Letter

Minimum principle for the flow of inelastic non-Newtonian fluids in macroscopic heterogeneous porous media

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The minimization of dissipation is a general principle in physics. It stipulates that a nonequilibrium system converges toward a state minimizing the energy dissipation. In fluid mechanics, this principle is well known for Newtonian fluids governed by the Stokes equation. It can be formulated as follows: Among all admissible velocity fields, the solution of the Stokes equation is the one that minimizes the total viscous dissipation. In this Letter, we extend these approaches to non-Newtonian fluids in macroscopic heterogeneous porous media or fractures. The flow is then governed by a nonlinear Darcy equation that can vary in space. In this case, a minimization principle can still be written depending on the boundary conditions. Moreover, such a minimization principle can be derived either for the velocity or the pressure field.

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I. INTRODUCTION

The minimization of dissipation is a general principle in physics. It stipulates that a nonequilibrium system converges toward a state minimizing the energy dissipation. In fluid mechanics [1], this principle was stated by Helmholtz [2] and demonstrated by Korteweg [3] for Newtonian fluids governed by Stokes equation and with a velocity imposed on its boundary. It can be formulated as follows: Given a fluid volume where the velocity is prescribed at its boundary, among all admissible velocity fields, the solution of the Stokes equation is the one that minimizes the viscous dissipation. Since then, this principle has been generalized to different boundary conditions [4]. Another important result is the generalization of this principle to non-Newtonian fluids proposed by Bird [5], which is also the basis of the augmented Lagrangian numerical method [6–9]. In this paper, we aim to extend these approaches to non-Newtonian fluids in macroscopic heterogeneous porous media.

Non-Newtonian fluids are found in many applications related to porous or fractured media. An important industrial application is, for example, enhanced oil recovery (see Refs. [10–12]). Oil is usually recovered by displacing it with another fluid (e.g., water). The main problem lies in the fact that the displacing fluid is often less viscous, resulting in viscous fingering. The fingering tends to create preferential flow paths, leaving a large fraction of the oil. The idea is then to inject a non-Newtonian fluid in order to prevent fingering. Another interesting application is the description of blood in the capillary network, which should be regarded as a suspension and, thus, with a non-Newtonian viscosity: shear thinning or yield stress [13,14]. Non-Newtonian fluids (cements, polymers, etc.) are also commonly used for fracture sealing [15].

A very recurrent problem when dealing with porous media is that of up scaling. If the equations of motion are generally well known at the pore scale (typically $\sim 10^{-3}$ m), a particular interest is to understand the flow at much larger scales (~ 1–10^3 m). This is usually performed by deriving constitutive equations for average quantities at an intermediate scale. This is illustrated by the famous

Darcy's law for Newtonian fluids, which relates linearly the mean flow rate to the macroscopic gradient of pressure.

At the microscopic level, Newtonian fluids obey the Stokes equation (neglecting inertia and compressibility),

$$\vec{0} = -\vec{\nabla} p + \mu \, \mathbf{\Delta} \vec{v} \quad \text{and} \quad \vec{\nabla} \cdot \vec{v} = \vec{0}, \tag{1}$$

where $\vec{v}$ is the fluid velocity, p is the pressure, and μ is the viscosity. After averaging over a large number of pores, it results in the Darcy's law [16–18],

$$\vec{u} = -\frac{\kappa}{\mu} \vec{\nabla} P, \tag{2}$$

where $\vec{u}$ is the volume average of the microscopic velocity field, P is a macroscopic pressure field, and κ is the permeability of the porous medium which depends on its nature (rock, sand, clay, etc.).

At the geological scale the type of material may, however, spatially vary leading to a macroscopic heterogeneous permeability field. The understanding of the large-scale flow then requires the resolution of the heterogeneous Darcy's law,

$$\vec{u} = -\frac{\kappa(\vec{r})}{\mu} \vec{\nabla} P \quad \text{and} \quad \vec{\nabla} \cdot \vec{u} = 0. \tag{3}$$

The $\kappa(\vec{r})$ field can be determined experimentally using different methods(borehole and pumping tests). Many models have also been proposed in the literature, the most popular being parallel strata of different natures or the log-normal distribution (see, for example [19–22]). This equation is two dimensional is also equivalent to the "cubic law" commonly used to solved the flow in heterogeneous fractures [23,24].

It is also important to remember that this equation is also used to solve the flow in fractures with heterogeneous openings, generally referred to as the Reynolds equation [25–29].

