suppressed by the particles in a dense shearing flow;

(iii) We solve not for the average volume flux, as do Berzi and Fraccarollo [16,17] and Jenkins and Larcher [18], but obtain numerical solutions for the velocity profiles of particle concentration, the strength of the particle velocity fluctuations, and the mean velocities of both the particles and the fluid; and

(iv) We compare these profiles to those measured in repeated runs in a single experimental configuration for dense, fully saturated, fluid-particle flows [19].

II. MODEL

We consider a steady, dense, fully developed flow of identical, frictional, inelastic spheres of mass m, mass density ρ_p, diameter d, with coefficient of restitution, e, and sliding friction, μ in a fluid of mass density ρ_f and viscosity η. The flow is driven by gravity, over an erodible bed, inclined at an angle of ϕ, in the presence of sidewalls a distance W apart. We focus on flows that are fully saturated; that is, those in which the height of the fluid and particle phases are the same.

The x axis is in the direction of flow, the y axis is normal to the flow, directed opposite to gravity from the base, and g is the gravitational acceleration. Primes denote derivatives with respect to y. The particle concentration is c, the particle velocity is u, the fluid velocity is U, and the measure of the strength of the particle velocity fluctuations, T, is one-third their mean-squared value. We introduce the density ratio, $\sigma \equiv \rho_p/\rho_f$, buoyant gravity, $\hat{g} \equiv (1-1/\sigma)g$, and make lengths dimensionless by d, velocity by $(\hat{g}d)^{1/2}$, stresses by $\rho_p\hat{g}d$, and energy flux by $\rho_p(\hat{g}d)^{3/2}$.

We introduce an effective coefficient of restitution, ε, that incorporates the coefficient particle friction, μ [23,24], $e_\mu = e - (3/2)\mu e^{-3\mu}$, and the fluid viscosity, η, through the Stokes number $St = \sigma T^{1/2}R/9$, in which $R = \rho_f d(d\hat{g})^{1/2}/\eta$ is the Reynolds number [12,25]:

$$\varepsilon = \max\left[0, e_\mu - \frac{6.96(1+e_\mu)}{St}\right]. \tag{1}$$

The Reynolds number is based on the terminal velocity of a particle falling under gravity, while the Stokes number is based on the strength of the particle velocity fluctuations, as it is these that are responsible for the particle interactions.

Then, we employ extended kinetic theory for the particle phase and phrase a boundary-value problem for the variations of the average fields across the flow, solve these equations numerically with appropriate boundary conditions, and compare the numerical solutions to the profiles measured in physical experiments of Armanini *et al.* [19] and presented as a dataset by Larcher *et al.* [26]. Details of the development of the kinetic theory employed here, including the incorporation of friction into collisional restitution and velocity correlations into the rate of collisional dissipation, can be found in the Introduction of Berzi *et al.* [27].

A. Flow momentum

The component of the fluid-momentum balance normal to the flow governs the variation of the pure fluid pressure, $\tilde{P}$:

$$0 = -(1-c)\tilde{P}' - \frac{(1-c)}{\sigma - 1}\cos\phi, \tag{2}$$

while the component of the fluid-momentum balance along the flow governs the variation of the average fluid shear stress, S:

$$0 = S' + \frac{1-c}{\sigma - 1}\sin\phi - \frac{c}{\sigma}D(\bar{U} - u), \tag{3}$$

where D is the drag coefficient, taken to be the simple form for a single particle introduced by Dallavalle [28], with the concentration dependence of Richardson and Zaki [29] that has been well tested in situations in which the high concentration of the particles suppresses the fluid turbulence:

$$D = \left[\frac{3}{10}|\bar{U} - u| + \frac{18}{R}\right]\frac{1}{(1-c)^{3.1}}, \tag{4}$$

in which $\bar{U}$, the average of U across the flow, is employed because the fluid velocity is brought to zero at the walls, while the particle velocity is not.

