Cascade of circulicity in compressible turbulence

O. Soulard[*] and A. Briard ●

CEA, DAM, DIF, F-91297 Arpajon, France

The purpose of this work is to investigate whether a cascading process can be associated with the rotational motions of compressible three-dimensional turbulence. This question is examined through the lens of circulicity, a concept related to the angular momentum carried by large turbulent scales. By deriving a Monin-Yaglom relation for circulicity, we show that an "effective" cascade of this quantity exists, provided the flow is stirred with a force having a solenoidal component. This outcome is obtained independently from the expression of the equation of state. To supplement these results, a coarse-graining analysis of the flow is performed. This approach allows us to separate the contributions of the transfer and production terms of circulicity and to discuss their respective effects in the inertial range.

I. INTRODUCTION

In incompressible turbulence, the behavior of small inertial scales is commonly described by invoking the physical image of the Richardson cascade [1–3]: under the stirring action of velocity gradients, inertial eddies break down into smaller and less energetic ones, that break down into even smaller and less energetic ones, and so on until viscous dissipation occurs. This phenomenological description is intimately linked with two sets of formal results stemming respectively from Kolmogorov's [4,5] and Onsager's [6,7] works.

The first approach focuses on the statistical properties of the velocity field taken at different points, a technique sometimes called "point splitting." Kolmogorov showed that, in the inertial range of homogeneous turbulence, kinetic energy flows on average from large to small scales with a constant flux given by its mean dissipation rate $\overline{\varepsilon}$. When turbulence is isotropic, this result takes the form of Kolmogorov's 4/5th law, while in anisotropic turbulence, it is expressed with the Monin-Yaglom relation [8]. Both expressions relate the third-order structure function of the velocity field to $\overline{\varepsilon}$. They are both derived from the incompressible Navier-Stokes equations under a crucial assumption: the mean dissipation $\overline{\varepsilon}$ must remain finite when the Reynolds number Re_t tends to infinity. This hypothesis is sometimes referred to as the zeroth law of turbulence and is related to the notion of dissipative anomalies [4,6,9–11].

Dissipative anomalies are also at the heart of the second set of formal results, the one originating from Onsager's work. The concept of anomaly arises from the observation that, when the Reynolds number Re_t increases, dissipative events take place in thinner and thinner structures while becoming more and more intense, in such a way that their average $\overline{\varepsilon}$ is finite. In the limit $\mathrm{Re}_\mathrm{t} \to \infty$, several authors, starting with Onsager [6,7,12,13], conjectured that this phenomenology could be described in the form of a weak singular ("anomalous") solution of Euler equations. As opposed to Kolmogorov's approach, Onsager's does not necessarily require statistical averages of two-point

---

[*]olivier.soulard@cea.fr

quantities. Instead, it can be expressed with the local properties of a regularized "coarse-grained" velocity field. Despite their inherent differences, Onsager's coarse-graining and Kolmogorov's point-splitting visions share the same core phenomelogical ideas. Accordingly, the coarse-graining approach has given rise to a series of predictions closely related to Kolmogorov's 4/5th law and to the idea that energy is transferred from scale to scale at a constant rate [7]. Both the "point-splitting" and "coarse-graining" approaches sustain the idea that an energy cascade takes place in the inertial range of incompressible turbulence.

When density inhomogeneities or compressible effects are present, the physical image of the Richardson cascade is harder to uphold. In that case, eddies are not only stirred and broken by velocity gradients, they are also affected by baroclinic production and by local dilatations and compressions. In principle, these additional processes may very well lead to an inverse transfer of energy, at the very least locally and transiently. These processes also cast their shadow on the point-splitting and coarse-graining approaches. Their adaptations to compressible turbulence face several hurdles, some of which have only been recently overcome. In particular, Aluie [14–16] and Eyink and Drivas [17,18] have produced two seminal works based on the coarse-graining approach. They have shown that in compressible turbulence, an inertial range exists where a local conservative cascade of kinetic energy takes place, despite the fact that kinetic energy is no longer an inviscid invariant of the flow. This result is conditioned on the presence of dissipative anomalies and on the sufficient roughness of the turbulent fields.

From the point-splitting side, no equivalent predictions have so far been derived. Still, several adaptations of the Monin-Yaglom relation to compressible turbulence have been proposed [19–30]. But as opposed to the coarse-graining approach, most of these efforts were not aimed at proving the existence of a cascading process. Instead, they were mostly driven by a practical goal: being able to evaluate the dissipation and large-scale injection of energy knowing two-point correlations. This information is indeed useful for interpreting several astrophysical phenomena [28,30]. As it turns out, one of the main difficulties for establishing Monin-Yaglom relations in compressible flows comes from the pressure field and the way it is coupled with other thermodynamical variables. To alleviate this difficulty, most of the mentioned works [19–30] make a simplifying assumption on the thermodynamical state of the flow: it is either isothermal [20–23,25–29], polytropic [19,24], or isentropic [30]. These assumptions are physically sound: depending on the astrophysical application considered [19–33], they do provide an accurate thermodynamical approximation of the flow. However, their use is not completely innocuous either. In particular, they *de facto* prevent the existence of several physical phenomena. For instance, the first two assumptions suppress the baroclinic torque and the development of convective instabilities such as the Rayleigh-Taylor and Richtmyer-Meshkov ones [34–37]. Besides, the third one is not compatible with the entropy cascade predicted in [17]. Therefore, despite their great practical use, the Monin-Yaglom relations derived so far are in principle restricted to particular flows. In this regard, it is worth highlighting that all the mentioned Monin-Yaglom relations are coupled to the energetic and mixing content of the flow, either through the presence of the pressure field, either because they purposely consider the inertial transfer of the total energy. None of them is purely dynamical, in the sense of involving only velocity and density correlations. Whether such a relation exists remains an open question. But its answer is tightly linked with the possibility of finding a Monin-Yaglom equation that is not restricted to a particular thermodynamical assumption. Besides, it can already be stressed that, if it exists, this answer would require finding a cascading quantity different from the energy: as shown with the coarse-graining approach in [14–17], the pressure is indeed one of the components acting in the interscale transfer of kinetic energy.