All the above mentioned equations apply to Newtonian fluids. A question that naturally arises is: How should this approach be modified when considering non-Newtonian fluids? Whereas there is a very large variety of non-Newtonian fluids [30–32], there are classical approaches in the case where there is a relationship between the shear rate $\dot{\gamma}$ and the shear-stress $\tau(\dot{\gamma})$. The approach (see, for instance, Refs. [33–42]) consists in determining an effective shear rate $\dot{\gamma}_{pm}$ in order to relate it to an effective shear stress (or viscosity). By defining a typical length scale λ (pore size, grain diameter, $\sqrt{\kappa}$, etc.) and using the average flow velocity u, a typical shear rate $\dot{\gamma}_{eff} \propto u/\lambda$ can be defined. Using the mean pressure gradient, a typical shear stress $\tau_{eff} \propto \lambda \nabla P$ can be defined. The idea is then to use these quantities in the rheological function $\dot{\gamma} = f(\tau)$ to derive a generalization of Darcy's law in the form $u \propto f(\mathbf{\nabla} P)$ where the prefactors must be determined (experimentally, numerically, or theoretically). It is, therefore, expected that the flow/pressure curve will keep the overall shape of the rheological curve. Similar to the permeability, it is also expected that the prefactors should depend on the local structure of the medium. This function should, thus, vary spatially.

In the present Letter, we aim to demonstrate that in this case, the flow also obeys a minimum principle. We need to make two hypotheses. First, the local porous medium is assumed to be isotropic, which implies that the velocity field is collinear and opposite to the pressure gradient. The second hypothesis is that the rheological function is an increasing monotonic function and, thus, inversible.

In this case, it can be assumed that the nonlinear heterogeneous Darcy's law can be written in the form

$$\vec{u} = -f(\vec{r}; \|\vec{\nabla} P\|) \frac{\vec{\nabla} P}{\|\vec{\nabla} P\|} \quad \text{or} \quad \vec{\nabla} P = -g(\vec{r}; \|\vec{u}\|) \frac{\vec{u}}{\|u\|}, \tag{4a}$$

and

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{4b}$$

with $f(\vec{r}; y)$ and $g(\vec{r}; y)$ positive monotonically increasing functions of y and $\|\cdot\|$ is the norm operator. It is important to mention that both $f()$ and $g()$ are not necessarily continuous.

In the literature, few works address the principle of minimization in porous media. Matheron [19] (in French) proved this principle for the linear Darcy law and the heterogeneous permeability field (see also Ref. [43]). Regarding the nonlinear Darcy equation, a variational principle has been proposed by Knupp and Lage [44] for the pressure field, the solution of an anisotropic Darcy-Forchheimer equation in a homogeneous permeability field. The pressure field of the nonlinear Darcy solution, thus, corresponds to the zero derivative of a certain functional. The main limitations of this approach are that, on the one hand, it assumes that the function is differentiable, which is not always the case. And on the other hand, if it shows that the solution is a local extremum, it does not necessarily the uniqueness of the solution.

This paper represents an extension of this Letter. We will show that the pressure field but also the velocity field obey a minimization principle for any monotonic nonlinear Darcy law, including the presence of permeability heterogeneities. It is important to note that the present principle, following the approach of Ref. [6] for Stokes flow, does not involve the derivative of the functional. This has two consequences. First, the function does not have to be differentiable. The f or g function can, therefore, be discontinuous as in the case of yield stress fluids [45] or discontinuous shear thickening [46,47], for example. Second, the principle of the minimum is not only local, which allows to prove the uniqueness of the solution depending on the different boundary conditions.

II. MINIMUM PRINCIPLE FOR THE VELOCITY FIELD

A. Pressure imposed boundary condition

We consider here a parallelepipedic domain $\mathcal{V}$ where the pressure is homogeneously imposed on the two opposite sides P_{in} and P_{out}. For simplicity, we will assume a periodic boundary condition on the lateral sides. In this case, the minimum principle states that the field $\vec{u}$, solution of the nonlinear Darcy equation Eqs. (4a) with an imposed pressure difference ΔP, is also the minimum among all admissible velocity fields $\vec{v}$ of a functional $\Phi[\vec{v}; \Delta P]$,

$$\vec{u} = \arg\min_{\vec{v}\in\Omega} \Phi[\vec{v}; \Delta P], \tag{5}$$

with

$$\Phi[\vec{v}; \Delta P] = \int G(\vec{r}; \|\vec{v}\|) dr^3 - \Delta P\, Q[\vec{v}], \tag{6}$$

where the admissible velocities are fields satisfying the divergence free and the periodic lateral boundary condition. $G(\vec{r}; \|\vec{v}\|)$ is defined as

$$G(\vec{r}; \|\vec{v}\|) = \int_0^{\|\vec{v}\|} g(\vec{r}; y) dy. \tag{7}$$

The functional $Q[\vec{v}] = \int_{\text{outlet}} \vec{v} \cdot d\vec{S}$ is the total flow rate associated with the field $\vec{v}$ ($d\vec{S}$ is directed towards the exterior of the domain).