The component of the particle-momentum balance normal to the flow governs the average collisional pressure, $p = c(\tilde{p} - \tilde{P})$, the product of the concentration and the excess of the total pure particle pressure, $\tilde{p}$, over the pure fluid pressure exerted by the fluid upon the particle:

$$0 = -p' - c\tilde{P}' - c\frac{\sigma}{\sigma - 1}\cos\phi, \tag{5}$$

or, with (2),

$$0 = -p' - c\cos\phi. \tag{6}$$

The component of the particle-momentum balance tangential to the flow governs the average particle collisional shear stress, s:

$$0 = s' + \frac{\sigma}{\sigma - 1}c\sin\phi + \frac{c}{\sigma}D(\bar{U} - u) - 2\mu_w\frac{p}{W}, \tag{7}$$

where the last term is the contribution of sidewall friction [27]. The factor that involves σ in Eq. (7) is due to the fact that in the steady, uniform, inclined flow, buoyancy influences only the particle-momentum balance, Eq. (6), across the flow, while the stresses have all been made dimensionless using the buoyant gravity.

The particle phase is assumed to be so dense that c is greater than 0.49 everywhere within the flow. Then, in the expressions for the particle stress, we retain only terms that result from collisional transfer of momentum and ignore those that result from transport. We include a term in the denominator that is associated with the influence of the elasticity of the spheres on the duration of a collision and, consequently, on their frequency. Then [27],

$$p = \frac{2\rho_s c(1+\varepsilon)GT}{1 + (12/5)G(T/k)^{1/2}}, \tag{8}$$

where k is the stiffness of the assumed linear elastic contact and

$$G = 5.69c\frac{c_c - 0.49}{c_c - c}, \tag{9}$$

in which the critical volume fraction, c_c, at which particles first touch along the axes of greatest compression and the collisional interactions become singular, is given in terms of the coefficient of sliding friction by [23]

$$c_c = 0.58 + (0.64 - 0.58)e^{-4.5\mu}. \tag{10}$$

For concentrations in the neighborhood of this singularity, those of our interest, transfer of momentum and energy between collisions, is negligible compared to that transferred in collisions.

With Eq. (8), Eq. (5) may be writen as an equation for the determination of c:

$$c' = -\left(c\frac{\cos\phi}{p} + \frac{T'}{T}\right)\left(\frac{2}{c} + \frac{1}{c_c - c}\right)^{-1}. \tag{11}$$

Also, upon including the influence of the contact elasticity on the frequency of collision,

$$s = \frac{4\hat{J}}{5\pi^{1/2}}\frac{2c(1+\varepsilon)GT^{1/2}}{1+(12/5)G(T/k)^{1/2}}u', \tag{12}$$

where [14]

$$\hat{J} = \frac{1}{2} + \frac{\pi}{4}\frac{(3\varepsilon - 1)(1+\varepsilon)}{24 - (1-\varepsilon)(11-\varepsilon)}. \tag{13}$$

The influence of elasticity on the frequency of collision is negligible for most of the flow; but, because of the singularity of G at c_c in (9), it becomes crucial as the bed is approached.

The dependence of the shear stress on the volume fraction may be eliminated in favor of a dependence on the pressure:

$$s = \frac{4\hat{J}}{5\pi^{1/2}}\frac{p}{T^{1/2}}u', \tag{14}$$

and this may be inverted to give the first-order differential equation to be solved for the average particle velocity:

$$u' = \frac{5\pi^{1/2}}{4\hat{J}}T^{1/2}\frac{s}{p}. \tag{15}$$

In a pure fluid, velocity fluctuations associated with fluid turbulence contribute to an eddy viscosity. However, Bagnold [22] and others [21] have observed that the presence of dense suspensions of particles eliminate the turbulent velocity fluctuations. The fluid velocity fluctuations that remain are induced by the velocity fluctuations of the sheared particles. Consequently, we assume that the fluid viscosity results from fluctuations that are correlated with the particle fluctuations, and incorporate the added mass of the fluid as its resistance to the induced fluctuations [21]:

$$S = \frac{1}{\sigma}\frac{1+2c}{2(1-c)}\eta U'. \tag{16}$$

Because the fluid does not slip at the walls, we give up the approximation that the fluid velocity is uniform across the flow and approximate it using the trapezoidal rule, in which case, the average fluid velocity across the flow is, roughly, half the uniform velocity. Then,