Before exploring further these issues, let us briefly return to the incompressible case. In this context, the velocity field is purely solenoidal. For the infinite domains considered in homogeneous turbulence, this entails the existence of a one-to-one correspondence between the velocity and vorticity fields: vorticity is given by the curl of velocity, while velocity is retrieved from vorticity using a Biot and Savart integral expression [38]. As a result of this one-to-one relation, the kinetic energy spectrum becomes equal to the circulicity spectrum in homogeneous turbulence. Circulicity

is a term coined by Kassinos *et al.* [39] in order to refer to a tensor describing the "large-scale" structure of the vorticity field. In this work, we will slightly tweak this definition and use circulicity to refer to the "large-scale" structure of the angular momentum. Indeed, when density is constant, vorticity becomes a proxy for measuring the angular momentum of a Eulerian control volume, so that the two notions become indistinguishable (see Appendix A). Piecing these statements together, one arrives at the conclusion that the energy cascade predicted by the point-splitting and coarse-graining approaches also coincides with a cascade of circulicity. This co-occurrence agrees with the physical picture of the Richardson cascade in which energy is transferred to smaller scales by the motion of turbulent eddies, which can be thought of as coherent regions of angular momentum (see [40,41] and references therein).

Coming back to compressible flows, the velocity field now possesses a dilatational component and is not solely representative of vortical motions. But most importantly, vorticity itself loses one of its key physical meaning. It is not connected to the angular momentum of Eulerian control volumes any longer. Indeed, angular momentum is defined with respect to the curl of the linear momentum and is only related to vorticity when density is constant (see [42,43] and Appendix A). Therefore, contrary to the incompressible case, the spectral properties of kinetic energy and circulicity dissociate in compressible turbulence. In particular, in homogeneous turbulence, the identity between the circulicity and kinetic energy spectra ceases to be valid. Instead, the circulicity spectrum becomes equal to the spectrum of the solenoidal component of the linear momentum. The latter is indeed the Helmholtz dual of the angular momentum, just as the solenoidal velocity is with respect to vorticity. As a consequence of this dissociation, the existence of the energy cascade predicted with the coarse-graining approach in [14,17] does not automatically extend to circulicity: even if energy cascades to small scales, it is not known whether this transfer goes along with the apparition of smaller whirls. Thus, the question arises as to whether a circulicity cascade exists in compressible turbulence. This question sends us back to the issues we already raised when commenting existing Monin-Yaglom relations. Indeed, as shown in [42,43], the evolution of the angular momentum and of the solenoidal linear momentum are not directly impacted by the pressure field. Hence, the study of their properties opens a door for deriving a Monin-Yaglom relation that is not bound by the equation of state and that is decoupled from the thermodynamics and mixing content of the flow.

Given these considerations, the purpose of this work is to study the transfer of circulicity in the inertial range of a compressible flow. More particularly, we aim to discuss whether a circulicity cascade exists and, if it does, what it implies for inertial scales. This study is amenable to both the point-splitting and coarse-graining approaches. For the sake of completeness, we will use the two techniques. To begin with, we will derive a Monin-Yaglom relation for the solenoidal component of the linear momentum, an appellation hereafter abridged to solenoidal momentum. We will then discuss how this relation depends on the different variables describing the flow. We will also discuss how it may potentially provide information about the scaling of the turbulent spectra. These different elements will also be examined through the lens of the coarse-graining approach. To this end, spatial filtering will be applied to the solenoidal momentum equation.

Before closing this introduction, we would like to emphasize that the word "cascade" has been used throughout this introduction in order to refer to physical processes occurring at asymptotically small scales and characterized by point-splitting and coarse-graining relationships. This implicit meaning will also be kept in the remaining of this paper. Establishing the existence of a cascading process over a finite range of scales is beyond the scope of this work.

II. GOVERNING EQUATIONS

A. Compressible Navier-Stokes equations

We consider a flow which state is defined by its density ρ, velocity $\mathbf{v}$, internal energy e and species mass fractions $\{c_\alpha, \ \alpha = 1, \ldots, N_s\}$. These variables evolve according to the compressible

multicomponent Navier-Stokes equations [44–46]:

$$\partial_t \rho + \partial_j(\rho v_j) = 0, \tag{1a}$$

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) = -\partial_j(p\delta_{ij} + \sigma_{ij}) + f_i, \tag{1b}$$

$$\partial_t(\rho e) + \partial_j(\rho v_j e) = \rho\varepsilon - p\partial_j v_j - \partial_j\sigma_j^{(e)}, \tag{1c}$$

$$\partial_t(\rho c_\alpha) + \partial_j(\rho v_j c_\alpha) = -\partial_j\sigma_j^{(\alpha)}, \tag{1d}$$

where p is the pressure, $\boldsymbol{f}$ a volumetric force, $\boldsymbol{\sigma}$ the viscosity tensor, $\boldsymbol{\sigma}^{(e)}$ the molecular heat flux, $\boldsymbol{\sigma}^{(\alpha)}$ the molecular diffusion flux of species α and $\rho\varepsilon = -\sigma_{ij}\partial_j v_i$ the dissipation rate of the kinetic energy. For simplicity, we assume that the viscosity tensor is defined by

$$\sigma_{ij} = -\mu S_{ij} - \eta\partial_k v_k\delta_{ij} \quad \text{with} \quad S_{ij} = \partial_j v_i + \partial_i v_j - \tfrac{2}{3}\partial_k v_k\delta_{ij}, \tag{1e}$$

where the shear viscosity μ and the bulk viscosity η are chosen to be constant. This hypothesis is only made for the sake of simplicity. In the general case, μ and η would depend on the density, temperature and concentration fields. The molecular heat and diffusion fluxes, $\boldsymbol{\sigma}^{(e)}$ and $\boldsymbol{\sigma}^{(\alpha)}$, will not be used in the remainder of this text and do not need to be detailed here. Their expressions are briefly recalled in Appendix E and we refer the reader to Refs. [44–46] for more information on the subject.