Demonstration. First, we define Ω the set of admissible velocity fields satisfying the divergence free conditions and the periodic lateral boundary condition. Multiplying Eq. (4a) by any $\vec{v} \in \Omega$, and integrating over the domain, yields to

$$\forall\, \vec{v} \in \Omega, \qquad \int_{\mathcal{V}} \vec{\nabla} P \cdot \vec{v}\, dr^3 = -\int_{\mathcal{V}} g(\vec{r}; \|\vec{u}\|) \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|} dr^3. \tag{8}$$

Using the divergence free of $\vec{v}$ and the divergence theorem, it follows:

$$\forall\,\vec{v}\in\Omega,\qquad -\int_{\mathcal{V}} g(\vec{r};\|\vec{u}\|)\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}dr^3 = (P_{\rm out}-P_{\rm in})\int_{\rm outlet}\vec{v}\cdot\vec{dS}$$
$$= -Q[\vec{v}]\Delta P, \tag{9}$$

where $Q[\vec{v}]=\int_{\rm outlet}\vec{v}\cdot\vec{dS}=-\int_{\rm inlet}\vec{v}\cdot\vec{dS}$ and $\Delta P=P_{\rm in}-P_{\rm out}$.

Now, it will be demonstrated that the field $\vec{u}$, solution of the Darcy's equation Eqs. (4a), is also the minimum among all $\vec{v}\in\Omega$ of $\Phi[\vec{v};\Delta P]$. It is equivalent to prove that

$$\forall\,\vec{v}\in\Omega,\qquad \Phi[\vec{v}+\vec{u};\Delta P]-\Phi[\vec{u};\Delta P]\geqslant 0. \tag{10}$$

Combining Eq. (6) and (9) leads to

$$\forall\,\vec{v}\in\Omega,\qquad \Phi[\vec{v}+\vec{u};\Delta P]-\Phi[\vec{u};\Delta P]$$
$$=\int_{\mathcal{V}}\{G(\vec{r};\|\vec{v}+\vec{u}\|)-G(\vec{r};\|\vec{u}\|)\}dr^3-\Delta PQ[\vec{v}] \tag{11a}$$
$$=\int_{\mathcal{V}}\left\{G(\vec{r};\|\vec{v}+\vec{u}\|)-G(\vec{r};\|\vec{u}\|)-\frac{g(\|\vec{u}\|)}{\|\vec{u}\|}\vec{u}\cdot\vec{v}\right\}dr^3 \tag{11b}$$

Since $g(\vec{r};y)$ is an increasing function of y, $G(\vec{r};y)$ is convex, and, thus,

$$G(\vec{r};\|\vec{v}+\vec{u}\|)-G(\vec{r};\|\vec{u}\|)\geqslant g(\vec{r};\|\vec{u}\|)(\|\vec{u}+\vec{v}\|-\|\vec{u}\|). \tag{12}$$

It follows the required property:

$$\forall\,\vec{v}\in\Omega,\qquad \Phi[\vec{v}+\vec{u};\Delta P]-\Phi[\vec{u};\Delta P]$$
$$\geqslant\int_{\mathcal{V}}\frac{g(\vec{r};\|\vec{u}\|)}{\|\vec{u}\|}\left[\|\vec{u}+\vec{v}\|\|\vec{u}\|-\|\vec{u}\|^2-\vec{u}\cdot\vec{v}\right]dr^3$$
$$\geqslant 0, \tag{13}$$

where the Cauchy-Schwartz inequality has been used: $(\vec{u}+\vec{v})\cdot\vec{u}\leqslant\|\vec{u}+\vec{v}\|\|u\|$.