$$S = 2\frac{1}{\sigma}\frac{1+2c}{2(1-c)}\eta\bar{U}', \tag{17}$$

and

$$\bar{U}' = \frac{\sigma}{\eta}\frac{1-c}{1+2c}S. \tag{18}$$

B. Particle fluctuation energy

The balance of fluctuation energy in the flow relates the gradient of the energy flux, q, the working of the shear stress, and the rate of collisional dissipation, γ:

$$-q' + su' - \gamma = 0. \tag{19}$$

Upon retaining only the collisional transfers [14],

$$q = -\frac{2\hat{M}}{\pi^{1/2}}\frac{2c(1+\varepsilon)GT^{1/2}}{1+(6/5)(1+\varepsilon)G(T/k)^{1/2}}dT', \tag{20}$$

where

$$\hat{M} = \frac{1}{2} + \frac{9\pi}{8}\frac{(2\varepsilon-1)(1+\varepsilon)}{16-7(1-\varepsilon)}, \tag{21}$$

or

$$q = -\frac{2\hat{M}}{\pi^{1/2}}\frac{p}{T^{1/2}}T', \tag{22}$$

which, upon inversion, is

$$T' = -\frac{\pi^{1/2}}{2\hat{M}}\frac{T^{1/2}}{p}q, \tag{23}$$

and [14]

$$\gamma = \frac{6}{\pi^{1/2}}(1-\varepsilon^2)\frac{1}{L}\frac{2cGT^{3/2}}{1+(6/5)(1+\varepsilon)G(T/k)^{1/2}}, \tag{24}$$

or

$$\gamma = \frac{6}{\pi^{1/2}}(1-\varepsilon)\frac{pT^{1/2}}{L}, \tag{25}$$

where, because in such dense flows, the collisions are seen to be correlated [30] and the rate of collisional dissipation is first influenced; L is the correlation length

$$\frac{L}{d} = f\frac{du'}{T^{1/2}} = f\frac{5\pi^{1/2}}{4\hat{J}}\frac{s}{p}, \tag{26}$$

in which [31]

$$f = \left[\frac{2\hat{J}}{15(1-\varepsilon)}\right]^{1/2}\left[1+\frac{26}{15}(1-\varepsilon)\left(\frac{c-0.49}{0.64-c}\right)\right]^{3/2}. \tag{27}$$

The correlation length is determined in the phenomenological balance, Eq. (36), between the orienting influence of the flow and the randomizing influence of collisions. With Eqs. (18), (22), and (24), the energy Eq. (19) becomes

$$q' = \frac{5\pi^{1/2}}{4\hat{J}}\frac{s^2}{p}T^{1/2} - \frac{6}{\pi^{1/2}}(1-\varepsilon)\frac{pT^{1/2}}{L}. \tag{28}$$

We note that when Eq. (22) is employed in this and the result written in terms of the fluctuation velocity, $w = T^{1/2}$, it becomes

$$w'' + \frac{p'}{p}w' - \frac{\pi^{1/2}}{4\hat{M}}\left[\frac{6}{\pi^{1/2}}\frac{1-\varepsilon}{L} - \frac{5\pi^{1/2}}{4\hat{J}}\left(\frac{s}{p}\right)^2\right]w = 0, \tag{29}$$

a linear, second-order differential equation with a variable coefficient, introduced by the variation of *s/p* and L with c.

C. Boundary conditions

1. Bed

At volume fractions larger than the critical, there are more persistent elastic interactions between spheres that occur in an ever-changing network of contacts. This provides rate-independent components of the stresses that are associated with force transmission through the contact network. However, collisions do occur, some associated with the release of elastic energy as contacts are broken, and are responsible for the rate-dependent components of the stresses.

The energy of the velocity fluctuations supplied at the surface of the erodible bed is conducted away from the surface, dissipated in collisions, and eventually disappears. Because the shear rate is negligible in the bed, we ignore the production of fluctuation energy associated with it, and assume that the divergence of the flux, q, of fluctuation energy is the negative of its rate of dissipation, γ :

$$-q' - \gamma = 0, \tag{30}$$

where

$$q = -c\frac{5(1+\varepsilon)\hat{M}}{3\pi^{1/2}}k^{1/2}T' \tag{31}$$

and

$$\gamma = c\frac{5}{\pi^{1/2}}\frac{(1-\varepsilon^2)}{L_c}k^{1/2}T, \tag{32}$$

in which L_c is the correlation length evaluated at c_c. When we ignore the variation of c and ε within the bed, Eq. (30) is