To close the flow description, two equations of states must be added, a thermal and a caloric one. We assume that they can be expressed as functions of density, temperature, and mass fractions:

$$p \equiv p(\rho, T, \boldsymbol{c}) \quad \text{and} \quad e \equiv e(\rho, T, \boldsymbol{c}), \tag{1f}$$

where T is the temperature. These functions can be general and are not constrained by an isothermal, polytropic, or isentropic assumption. Similar to $\boldsymbol{\sigma}^{(e)}$ and $\boldsymbol{\sigma}^{(\alpha)}$, the equations of state do not play any role in the forthcoming analysis and do not need to be made explicit.

B. Helmholtz decomposition of the momentum

Using an Helmholtz decomposition, along with the fact that the domain considered in this study is unbounded, the momentum $\boldsymbol{j} = \rho\boldsymbol{v}$ can be split into a solenoidal and a dilatational component respectively denoted by $\boldsymbol{s}$ and $\boldsymbol{d}$ [42,43]:

$$\boldsymbol{j} = \rho\boldsymbol{v} = \boldsymbol{s} + \boldsymbol{d} \quad \text{with} \quad d_i = -\partial_i\Phi \quad \text{and} \quad s_i = \epsilon_{ijk}\partial_j\Psi_k, \tag{2}$$

where ϵ_{ijk} is the Levi-Civita tensor. The scalar potential Φ and the divergence-free vector potential $\boldsymbol{\Psi}$ are given by

$$\partial_{jj}^2\Phi = -\Xi \quad \text{and} \quad \partial_{jj}^2\Psi_i = -\Omega_i, \tag{3}$$

where Ξ is divergence of the momentum and $\boldsymbol{\Omega}$ is the angular momentum of a Eulerian control volume (see Appendix A):

$$\Xi = \partial_j(\rho v_j) = -\partial_t\rho \quad \text{and} \quad \Omega_i = \epsilon_{ijk}\partial_j(\rho v_k). \tag{4}$$

It is important to stress that the angular momentum $\boldsymbol{\Omega}$ does not carry the same information as the vorticity $\boldsymbol{\omega}$. The latter is defined by

$$\omega_i = \epsilon_{ijk}\partial_j v_k = \Omega_i/\rho - \epsilon_{ijk}v_k\partial_j\rho/\rho.$$

Similarly, the momentum divergence Ξ is different from the velocity divergence Θ defined as

$$\Theta = \partial_j v_j = \Xi/\rho - v_j\partial_j\rho/\rho.$$

Accordingly, the Helmholtz decomposition of the momentum $\rho\boldsymbol{v}$ is also different from the Helmholtz decomposition of the velocity field $\boldsymbol{v}$, which can be written as

$$\boldsymbol{v} = \boldsymbol{v}^s + \boldsymbol{v}^d \quad \text{with} \quad v_i^d = -\partial_i\varphi^v, \quad v_i^s = \epsilon_{ijk}\partial_j\psi_k^v \quad \text{and} \quad \partial_{jj}^2\varphi^v = -\Theta, \quad \partial_{jj}^2\psi_i^v = -\omega_i. \tag{5}$$

Except when density is constant, one generally has

$$\mathbf{\Omega} \neq \rho\boldsymbol{\omega}, \quad \Xi \neq \rho\Theta, \quad \Phi \neq \rho\varphi^v, \quad \text{and} \quad \mathbf{\Psi} \neq \rho\boldsymbol{\psi}^v.$$

C. Circulicity

Given the Helmholtz decomposition (2) and (3), the solenoidal momentum s characterizes the rotational motions of the flow, "rotational" being used here as a reference to angular momentum and not vorticity. This link between s and $\mathbf{\Omega}$ can be formalized further with the notion of circulicity. More precisely, in this work, we define the local value of the circulicity field $\mathcal{C}$ by

$$\mathcal{C} = \tfrac{1}{2}s_i s_i. \tag{6}$$

The quantity $\mathcal{C}/\rho$ can be interpreted as the contribution to the kinetic energy coming exclusively from rotational motions. More precisely, one has $\rho v_i v_i/2 = \mathcal{C}/\rho + s_i d_i/\rho + d_i d_i/(2\rho)$, so that two other contributions must be accounted for in addition to $\mathcal{C}/\rho$: a purely dilatational one and a cross-rotational-dilatational one. But $\mathcal{C}$ also has another interpretation that is not linked to energy and that is more easily discussed in homogeneous turbulence. In this context, the mean of $\mathcal{C}$, which is half the variance of s, is equal to the integral of the solenoidal momentum spectrum E_{ss}:

$$\text{Homogeneous turbulence:} \quad \overline{\mathcal{C}} = \frac{1}{2}\overline{s_i s_i} = \int E_{ss}\,dk, \tag{7}$$

where $\overline{\cdot}$ denotes the ensemble mean and where k is the wave number. Besides, given the Poisson equations (3), the spectrum E_{ss} of s is linked to the spectrum $E_{\Omega\Omega}$ of $\mathbf{\Omega}$ by

$$\text{Homogeneous turbulence:} \quad E_{ss} = \frac{E_{\Omega\Omega}}{k^2}. \tag{8}$$

Equation (8) indicates that the solenoidal momentum carries the same two-point information as the angular momentum, with the difference that a greater weight is given to large scales. When combined with Eq. (7), this result shows that the mean circulicity $\overline{\mathcal{C}}$ accounts for the large-scale content of the angular momentum. This conclusion has been illustrated for homogeneous turbulence, but the overall correspondence between circulicity and large-scale angular momentum extends to the inhomogeneneous case.