We can make several remarks:

Remark 1. It is instructive to put the solution $\vec{u}$ in Eq. (9). Leading to

$$Q[\vec{u}]\Delta P=\int_{\mathcal{V}}g(\vec{r};\|\vec{u}\|)\|\vec{u}\|dr^3>0. \tag{14}$$

This expression represents an energy balance. Since pressure is a potential energy per unit volume, the term on the left is the difference between the input and the output fluxes of this energy. And the right-hand term is the total viscous energy dissipation rate in the domain (see Appendix B). It also shows the expected results that the mean flow rate is always opposed to the mean gradient of pressure.

Remark 2: Reversibility. $\Phi[\vec{r};\Delta P]$ has the following symmetry property:

$$\Phi[\vec{v};-\Delta P]=\Phi[-\vec{v};\Delta P]. \tag{15}$$

It follows that changing the sign of the pressure difference only changes the direction of the velocity field, not its amplitude distribution. Fluid elements will then follow the same stream lines in the opposite direction.

Remark 3: Reciprocal theorem. It is worth noting that, in Eq. (9), $\vec{v}$ can be any diverging free field. An interesting application of this equation, can be the use of a particular solution (e.g., the Newtonian solution) in order to obtain the flow rate-pressure drop relation as in Day and Stone [48] and Boyko and Stone [49].

Remark 4: Nonuniform imposed pressure. For convenience, it has been assumed that the pressure is imposed uniformly at the edges of the inlet and outlet as this is what is most natural from an experimental perspective. For more complex pressure distributions, it is then necessary to replace $-\Delta P\, Q[\vec{v}]$ by $\int_{\partial\mathcal{V}} P\ \vec{v}\cdot\vec{d}S$ in Eq. (6), where $\partial\mathcal{V}$ represents the boundary surface.

B. Examples

Although there is a very large variety of different rheological models, we can explicitly write the functional for the most common ones.

Newtonian (Darcy). In the case of a Newtonian fluid in heterogeneous porous media, the flow satisfies Darcy's law,

$$\nabla P = -\frac{\mu}{\kappa(\vec{r})}\vec{u}. \tag{16}$$

In this case, $g(\vec{r};\|\vec{u}\|) = \frac{\mu}{\kappa(\vec{r})}\|\vec{u}\|$ yields to

$$\Phi[\vec{v};\Delta P] = \int_{\mathcal{V}} \frac{\mu}{2\kappa(\vec{r})}\|\vec{v}\|^2 dr^3 - \Delta P\, Q[\vec{v}], \tag{17}$$

where we find the result proposed by Matheron [19] with the last additional term imposing the boundary pressure. This expression is interesting because it shows the expected result that, to minimize dissipation, it is more favorable to have a higher velocity where the permeability is high. However, it is important to note that the admissible field $\vec{v}$ must satisfy the divergence free condition. This constraint can cause low permeability regions to have high velocity (and vice versa).

Power-law rheology. Another very common rheology is the power law, where $\tau \propto \dot{\gamma}^n$, with n the flow index. In this case, the heterogeneous Darcy's law [33,50] can be written

$$-\vec{\nabla}P = c(\vec{r})\|\vec{u}\|^{n-1}\vec{u}. \tag{18}$$

This leads to

$$\vec{u} = \arg\min_{\vec{v}\in\Omega}\left\{\int_{\mathcal{V}} \frac{c(\vec{r})}{n+1}\|\vec{v}\|^{n+1} dr^3 - \Delta P Q[\vec{v}]\right\}. \tag{19}$$

From this relation, one can recover a scaling analysis for the solution. Indeed for any positive ϵ, multiplying by ϵ^{n+1} does not change the argument of the minimum. It gives then,

$$\vec{u}(\Delta P) = \arg\min_{\vec{v}\in\Omega}\left\{\int_{\mathcal{V}} \frac{c(\vec{r})}{n+1}\|\epsilon\vec{v}\|^{n+1} dr^3 - \epsilon^n \Delta P Q[\epsilon\vec{v}]\right\} = \frac{1}{\epsilon}\vec{u}(\epsilon^n\Delta P). \tag{20}$$

It follows that the field $\frac{\vec{u}(\vec{r})}{Q}$ is a constant field, independent of the applied pressure difference. Combining this with the symmetry discussed earlier, it follows:

$$Q(\Delta P) \propto \|\Delta P\|^{1/n-1}\Delta P. \tag{21}$$

Herschel-Bulkley. Yield stress fluids are often described by the Herschel-Bulkley rheology, $\tau = \tau_0 + K\dot{\gamma}^n$, where τ_0 is the yield stress. At Darcy's scale, the velocity field can be described by (see Refs. [22,51,52]),

$$-\nabla P = c(\vec{r})\|\vec{u}\|^{n-1}\vec{u} + g_c(\vec{r})\frac{\vec{u}}{\|\vec{u}\|}, \tag{22}$$

where $g_c(\vec{r})$ is the local critical pressure gradient below which there is no flow, $c(\vec{r})$ is a prefactor that depend on the consistency, the local geometry, and n is the flow index.