$$T'' - \frac{3}{\hat{M}}\frac{(1-\varepsilon)}{L_c}T = 0. \tag{33}$$

So, in the bed, where y is negative, $w = w_0 e^{\lambda y}$, in which $w_0 = w(0)$ and $\lambda^2 = 3(1-\varepsilon)/(4\hat{M}L_c)$. Then, at the surface of the bed,

$$w'|_0 = \lambda w_0. \tag{34}$$

We use this as a boundary condition for the flow above the bed. Also, at the bed, $c = c_c$ and, as approximations, we take both the particle and fluid velocities to be zero.

2. Free surface

The top layer of particles is assumed to be submerged and being agitated by collisions from the particles below it. The buoyant weight of this layer, its dry weight reduced by the amount of water it displaces, is taken to be supported by the pressure of the momentum transferred in these collisions. That is,

$$p = c\cos\phi. \tag{35}$$

Because the agitated particles in the top layer do not fly high enough to obtain tangential momentum or energy from gravity, there is no source of particle shear stress or fluctuation energy at the free surface [32].

D. Boundary-value problem

The system of differential equations are Eqs. (11), (29), (15), (7), (18), and (3):

$$c' = -\left(c\frac{\cos\phi}{p} + \frac{2w'}{w}\right)\left(\frac{2}{c} + \frac{1}{c_c - c}\right)^{-1}, \tag{36}$$

where $p = 2c(1+\varepsilon)Gw^2/[1+(12/5)Gw/k^{1/2}]$, with $G = 5.69c(c_c{-}0.49)/(c_c{-}c)$;

$$w'' - \frac{\cos\phi}{p}w' + \frac{\pi^{1/2}}{4\hat{M}}\left[\frac{5\pi^{1/2}}{4\hat{J}}\left(\frac{s}{p}\right)^2 - \frac{6}{\pi^{1/2}}\frac{1-\varepsilon}{L}\right]w = 0, \tag{37}$$

where $L = [(2/15)\hat{J}/(1{-}\varepsilon)]^{1/2}[1+(26/15)(1{-}\varepsilon)(c{-}0.49)/(0.64{-}c)]^{3/2}(5\pi^{1/2}/4\hat{J})s/p$;

$$u' = \frac{5\pi^{1/2}}{4\hat{J}}\frac{w}{p}s \tag{38}$$

$$s' = -\frac{\sigma}{\sigma - 1}c\sin\phi - \frac{c}{\sigma}D(\bar{U} - u) - 2\mu_w\frac{p}{W}, \tag{39}$$

where $D = [(3/10)|\bar{U} - u| + 18/R]/(1{-}c)^{3.1}$;

$$\bar{U}' = \frac{\sigma}{2}\frac{2(1-c)}{1+2c}\frac{5\pi^{1/2}}{4\hat{J}}\frac{w}{p}S \tag{40}$$

and

$$S' = -\frac{1-c}{\sigma - 1}\sin\phi + \frac{c}{\sigma}D(\bar{U} - u). \tag{41}$$

The boundary conditions at the bed, $y = 0$, are

$$c = c_c, \quad u = 0, \quad w' = \left(\frac{3}{2\hat{M}}\frac{1-\varepsilon}{L}\right)^{1/2} w, \quad \text{and} \quad U = 0; \tag{42}$$

while those at the free surface, $y = h$, are

$$p = c\cos\phi, \quad s = 0, \quad q \propto w' = 0, \quad \text{and} \quad S = 0. \tag{43}$$

Equations (36) to (41) are a system of seven first-order differential equations for c, q,w, u, s, U, and S with the eight boundary conditions Eqs. (42) and (43) at the bed and the free surface. Because there is one more boundary condition than differential equations the thickness of the flow may be determined as a parameter. The analytical solution of dense, inclined dry flow [20] indicates that the flow thickness is the eigenvalue of the two-point boundary-value problem.