Note also that in a constant-density flow, one would simply have $\overline{\mathcal{C}} = \rho^2\overline{v_i v_i}/2$. Hence, circulicity would be redundant with kinetic energy and there would be no need to study its properties, nor those of the solenoidal momentum two-point correlations. However, this is no longer the case when density is variable. As mentioned in the introduction, the concepts of circulicity and energy dissociate in compressible flows. Finally, it is worth stressing that the notion of circulicity used here is only loosely based on the one proposed by Kassinos *et al.* [39]. The latter introduced this concept to measure different contributions of the Reynolds-stress tensor of an incompressible velocity field. Here we only preserved the idea that circulicity gives access to the large-scale structure of the angular momentum.

D. Evolution of the solenoidal momentum s

The solenoidal momentum s is the main focus of this work because of its connection to angular momentum and circulicity. To derive its evolution, we start from the linear momentum equation (1b) and note that $\partial_t d_i = -\partial_i\partial_t\Phi$. This term can then be added with $\partial_i p$ and the dilatational parts of the forcing and viscous terms in order to form the gradient of a pseudopressure π^s. This procedure yields $\pi^s = p - \partial_t\Phi + \phi^f - (\eta + \frac{4}{3}\mu)\partial_j v_j$ with ϕ^f the scalar potential of the force $\boldsymbol{f}$ (see also Appendix B). This particular expression is only given for the sake of completeness and serves no further purpose. Indeed, the value of π^s is set in practice by the fact that s is divergence-free, as detailed below. The evolution of s is given by

$$\partial_t s_i + \partial_j(\rho v_i v_j) = -\partial_i \pi^s - \partial_j \sigma_{ij}^s + f_i^s, \tag{9a}$$

where $\boldsymbol{\sigma}^s$ and $\boldsymbol{f}^s$ are the solenoidal parts of the forcing and viscous terms:

$$\sigma_{ij}^s = -\mu(\partial_j v_i + \partial_i v_j) + 2\mu\partial_k v_k \delta_{ij} \quad \text{and} \quad f_i^s = \epsilon_{ijk}\partial_j \psi_k^f \quad \text{with} \quad \partial_{jj}^2 \psi_i^f = -\epsilon_{ijk}\partial_j f_k. \quad (9b)$$

The pseudopressure π^s enforces the divergence-free constraint of $\boldsymbol{s}$ for the advection term $\partial_j(\rho v_i v_j)$:

$$\partial_{jj}^2 \pi^s = -\partial_{ij}^2(\rho v_i v_j). \quad (9c)$$

It is worth highlighting that π^s is different from the actual pressure p:

$$\pi^s \neq p.$$

The pressure p is set by the equation of state (1f). By contrast, π^s is the solution of a Poisson equation and is set by the dynamics of the flow. This remains true whether the flow displays strong compressibility effects or not. Another significant point is that p does not appear explicitly in Eq. (9a). Thus, even though the equation of state is important for determining the overall evolution of the flow, its influence on the solenoidal momentum $\boldsymbol{s}$ is only indirect. This is not the case for all solenoidal fields. For instance, pressure appears explicitly in the evolution equation of the solenoidal velocity field $\boldsymbol{v}^s$.

E. Evolution of the circulicity $\mathcal{C}$

Starting from the evolution equation (9a) of the solenoidal momentum and using the Poisson equations (3), the following evolution of the circulicity can be deduced:

$$\partial_t \mathcal{C} + \partial_j \mathcal{F}_j^c = \mathcal{P}_c - \varepsilon_c + s_i f_i^s, \quad (10)$$

where $\mathcal{F}^c$ is the flux of circulicity:

$$\mathcal{F}_i^c = \tfrac{1}{2}|\boldsymbol{j}|^2 v_i + \pi^s j_i + \Phi\partial_j(\rho v_i v_j + \pi^s \delta_{ij}) + \mu s_j(\partial_j v_i - \partial_i v_j), \quad (11)$$

where $\mathcal{P}_c$ and ε_c are terms associated with the local production and dissipation of circulicity:

$$\mathcal{P}_c = -\tfrac{1}{2}|\boldsymbol{j}|^2 \Theta + \pi^s \Xi \quad \text{and} \quad \varepsilon_c = \mu\Omega_i\omega_i, \quad (12)$$

where we recall that $\boldsymbol{j} = \rho\boldsymbol{v}$, $\Theta = \partial_i v_i$ and $\Xi = \partial_i(\rho v_i)$.

This formulation of the circulicity equation allows us to highlight the role played by the dilatational components of the velocity and momentum: it is the divergences of these fields that control the circulicity production term $\mathcal{P}_c$. The other interest of this formulation is to allow for a direct comparison with the evolution of the total momentum norm. Indeed, one has

$$\partial_t\left(\tfrac{1}{2}|\boldsymbol{j}|^2\right) + \partial_j\left[\tfrac{1}{2}|\boldsymbol{j}|^2 v_j + p j_j + \sigma_{ij} j_i\right] = -\tfrac{1}{2}|\boldsymbol{j}|^2\Theta + p\Xi + \sigma_{ij}\partial_j j_i + j_i f_i. \quad (13)$$

The right-hand sides of Eqs. (10) and (13) share the same structure, with similar source, dissipation and forcing terms. The main differences are that, in Eq. (10), the dynamic pressure π^s replaces the actual pressure p and that the solenoidal force replaces the total one. Besides, the dissipation term ε_c only accounts for shear viscosity effects and not bulk ones. It is worth stressing that ε_c is not necessarily positive. The angular momentum and vorticity fields can indeed be misaligned. This misalignment is linked to the density and velocity gradients as follows:

$$\varepsilon_c = \mu\,\rho\omega_i\omega_i + \mu\,v_i(\partial_j v_i - \partial_i v_j)\partial_j\rho.$$