It then follows that $g(\vec{r};\|\vec{u}\|) = c(\vec{r})\|\vec{u}\|^n + g_c(\vec{r})$. Thus,

$$\Phi[\vec{v};\Delta P] = \int_{\mathcal{V}} \left[\frac{c(\vec{r})}{n+1}\|\vec{v}\|^{n+1} + g_c(\vec{r})\|\vec{v}\|\right] dr^3 - \Delta P Q[\vec{v}]. \tag{23}$$

It is important to note that this function is not differentiable where $\|\vec{v}\| = 0$.

Forchheimer. Forchheimer's law corresponds to the generalization of Newtonian Darcy's law including the influence of inertia. A relation of the form is generally proposed [53]

$$\|\vec{\nabla}P\| = A\|\vec{u}\|^3 + B\|\vec{u}\|^2 + C\|\vec{u}\|. \tag{24}$$

Although Forchheimer's law applies to Newtonian fluids, it is, thus, equivalent to a non-Newtonian fluid (shear thickening). In heterogeneous porous media the constants should depend on the local properties, thus, $g(\vec{r}; \|\vec{u}\|) = A(\vec{r})\|\vec{u}\|^3 + B(\vec{r})\|\vec{u}\|^2 + C(\vec{r})\|\vec{u}\|$. It follows:

$$\Phi[\vec{v}; \Delta P] = \int_{\mathcal{V}} \left\{A(\vec{r})\frac{1}{4}\|\vec{v}\|^4 + B(\vec{r})\frac{1}{3}\|\vec{v}\|^3 + \frac{1}{2}C(\vec{r})\|\vec{v}\|^2\right\} dr^3 - \Delta P\, Q[\vec{v}]. \tag{25}$$

C. Velocity imposed boundary condition

A similar result can be demonstrated in the case where the normal velocity is prescribed at the boundary. Defining Ω_V as the ensemble of velocity satisfying the conservation of mass and sharing the same normal flow rate on the boundary $\partial\mathcal{V}$, one has

$$\vec{u} = \arg\min_{\vec{v}\in\Omega_V} \Phi[\vec{v}] \quad \text{with } \Phi[\vec{v}] = \int_{\mathcal{V}} G(\vec{r}; \|\vec{v}\|) dr^3. \tag{26}$$

Demonstration. For any $\vec{v} \in \Omega_V$,

$$\Phi[\vec{v}] - \Phi[\vec{u}] \geqslant \int_{\mathcal{V}} g(\vec{r}; \vec{u})(\|\vec{v}\| - \|\vec{u}\|) dr^3 \tag{27a}$$

$$\geqslant \int_{\mathcal{V}} \frac{g(\vec{r}; \vec{u})}{\|\vec{u}\|} \vec{u} \cdot (\vec{v} - \vec{u})\, dr^3 \tag{27b}$$

$$\geqslant -\int_{\mathcal{V}} \vec{\nabla}P \cdot (\vec{v} - \vec{u}) dr^3 \tag{27c}$$

$$\geqslant \oint_{\partial\mathcal{V}} P\, (\vec{v} - \vec{u}) \cdot d\vec{S} \tag{27d}$$

$$= 0, \tag{27e}$$

because $\vec{u}$ and $\vec{v}$ are sharing the same normal velocity at the boundary.

III. MINIMUM PRINCIPLE FOR THE PRESSURE FIELD

Finally, it is also interesting that a similar results exists for the pressure field with a prescribed value at the boundary. Indeed, calling Θ_P the ensemble of field with a given distribution at the boundary, the pressure field P solution of Eq. (4a) minimizes the functional $\Psi[H]$: $P = \arg\min_{H\in\Theta_P} \Psi[H]$ with

$$\Psi[H] = \int_{\mathcal{V}} F(\vec{r}; \|\vec{\nabla}H\|) dr^3 \quad \text{and} \quad F(\vec{r}; y) = \int_0^y f(\vec{r}; y) dy. \tag{28}$$