III. COMPARISONS AND CONCLUSIONS

We compare predictions of the model with the results of experiments conducted by Armanini *et al.* [19] on steady, fully developed flows of a particle-fluid mixture over an inclined particle

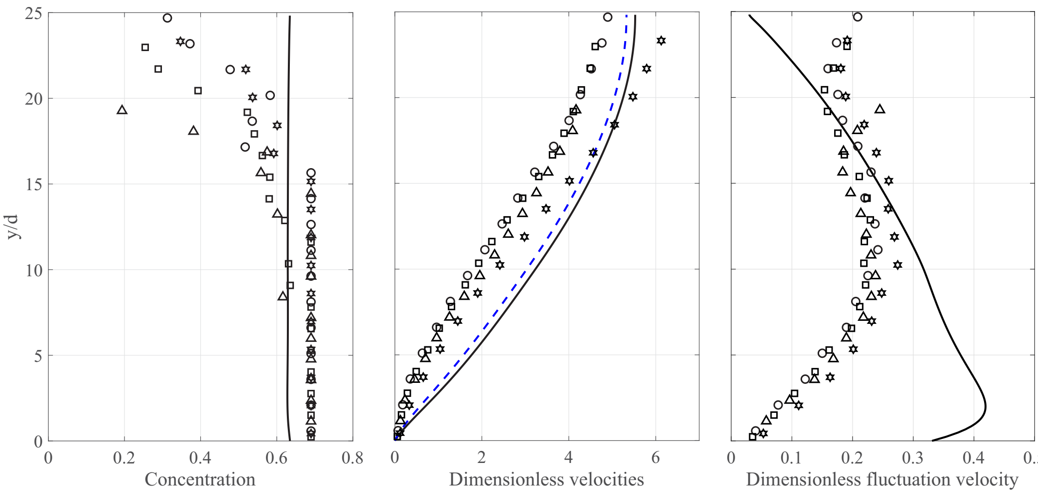

FIG. 1. Dimensionless flow height vs (a) concentration, c; (b) dimensionless mean fluid velocity, U (dashed), and mean particle velocity, u (solid); and (c) dimensionless fluctuation velocity, w, for $d = 0.37$ cm, $\rho_p/\rho_f = 1.54$, $\phi = 0.147$, $W = 54d$, and choices of the parameter values of $e = 0.45$, $\mu = 0.005$, and $\mu_w = 0.045$. The symbols are the measurements of Armanini *et al.* [19] and refer to their runs 005 (✡), 088 (△), 091 (□), and 090 (○), respectively.

bed in a chute with vertical sidewalls separated by a distance of 20 cm. The particles were plastic cylinders with a specific gravity of 1.54 and an effective radius of 0.37 cm and the fluid was water. The width, W, of the chute for these particles was $54d$.

As mentioned earlier, we focus on those flows in which the height of the fluid and particle phases above the bed were the same. There were four such flows reported. In these, the angle of inclination ranged from 8.1° to 8.5°, or 0.141 to 0.148 radians, the depth-averaged flow velocity, recalculated by us, ranged from 40.6 to 60.6 cm/s or, in dimensionless terms, from 2.13 to 3.18, and the thickness of the flow ranged from 7.0 to 9.2 cm, or $18.9d$ to $24.9d$. These values rather strongly constrain the angle of inclination, the velocity fields, and the flow thickness of the predictions. In particular, run 005 is characterized by an angle of inclination of 8.43° and a dimensionless depth-averaged flow velocity of 3.18; run 088 by an angle of inclination of 8.07° and a dimensionless depth-averaged flow velocity of 2.13; run 090 by an angle of inclination of 8.47° and a dimensionless depth-averaged flow velocity of 2.55; run 091 by an angle of inclination of 8.43° and a dimensionless depth-averaged flow velocity of 2.41.

We employ the MATLAB solver *bvp4c* to obtain solutions to the two-point boundary-value problem where the thickness of the flow is determined as a parameter. At a given angle of inclination, we make appropriate choices of the parameters e, μ, and μ_w to obtain a flow depth and an average particle flow velocity within the ranges of the experiment. All of the parameters have an influence on the depth of the flow and the velocities. The sensitivity of the solution to μ enters because of its influence on c_c, and, as indicated in Eq. (7), the angle of inclination and the wall friction have competing influences on the solution.