However, if we assume that the density and velocity fields decorrelate at small scales, the mean of ε_c will be dominated by the mean of the first term. In that case, the mean of ε_c remains positive, even though local values of ε_c can be negative. In the remaining of this work, we will assume that such an hypothesis holds and that

$$\overline{\varepsilon_c} > 0.$$

III. MONIN-YAGLOM RELATION AND TRANSFER OF CIRCULICITY IN HOMOGENEOUS TURBULENCE

Our purpose in this section is to study how circulicity is transferred from scale to scale in the inertial range of a homogeneous compressible flow. Given the relation between circulicity and solenoidal momentum (see Sec. II C), we start by deriving a Kármán-Howarth equation for the solenoidal momentum s. Then we use this equation to express a Monin-Yaglom relation describing the interscale transfer of this quantity and by extension of circulicity. This relation is compared against existing ones [19–30] and is also used to discuss inertial range scalings. This section only makes use of the point-splitting approach. The coarse-graining method will be discussed in the next section.

A. Kármán-Howarth equation for the solenoidal momentum

The first step in our study consists in deriving a Kármán-Howarth equation for the solenoidal momentum, or, in other words, an evolution equation for the second-order structure function of the solenoidal momentum. The latter is defined by

$$\overline{\Delta s_i \Delta s_i}(\boldsymbol{r}, t) = 2\overline{s_i s_i} - 2\overline{s_i(\boldsymbol{x})s_i(\boldsymbol{x} + \boldsymbol{r})},$$

where for any quantity X, ΔX refers to the difference between the values of X taken at two different points $\boldsymbol{x}$ and $\boldsymbol{x}'$, separated by the vector $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$. For later purposes, it will also be useful to introduce the median value $\{X\}_{\frac{1}{2}}$ of X at these two points. These two-point difference and median values are defined by

$$\Delta X(\boldsymbol{x}, \boldsymbol{r}, t) = X(\boldsymbol{x}', t) - X(\boldsymbol{x}, t) \quad \text{and} \quad \{X\}_{\frac{1}{2}}(\boldsymbol{x}, \boldsymbol{r}, t) = [X(\boldsymbol{x}', t) + X(\boldsymbol{x}, t)]/2 \quad \text{with} \quad \boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}.$$

We would like to stress again that $\overline{\Delta s_i \Delta s_i}$ is related to circulicity and that, as such, it accounts for the large-scale content of the angular momentum. Indeed, one has $\overline{\Delta s_i \Delta s_i}(\boldsymbol{r}, t) = 4\overline{\mathcal{C}} - 4 \int e^{\imath \boldsymbol{k} \cdot \boldsymbol{r}} \frac{E_{\Omega\Omega}}{2\pi k^4} \, d\boldsymbol{k}$. Therefore, studying the inertial range transfer of $\overline{\Delta s_i \Delta s_i}$ is equivalent to studying the transfer of circulicity.

Starting from Eq. (9a) and using some of the relations detailed in Appendix C, we find after some algebra that

$$\partial_t \overline{\Delta s_i \Delta s_i}(\boldsymbol{r}, t) = -\partial_{r_j} \mathcal{F}_j(\boldsymbol{r}, t) - 4\mathcal{E}_{\text{inj}}(t) + 2\mu \partial^2_{r_j r_j} \overline{\Delta v_i \Delta s_i}(\boldsymbol{r}, t) + 2\overline{\Delta s_i \Delta f_i^s}(\boldsymbol{r}, t), \tag{14a}$$

where the circulicity flux $\mathcal{F}$ and injection rate $\mathcal{E}_{\text{inj}}$ are defined by

$$\mathcal{F}_j(\boldsymbol{r}, t) = \overline{\Delta v_j \Delta j_i \Delta j_i}(\boldsymbol{r}, t) - 2\overline{\Delta v_j^d \{j_i j_i\}_{\frac{1}{2}}}(\boldsymbol{r}, t) - 4\overline{\Delta d_i \{\rho v_i v_j\}_{\frac{1}{2}}}(\boldsymbol{r}, t) \tag{14b}$$

$$\text{and} \quad \mathcal{E}_{\text{inj}}(t) = \overline{s_i f_i^s} - \partial_t \overline{\mathcal{C}} = \overline{\varepsilon_c}(t) - \overline{\mathcal{P}_c}(t). \tag{14c}$$

The first term in the right-hand side of Eq. (14a) describes the nonlinear transfer of circulicity from scale to scale. It is expressed as the divergence of a flux. The second term accounts for the unforced evolution of the mean circulicity $\overline{\mathcal{C}} = \overline{s_i s_i}/2$ and is scale-independent. The third one expresses the transfer of circulicity induced by viscous effects. The last one corresponds to the correlation between s and the external force $\boldsymbol{f}$ or more precisely its solenoidal component $\boldsymbol{f}^s$. Overall, the evolution equation of $\overline{\Delta s_i \Delta s_i}$ retains the same form as the equation of the second-order structure function of the velocity field classically derived in incompressible flows [5,8]. However, the flux and dissipation terms of Eq. (14a) are affected by the presence of density inhomogeneities and by compressibility.