Demonstration. The demonstration is very similar to the previous ones. Using the convexity of the function $F(\vec{r}; y)$, one has for any $H \in \Theta_P$,

$$\Psi[H] - \Psi[P] = \int_{\mathcal{V}} [F(\vec{r}; \|\vec{\nabla}H\|) - F(\vec{r}; \|\vec{\nabla}P\|)] dr^3, \tag{29a}$$

$$\geqslant \int_{\mathcal{V}} f(\vec{r}; \|\vec{\nabla}P\|)(\|\vec{\nabla}H\| - \|\vec{\nabla}P\|) dr^3, \tag{29b}$$

$$\geqslant \int_{\mathcal{V}} \frac{f(\vec{r}; \|\vec{\nabla}P\|)}{\|\vec{\nabla}P\|} (\|\vec{\nabla}H\|\|\vec{\nabla}P\| - \|\vec{\nabla}P\|^2) dr^3, \tag{29c}$$

$$\geqslant \int_{\mathcal{V}} \frac{f(\vec{r};\|\vec{\nabla}P\|)}{\|\vec{\nabla}P\|}\vec{\nabla}P\cdot(\vec{\nabla}H-\vec{\nabla}P)dr^3, \tag{29d}$$

$$\geqslant -\int_{\mathcal{V}} \vec{u}\cdot(\vec{\nabla}H-\vec{\nabla}P)dr^3, \tag{29e}$$

$$\geqslant -\oint_{\partial\mathcal{V}} (H-P)\vec{u}\cdot d\vec{S} = 0, \tag{29f}$$

because H and P have the same value at the boundary.

Remark. It is interesting to note that any minimum of a differentiable functional in the form of $\Psi[h] = \int_V D(\vec{r};\|\vec{\nabla}h\|)dr^3$, where $D(\vec{r};y)$ is a convex function of y, allows to define a nonlinear Darcy's law. Indeed, using the Euler-Lagrange formula, we have

$$\frac{\partial D}{\partial h} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial D}{\partial h_i} = 0, \tag{30}$$

where we use the notation $h_i \hat{=} \frac{\partial h}{\partial x_i}$.

Since $\frac{\partial D}{\partial h} = 0$, the vector field,

$$q_i = -\frac{\partial D}{\partial h_i} = -\frac{d(\vec{r};\|\vec{\nabla}h\|)}{\|\vec{\nabla}h\|}\frac{\partial h}{\partial x_i}, \qquad i = 1\cdots 3, \tag{31}$$

with $d(\vec{r};y) = \partial_y D(\vec{r};y)$ then satisfies the conservation of mass $\vec{\nabla}\cdot\vec{q} = 0$. This, thus, defines a system of equations in the form of Eq. (4a). We retrieve here the approach of Knupp and Lage [44] for the Forchheimer equation. The only main difference here is that the function $d(\vec{r};\|\vec{\nabla}h\|)$ may vary in space.

IV. CONCLUSION

In conclusion, we were able to establish a minimization principle for nonlinear heterogeneous Darcy flows. This principle can be applied either to the velocity field or to the pressure field. If the function to be minimized differs slightly according to the boundary conditions constraint, all are based on the integral of the flow-pressure relationship. This shows that the important quantity is not so much the instant energy dissipation rate given by $\vec{u}\cdot\vec{\nabla}P = g(\|\vec{u}\|)\|\vec{u}\|$ (see Appendix B) but rather the cumulative dissipation for the velocity to rise from zero to a given value $\int_0^{\|\vec{u}\|} g(y)dy$. For the Newtonian, the two functions are proportional, so the minimization principle represents also a minimization of viscous dissipation.

This principle can also be generalized where the flow is also driven by a body force as discussed in Appendix A.

With a little retrospect, it does not seem too surprising that a minimization principle exists at the Darcy scale. Indeed, if such a principle exists at a microscopic scale, it seems then quite natural that a similar one is applicable for locally averaged quantities. There is, however, a significant difference between the microscopic and the macroscopic aspects. At the macroscopic scale, the constitutive law and, thus, the energy function can be heterogeneous in space. For instance, if some regions are linear whereas others are nonlinear, this minimization principle is still applicable.

It is worth recalling the different assumptions performed in the present Letter. First of all, this approach is *a priori* limited to nonthixotropic and inelastic fluids because the local rheology has been assumed constant in time and not dependent on the history of the fluid element.