In Fig. 1, we show profiles predicted for c, u, U, and w for the values of e, μ, and μ_w indicated and the values measured for fully saturated flows in the experiments. The angle of inclination and the thickness of the flow are within the desired range, as is the dimensionless depth-averaged particle velocity of 2.86 calculated from the solution. The normalized concentration is relatively constant through the depth of the flow. The particle velocity exceeds the fluid velocity everywhere and the fluctuation velocity is not monotonic. The agreement between the predictions and the measurements of flow height, concentration, and velocity is relatively good, but the curvature of the velocity profiles is different.

The agreement between the predicted and measured values of fluctuation velocity is less good. However, the experiments by Armanini *et al.* [19] were performed using Polyvinyl Chloride (PVC) cylinders, which show a preferential alignment close to the bed (see their Fig. 1), where the critical volume fraction is approached and the radial distribution function becomes singular. In that situation, the experimental concentration can exceed the typical values measured for spheres, as shown in the first panel of our Fig. 1, and the velocity fluctuations can be damped, as shown in the third panel of our Fig. 1. When the volume fraction moves away from the critical value (e.g., for y/d greater than about 10 in Fig. 1), discrete element method (DEM) simulations performed by Berzi *et al.* (2016) [33] confirm the experimental evidence that there is no preferential alignment of the particles, and the cylinders are free to rotate in every direction, behaving like spheres. Larger experimental measurements of particle velocity fluctuation in the upper flow layers can be attributed to the stick and slip of PVC cylinders observed at the channel wall [19].

The extended kinetic theory employed here has the capability to predict profiles of concentration, and particle and fluid velocity, and particle velocity fluctuations measured in experiments on dense particle-fluid flows as solutions to a system of ordinary differential equations of a two-point boundary-value problem, with the height of the flow delivered as part of the solution. A question that remains is whether the values of e, μ, and μ_{w} that were employed in generating the solution are those appropriate for inelastic, frictional particles interacting with each other and with flat, frictional sidewalls while under water. This might be investigated using lattice Boltzmann method–discrete element method simulations of particle-particle and particle-wall interactions (e.g., Ref. [34]).

Finally, we should point out that that there are alternative formulations for fluid-particle flows based on variations of the μ–I rheology that have similarly been tested against the results of physical experiments and, in some instances, numerical simulations (e.g., Refs. [34–36]). Given the relationship between this rheology and the extended kinetic theory, the two approaches are not completely independent, but more work must be done to reveal the similarities and the differences.

[1] K. Kamrin and G. Koval, Nonlocal Constitutive Relation for Steady Granular Flow, Phys. Rev. Lett. **108**, 178301 (2012).

[2] D. L. Henann and K. Kamrin, A predictive, size-dependent continuum model for dense granular flows, Proc. Natl. Acad. Sci. USA **110**, 6730 (2013).

[3] K. Kamrin and D. L. Henann, Nonlocal modeling of granular flows down inclines, Soft Matter **11**, 179 (2015).

[4] G. D. R. MiDi, On dense granular flows, Eur. Phys. J. E **14**, 341 (2004).

[5] E. Guazzelli and O. Pouliquen, Rheology of dense granular suspensions, J. Fluid Mech. **852**, 1 (2018).

[6] J. T. Jenkins, Dense inclined flows of inelastic spheres, Gran. Matt. **10**, 47 (2007).

[7] J. T. Jenkins and D. Berzi, Dense inclined flows of inelastic spheres: Tests of an extension of kinetic theory, Gran. Matt. **12**, 151 (2010).

[8] D. Gollin, D. Berzi, and E. T. Bowman, Extended kinetic theory applied to inclined granular flows: Role of boundaries, Gran. Matt. **19**, 56 (2017).

[9] Q. Zhang and K. Kamrin, Microscopic Description of the Granular Fluidity Field in Nonlocal Flow Modeling, Phys. Rev. Lett. **118**, 058001 (2017).

[10] D. Berzi and J. T. Jenkins, Steady shearing flows of deformable, inelastic spheres, Soft Matter **11**, 4799 (2015).

[11] J. T. Jenkins and D. M. Hanes, Collisional sheet flows of sediment driven by a turbulent fluid, J. Fluid Mech. **370**, 20 (1998).

[12] D. Berzi, Analytical solution of collisional sheet flows, J. Hydraul. Eng. **137**, 1200 (2011).