More precisely, the nonlinear transfer of circulicity involves three different components. The first one, $\overline{\Delta v_j \Delta j_i \Delta j_i}$, is linked to the advection of s by the velocity field $\boldsymbol{v}$. It takes the form of a third-order structure function. As shown in Appendix D, its expression is identical to the one appearing in the transfer flux of $\overline{\Delta j_i \Delta j_i}$. Hence, this first component is representative of the transfer between

scales not only of the solenoidal momentum, but also of the total momentum. The remaining two contributions involve the increments of dilatational components of the momentum and velocity fields. Their presence is due to the fact that circulicity is driven by a production term $\mathcal{P}_c$ and is not a conserved quantity. However, Eq. (14a) shows that the effects of this nonconservative production can still be cast in the form of a conservative flux as far as the interscale transfer of circulicity is concerned. Note that both of these contributions vanish in constant-density turbulence. In that case, one is left with $\mathscr{F}_j = \rho^2 \overline{\Delta v_j \Delta v_i \Delta v_i}$, which is the flux obtained in the incompressible case weighted by ρ^2.

Concerning $\mathscr{E}_{\text{inj}}$, it has two components. The first one is the mean dissipation of circulicity $\overline{\varepsilon_c}$, the second its mean production rate $\overline{\mathcal{P}_c}$ [see Eq. (10)]. The latter has the same origin as the two dilatational contributions of the nonlinear transfer flux. In this regard, a balance between these two dilatational terms and $\overline{\mathcal{P}_c}$ would exist in the limit $r \to 0$ if the turbulent fields were smooth. For rough fields, this is not case. Note also that for constant-density turbulence, $\mathcal{P}_c$ vanishes and one is left with $\mathscr{E}_{\text{inj}}(t) = \mu \, \rho \overline{\omega_i \omega_i}$, which is the solenoidal dissipation weighted by the density. We would also like to emphasize that the definition of $\mathscr{E}_{\text{inj}}$ stems from the unforced evolution of $\overline{\mathcal{C}}$, as given by the first equality of Eq. (14c). This origin is important because in forced stationary turbulence, one gets

$$\mathscr{E}_{\text{inj}} = \overline{s_i f_i^s}. \tag{15}$$

In that case, the value of $\mathscr{E}_{\text{inj}}$ corresponds to the injection rate of circulicity. It is then set by the solenoidal component of the force and not by the total force. The implications of the latter property will be commented further in Sec. III C.

B. Monin-Yaglom relation

We now assume that the flow is stationary and consider the limit $\mu \to 0$ and $r \to 0$ of the Kármán-Howarth equation (14a). The notation $\mu \to 0$ is used as a shorthand for referring to the high Reynolds number limit. Then, provided the correlation of the force $\boldsymbol{f}$ falls off exponentially, the only two terms that do not vanish in this limit are the nonlinear flux divergence and $\mathscr{E}_{\text{inj}}$. Thus, for a stationary flow, in the limit $\mu \to 0$, $r \to 0$, we obtain

$$\partial_{r_j}\left(\overline{\Delta v_j \Delta j_i \Delta j_i} - 2\overline{\Delta v_j^d \{j_i j_i\}_{\frac{1}{2}}} - 4\overline{\Delta d_i \{\rho v_i v_j\}_{\frac{1}{2}}}\right) = -4\mathscr{E}_{\text{inj}}. \tag{16}$$

This equation is one of the main results of this work. It constitutes the Monin-Yaglom relation derived from the solenoidal momentum evolution equation (9). Interestingly, this relation can be written in several alternative forms that may help shed light on different aspects of the transfer of circulicity. For instance, using the Helmholtz decomposition (2) and (3) and the Poisson equation (9c), one may get rid of the dilatational components v_i^d and d_i. Equation (16) can indeed be rewritten as

$$\partial_{r_j}\left(\overline{\Delta v_j \Delta j_i \Delta j_i} - 2\overline{\Delta v_j \{j_i j_i\}_{\frac{1}{2}}} + 4\overline{\Delta j_j \{\pi^s\}_{\frac{1}{2}}}\right) = -4\mathscr{E}_{\text{inj}}. \tag{17}$$

In agreement with this formulation, one may also rewrite $\mathscr{E}_{\text{inj}}$ as $\mathscr{E}_{\text{inj}} = \overline{\varepsilon_c} + \frac{1}{2}\overline{j_i j_i \partial_j v_j} - \overline{\pi^s \partial_j (\rho v_j)}$. Another possibility is to forgo the flux divergence formulation and allow for the presence of a source term. In particular, one may reformulate Eq. (16) as

$$\partial_{r_j}(\overline{\Delta v_j \Delta j_i \Delta j_i}) + \overline{\Delta \Theta \Delta (j_i j_i)} - 2\overline{\Delta \Xi \Delta \pi^s} = -4\overline{\varepsilon_c}, \tag{18}$$

where we recall that $\Theta = \partial_j v_j$ and $\Xi = \partial_j (\rho v_j)$. This formulation is closer to the ones derived for instance in [20,24]. As in these references, the source terms can be associated with the dissipation term to provide a scale dependent effective dissipation ε_{eff}.

C. Effective circulicity cascade

The main physical interpretation of Eq. (16), and its variants Eqs. (17) and (18), is similar to the one put forward in the classical incompressible case, except for two important points. First, it applies to circulicity and not energy, and second, the flux $\mathscr{F}$ in Eq. (16) is an "effective" flux that accounts not only for the scale-to-scale transfer of circulicity but also for its production by the dilatational source term $\mathcal{P}_c$. The two processes are actually intertwined and setting a precise separation between the two bears some degree of arbitrariness. For this reason, the fact that these two effects can be collected within a single flux entity allows us to get rid of this distinction and to put forward a simple interpretation of the Monin-Yaglom equation (16). More precisely, the global flux Π of circulicity flowing from scales larger than r to scales smaller than r is defined by [47]

$$\Pi = -\frac{1}{4V_r} \int_{|\mathbf{r}'| \leqslant r} \partial_{r_j} \mathscr{F}_j(\mathbf{r}') \, d\mathbf{r}' = -\frac{3}{4r} \langle \mathscr{F}_\| \rangle_{4\pi}(r),$$

where the subscript $\|$ denotes the coordinate of a vector in the direction $\hat{\mathbf{r}} = \mathbf{r}/r$ and where $\langle \cdot \rangle_{4\pi}$ refers to the average over the surface of the unit sphere: $\mathscr{F}_\| = \mathscr{F}_j \hat{r}_j$ and $\langle \cdot \rangle_{4\pi} = \frac{1}{4\pi} \oint \cdot \, d\hat{\mathbf{r}}$. From the definition of $\mathscr{F}$, one can express this flux as

$$\Pi = -\frac{3}{4r} \langle \overline{\Delta v_\| \Delta j_i \Delta j_i} - 2\overline{\Delta v_\|^d \{j_i j_i\}_{\frac{1}{2}}} - 4\overline{\Delta d_i \{\rho v_i v_\|\}_{\frac{1}{2}}} \rangle_{4\pi}. \tag{19}$$