Second, the monotonicity of the flow-pressure curve, $g(\vec{r};y)$ [respectively $f(\vec{r};y)$, has been assumed, implying the convexity of the function $G(\vec{r};y)$ (respectively]. This assumption is indeed necessary to prove the uniqueness of the solution. For example, for a nonmonotonic $g(\,)$ function, imposing a pressure difference could lead to different velocity fields. However, if the function $\Phi[\,]$

is differentiable, a variational approach could still be used. Each solution of the nonlinear Darcy's law would then correspond to a local extremum of $\Phi[\vec{r}; \Delta P]$.

Another important assumption is the isotropy of the local nonlinear Darcy equation, leading to an alignment of the pressure gradient and the velocity. The first step to generalize to anisotropic media would be to determine a generic nonlinear anisotropic Darcy's law. Knupp and Lage [44] assumed a permeability tensor formulation for the Forchheimer equation. In this case, a variational formulation can be used. We note, however, that more generic and complex formulations have been proposed in the literature. For example, Auriault *et al.* [54] proposed a formulation involving three principal axes and four functions of the mean pressure gradient for power-law fluids. Here also, a generic Darcy remains to be formulated for any type of rheology and anisotropy to be able to generalize this Letter. In addition, one of the main difficulties for heterogeneous permeability fields is that the principal axes could potentially also vary in space.

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APPENDIX A: FLOW DRIVEN BY A BODY FORCE

Another possible condition to drive the flow is the presence of a body force $\vec{\mathcal{G}}$ (homogeneous or not). We assume also an imposed pressure difference ΔP between the outlet and the inlet as in Sec. II A. In this case Darcy's law can be written as

$$\vec{\nabla} P - \vec{\mathcal{G}}(\vec{r}) = -g(\vec{r}; \|\vec{u}\|)\frac{\vec{u}}{\|u\|}, \tag{A1}$$

and the minimum principle then reads

$$\vec{u} = \arg\min_{\vec{v}\in\Omega} \Phi_{\vec{\mathcal{G}}}[\vec{v}; \Delta P], \tag{A2}$$

with

$$\Phi_{\vec{\mathcal{G}}}[\vec{v}; \Delta P] = \int_{\mathcal{V}} G(\vec{r}; \|\vec{v}\|)dr^3 - \Delta P\, Q[\vec{v}] - \int_{\mathcal{V}} \vec{\mathcal{G}}(\vec{r}) \cdot \vec{v}\, dr^3. \tag{A3}$$

Indeed, from Eq. (A1), we have for any $\vec{v} \in \Omega$,

$$\forall \vec{v} \in \Omega, \qquad -\int_{\mathcal{V}} g(\vec{r}; \|\vec{u}\|)\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}dr^3 = \int_{\mathcal{V}} [\vec{\nabla} P - \vec{\mathcal{G}}(\vec{r})] \cdot \vec{v}\, dr^3 \tag{A4a}$$

$$= -Q[\vec{v}]\Delta P - \int_{\mathcal{V}} \vec{\mathcal{G}}(\vec{r}) \cdot \vec{v}\, dr^3. \tag{A4b}$$

It then follows:

$$\forall \vec{v} \in \Omega, \qquad \Phi_{\vec{\mathcal{G}}}[\vec{v}+\vec{u}; \Delta P] - \Phi_{\vec{\mathcal{G}}}[\vec{u}; \Delta P]$$

$$= \int_{\mathcal{V}} \{G(\vec{r}; \|\vec{u}+\vec{v}\|) - G(\vec{r}; \|\vec{v}\|) - \vec{\mathcal{G}}(\vec{r}) \cdot \vec{v}\}dr^3 - \Delta P Q[\vec{v}], \tag{A5a}$$

$$= \int_{\mathcal{V}} \left\{G(\vec{r}; \|\vec{u}+\vec{v}\|) - G(\vec{r}; \|\vec{v}\|) - g(\vec{r}; \|\vec{u}\|)\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\right\}dr^3, \tag{A5b}$$

$$\geqslant \int_{\mathcal{V}} \frac{g(\vec{r}; \|\vec{u}\|)}{\|\vec{u}\|}\{\|\vec{u}+\vec{v}\|\|\vec{u}\| - \|\vec{u}\|^2 - \vec{v}\cdot\vec{u}\}dr^3, \tag{A5c}$$

$$\geqslant 0. \tag{A5d}$$

APPENDIX B: ENERGY DISSIPATION RATE FOR NONLINEAR DARCY'S LAW

In this Appendix, we recall the relationship between the Darcy's law and the viscous energy dissipation rate at the microscopic level. We consider a parrallepipedic volume $V = L_1 L_2 L_3$ containing both solid and fluid regions. A uniform pressure is imposed on each side $p_i^{\text{in/out}}$ of each direction $i = 1, 2 \cdots 3$, and we assume the absence of any other stress at the boundary. A no-slip condition is assumed at the fluid/solid boundary.