[13] G. G. Joseph, R. Zenit, M. L. Hunt, and A. M. Rosenwinkel, Particle-wall collisions in a viscous fluid, J. Fluid Mech. **433**, 329 (2001).
[14] V. Garzó and J. W. Dufty, Dense fluid transport for inelastic hard spheres, Phys. Rev. E **59**, 5895 (1999).
[15] J. T. Jenkins and D. Berzi, Kinetic theory applied to inclined flows, Gran. Matt. **14**, 79 (2012).
[16] D. Berzi and L. Fraccarollo, Inclined, collisional sediment transport, Phys. Fluids **25**, 106601 (2013).
[17] D. Berzi and L. Fraccarollo, Intense sediment transport; Collisional to turbulent suspension, Phys. Fluids **28**, 023302 (2016).
[18] M. Larcher and J. T. Jenkins, Saturated, collisional flows of spheres over an inclined, erodible bed between vertical sidewalls, Adv. Water Res. **72**, 15 (2014).
[19] A. Armanini, H. Capart, L. Fraccarollo, and M. Larcher, Rheological stratification in experimental free-surface flows of granular–liquid mixtures, J. Fluid Mech. **532**, 269 (2005).
[20] J. T. Jenkins and D. Berzi, Analytical solutions for dense, inclined, granular flow over a rigid, bumpy base, EPJ Web Conf. **249**, 03039 (2021).
[21] D. Berzi and L. Fraccarollo, Turbulence Locality and Granularlike Fluid Shear Viscosity in Collisional Suspensions, Phys. Rev. Lett. **115**, 194501 (2015).
[22] R. A. Bagnold, Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear, Proc. R. Soc. London **225**, 49 (1954).
[23] S. Chialvo, J. Sun, and S. Sundaresan, Bridging the rheology of granular flows in three regimes, Phys. Rev. E **85**, 021305 (2012).
[24] M. Larcher and J. T. Jenkins, Segregation and mixture profiles in dense, inclined flows of two types of spheres, Phys. Fluids **25**, 113301 (2013).
[25] G. Barnocky and R. H. Davis, Elasto-hydrodynamic collision and rebound of spheres: Experimental verification, Phys. Fluids **31**, 1324 (1988).
[26] M. Larcher, L. Fraccarollo, A. Armanini, and H. Capart, Set of measurement data from flume experiments on steady uniform debris flows, J. Hydraul. Res. **45**, 59 (2007).
[27] D. Berzi, J. T. Jenkins, and P. Richard, Extended kinetic theory for granular flow over and within an inclined erodible bed, J. Fluid Mech**. 885**, A27 (2020).
[28] J. Dallavalle, *Micromeritics* (Pitman, New York, 1943).
[29] J. F. Richardson and W. N. Zaki, Sedimentation and fluidization: Part I, Trans. Inst. Chem. Eng. **32**, 35 (1954).
[30] N. Mitarai and H. Makanishi, Velocity correlations in dense granular shear flows: Effects on energy dissipation and normal stress, Phys. Rev. E **75**, 031305 (2007).
[31] D. Berzi and D. Vescovi, Different singularities in the functions of extended kinetic theory at the origin of the yield stress in granular flows, Phys. Fluids **27**, 013302 (2015).
[32] J. T. Jenkins and D. H. Hanes, The balance of momentum and energy at an interface between colliding and freely flying grains in a rapid granular flow, Phys. Fluids **A5**, 781 (1993).
[33] D. Berzi, N. Thai-Quang, Y. Guo, and J. Curtis, Stresses and orientational order in shearing flows of granular liquid crystals, Phys. Rev. E **93**, 040901(R) (2016).
[34] Q. Zhang, E. Deal, J. T. Perron, J. G. Venditti, S. J. Benavides, M. Rushlow, and K. Kamrin, Fluid-driven transport of round sediment particles: From discrete simulations to continuum modeling, J. Geophys. Res: Earth Sur. **127**, e2021JF006504 (2022).
[35] J. Chauchat, A comprehensive two-phase flow model for unidirectional sheet-flows, J. Hydraul. Res. **56**, 15 (2018).
[36] R. Maurin, J. Chauchat, and P. Frey, Revisiting slope influence in turbulent bedload transport: Consequences for vertical flow structure and transport rate scaling, J. Fluid Mech. **839**, 135 (2018).