In isotropic turbulence, the brackets $\langle \cdot \rangle_{4\pi}$ can be dropped from this expression. Integrating the Monin-Yaglom relation Eq. (16) over the sphere of radius r, one deduces that Π is constant and equal to $\mathscr{E}_{\text{inj}}$:

$$\Pi = \mathscr{E}_{\text{inj}}. \tag{20}$$

Alternatively, one can write that

$$\langle \mathscr{F}_\| \rangle_{4\pi} = \langle \overline{\Delta v_\| \Delta j_i \Delta j_i} - 2\overline{\Delta v_\|^d \{j_i j_i\}_{\frac{1}{2}}} - 4\overline{\Delta d_i \{\rho v_i v_\|\}_{\frac{1}{2}}} \rangle_{4\pi} = -\frac{4}{3} \mathscr{E}_{\text{inj}} r.$$

Now, let us assume that the flow is forced and stationary. Then the value of $\mathscr{E}_{\text{inj}}$ is determined by the injection of solenoidal momentum: equation (15) is valid and one has $\mathscr{E}_{\text{inj}} = \overline{s_i f_i^s} \geqslant 0$. From there, two situations can be distinguished:

The first one occurs when the solenoidal component of the force is not null. Then, we have

$$\Pi = \mathscr{E}_{\text{inj}} > 0.$$

This condition indicates the existence of an effective circulicity cascade: keeping in mind the comments made at the beginning of this section, circulicity may be thought to be transferred on average from large to small scales at a rate $\mathscr{E}_{\text{inj}}$ independent from the scale r. If we leave aside the composite nature of the effective flux Π, this description is similar to the forward energy cascade associated with the Kolmogorov-Obukhov theory in incompressible turbulence.

The second situation arises when $\boldsymbol{f}^s = 0$ and turbulence is only forced on dilatational modes. Then the flux of circulicity becomes null:

$$\Pi = \mathscr{E}_{\text{inj}} = 0.$$

In that case, there is no effective circulicity cascade in the inertial range. Instead, an effective nonlinear equilibrium [3] takes place in the inertial range, set by an exchange between the variances of the dilatational and solenoidal components of the momentum and constrained by the condition $\Pi = 0$.

Thus, the existence of an effective circulicity cascade is not systematic and depends on the way turbulence is forced at large scales. A parallel can be drawn between this conclusion and several observations which have been made in simulations of forced compressible turbulence [11,48–53]. Depending on whether the massic force is solenoidal or dilatational, significant differences were observed in the statistics of the simulated flows. However, one should not go beyond making a

parallel since a direct observation of the particular case $f^s = 0$ is not given in these references. Indeed, in Refs. [11,48–53] the solenoidal-dilatational separation is applied to the massic force f/ρ and not to the volumetric force f as required in this work. Despite this difference, a link between the distinct behaviors observed in [11,48–53] and the possible disappearance of the circulicity cascade would be worth investigating further.

We close this section as we started it, by stressing that circulicity is not a conserved quantity and that the notions of its cascade and equilibrium state are, to some extent, ill-posed. This is why we used the term "effective" to qualify them. The flux $\mathscr{F}$ and its integral Π combine nonlinear effects coming from conservative and nonconservative processes. In Sec. IV, we will use the coarse-graining approach to help disentangle the roles of cascading and production effects for circulicity in the inertial range.

D. Comparison with previous relations and practical use of Eq. (16)

The Monin-Yaglom equation (16), and its variants (17) and (18), are valid for compressible turbulence, whether in a highly compressible or weakly pseudocompressible regime. As opposed to other such relations, for instance those derived in [19–30], Eq. (16) does not involve the actual pressure field and does not make any assumption on the equation of state, i.e. there is no hypothesis on whether the flow is isothermal, isentropic or polytropic. What is more, all the terms appearing in Eq. (16) can be evaluated knowing only the density ρ and the velocity field $\boldsymbol{v}$ at a given time. In this restricted sense, it is entirely decoupled from the energetic and mixing content of the flow. Besides, it is also worth stressing that, in Eq. (16), the transfer between scales is written as the divergence of a flux $\mathscr{F}$. By contrast, in the relations proposed in [19–30], source terms are present which might not be easily recast in the form of a flux.

Compared to previous Monin-Yaglom relations there is also a major difference. Most of these relations have been derived with a practical goal in mind: estimating the total injection and dissipation rates of energy. The relation we derive cannot fulfill the same goal. Indeed, it can provide information about only the injection rate of solenoidal momentum $\mathscr{E}_{\mathrm{inj}} = \overline{j_i f_i^s} = \overline{s_i f_i^s}$. Thus, from a practical point of view, the Monin-Yaglom equation (16) cannot work as a substitute to the previously derived relations [19–30]. Instead, Eq. (16) should be viewed as providing a complementary information. When coupled with previous relations, it should help disentangle the solenoidal and dilatational contributions of the dissipation and energy injection.