In the fluid region, neglecting inertia, the flow satisfies the Cauchy equation at steady state,

$$\vec{\nabla} \cdot \mathbf{\Pi} - \vec{\nabla} p = \mathbf{0}, \tag{B1}$$

where p is the microscopic pressure and $\mathbf{\Pi}$ is the deviatoric stress tensor. The strain rate tensor is defined as

$$\mathbf{\Delta}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{B2}$$

with $\vec{v}$ as the microscopic velocity.

Applying the scalar product with $\vec{v}$ in Eq. (B1) and averaging over V gives

$$\frac{1}{V}\int_{V_F} \{(\vec{\nabla} \cdot \mathbf{\Pi}) \cdot \vec{v} - (\vec{\nabla} p) \cdot \vec{v}\} dr^3 = 0. \tag{B3}$$

Here, V_F stands for the volume of fluid inside V.

The two terms are analyzed separately. The first term reads

$$\int_{V_F} (\vec{\nabla} \cdot \mathbf{\Pi}) \cdot \vec{v}\, dr^3 = \int_{V_F} \sum_{ij} (\partial_i \Pi_{ij}) v_j dr^3 = \int_{V_F} \sum_{ij} \{\partial_i(\Pi_{ij} v_j) - \Pi_{ij} \partial_i v_j\} dr^3 \tag{B4}$$

$$= -\int_{V_F} \sum_{ij} \Pi_{ij} \Delta_{ij} dr^3 + \int_{\partial V_F} \sum_{ij} (\Pi_{ij} v_j) dS_i. \tag{B5}$$

The second term in this equation is a surface integral on the fluid boundary. There are two types of boundaries. At the boundary between solid and fluid, the velocity is zero due to the no-slip condition. And at the boundary of the domain, the deviatoric stress is zero. For these two reasons, the surface integral is zero. It results

$$\frac{1}{V}\int_{V_F} (\vec{\nabla} \cdot \mathbf{\Pi}) \cdot \vec{v}\, dr^3 = -\frac{1}{V}\int_{V_F} \sum_{ij} \Pi_{ij} \Delta_{ij} dr^3. \tag{B6}$$

The first term of Eq. (B3), thus, represents the average viscous dissipation within the porous medium.

The second term in this equation writes

$$\int_{V_F} (\vec{\nabla} p) \cdot \vec{v}\, dr^3 = \int_{V_F} \vec{\nabla} \cdot (p\vec{v}) dr^3 = \int_{\partial V_F} p\vec{v} \cdot d\vec{S}. \tag{B7}$$

This integral is zero at the solid/fluid boundary. Since the pressure is uniform on each side of the domain, it gives

$$\int_{V_F} (\vec{\nabla} p) \cdot \vec{v}\, dr^3 = \sum_i (q_i^{\text{out}} p_i^{\text{out}} - q_i^{\text{in}} p_i^{\text{in}}), \tag{B8}$$

with $q_i^{\text{in/out}}$ represents the velocity flux at the two boundary in the direction i. In the homogenization procedure, these flows are assumed to be equal at first order. This allows to define the mean velocity component $u_i = q_i/S_i$ and the mesoscopic pressure gradient $\vec{\nabla} P = \frac{p_i^{\mathrm{out}} - p_i^{\mathrm{in}}}{L_i}$. It follows that,

$$\frac{1}{V}\int_{V_F} (\vec{\nabla} p) \cdot \vec{v}\, dr^3 = \sum_i u_i \frac{p_i^{\mathrm{out}} - p_i^{\mathrm{in}}}{L_i} = \vec{u} \cdot \vec{\nabla} P. \tag{B9}$$

Combining Eqs. (B3), (B6), and (B9), thus, shows that at the Darcy's scale, the term

$$\vec{u} \cdot \vec{\nabla} P = -g(\vec{r}; \|\vec{u}\|)\|u\| = -f(\vec{r}; \|\vec{\nabla} P\|)\|\vec{\nabla} P\| = -\frac{1}{V}\int_{V_F} \sum_{ij} \Pi_{ij}\Delta_{ij} dr^3 \tag{B10}$$

represents the averaged microscopic energy dissipation rate.

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