E. Approximate Monin-Yaglom relation in the weakly dilatational regime

In the Monin-Yaglom equation (16), the different contributions of the flux $\mathscr{F}$ are respectively on the order of

$$\overline{\Delta v_j \Delta j_i \Delta j_i} = O[\delta v \, \delta(\rho v) \, \delta(\rho v)], \quad \overline{\Delta v_j^d \{j_i j_i\}_{\frac{1}{2}}} = O[\delta v^d \, \delta(\rho^2 v^2)],$$

$$\text{and} \quad \overline{\Delta d_i \{\rho v_i v_j\}_{\frac{1}{2}}} = O[\delta d \, \delta(\rho v^2)],$$

where the notation $\delta X(r)$ refers to the order of magnitude of the increment ΔX. Provided $\delta\rho/\rho \to 0$ when $r \to 0$, one may approximate $\delta(\rho v)$ by $\rho \delta v$, $\delta(\rho^2 v^2)$ by $\rho^2 \delta(v^2)$ and $\delta(\rho v^2)$ by $\rho\delta(v^2)$. Therefore, the two dilatational contributions $\overline{\Delta v_j^d \{j_i j_i\}_{\frac{1}{2}}}$ and $\overline{\Delta d_i \{\rho v_i v_j\}_{\frac{1}{2}}}$ become negligible compared to the first one whenever

$$\delta v^d \ll \delta v \frac{(\delta v)^2}{\delta(v^2)} \quad \text{and} \quad \delta d \ll \rho\delta v \frac{(\delta v)^2}{\delta(v^2)}. \tag{21}$$

These conditions define a particular regime that we call weakly dilatational. We note that when $\boldsymbol{v}$ is solenoidal, $\delta(v^2)$ should be on the order of $(\delta v)^2$, as discussed in [54]. Therefore, to reach the weakly dilatational regime, it could possibly be sufficient to verify

$$\delta v^d \ll \delta v \quad \text{and} \quad \delta d \ll \delta j. \tag{22}$$

An important point is that condition (21) or (22) needs to be verified only at small scales: large scales can still possess large dilatational components. Besides, conditions (21) and (22) are verified whenever δv^d and δd decrease sufficiently fast when $r \to 0$, in particular when they decrease faster than their total counterparts. In that case, the inertial range may be divided in two ranges by a limit scale r_s: below r_s, condition (21) or (22) is satisfied while above r_s they are not.

Simulations of subsonic and transsonic turbulence suggest that $\delta v^d \ll \delta v$ on most of the inertial range [52], with $\delta v^d \sim r^{1/2}$ and $\delta v \sim r^{1/3}$. Simulations of supersonic turbulence [29,48,55] may be compatible with the idea that $\delta v^d \ll \delta v$ but only below the sonic scale. We are not aware of simulation results showing the behavior of the solenoidal and dilatational components of the momentum.

In any case, whenever the weakly dilatational regime defined by Eq. (21) and possibly by Eq. (22) is reached, the Monin-Yaglom equation (16) can be simplified as

$$\partial_{r_j}\overline{\Delta v_j \Delta j_i \Delta j_i} = \partial_{r_j}\overline{\Delta v_j^s \Delta s_i \Delta s_i} = -4\mathscr{E}_{\text{inj}}. \tag{23}$$

This relation possibly applies to subsonic and transsonic turbulence and below the sonic scale for supersonic turbulence.

Another approximation of Eq. (16) can also be formulated, always based on the idea that dilatational contributions can be neglected. In [14,15] Aluie discarded the correlation between the pressure and velocity divergence by assuming that their cospectrum decays as $k^{-\beta}$ with $\beta > 1$. In terms of increments, Aluie made the assumption that $\overline{\Delta p \Delta \Theta} \propto r^{(\beta-1)/2}$, so that $\overline{\Delta p \Delta \Theta} \to 0$ when $r \to 0$. The same hypothesis can also be applied to the version (18) of the present Monin-Yaglom relation. As in [14,15], we can assume that the dilatational source terms $\overline{\Delta \Theta \Delta (j_i j_i)}$ and $\overline{\Delta \Xi \Delta \pi^s}$ go to 0 when $r \to 0$. In that case, the Monin-Yaglom relation (18) is approximated by

$$\partial_{r_j}\overline{\Delta v_j \Delta j_i \Delta j_i} = -4\overline{\varepsilon_c}. \tag{24}$$

The two approximations (23) and (24) coincide provided $\mathscr{E}_{\text{inj}} = \overline{\varepsilon_c}$. This occurs when $\overline{\mathcal{P}_c} \ll \overline{\varepsilon_c}$. This condition can be met when dilatational motions are negligible not only at small scales but also on the whole spectrum and in particular, when the dilatational forcing is weak. In that case, the circulicity injected at large scales is equal to the circulicity dissipated at small scales. Equations (23) and (24) are then indicative of a circulicity cascade.

F. Inertial range scalings in the weakly dilatational regime

In incompressible turbulence, the classical Monin-Yaglom relation can be used to determine the scaling of velocity increments [5]. To this end, it is assumed that velocity increments obey a self-similar scaling such that $\Delta v_i \overset{\text{law}}{\propto} r^{n_v}$. In particular, the velocity structure function of order p verifies

$$\overline{|\Delta \boldsymbol{v}|^p} \propto r^{n_v p}. \tag{25}$$

The incompressible Monin-Yaglom relation then constrains n_v to be equal to $1/3$. It also implies that the prefactor of equation (25) is the solenoidal dissipation $\overline{\varepsilon}^{1/3}$. These predictions must of course be corrected to account for internal intermittency when necessary [5].

In Sec. III E we argued that below a sonic-like scale, under conditions (22), the Monin-Yaglom relation (16) could be approximated by Eq. (23). In addition to this approximation, we assume that the increments of the different flow variables obey a self-similar scaling as in incompressible turbulence. For a given variable X, we write

$$\Delta X \overset{\text{law}}{\propto} r^{n_X} \tag{26}$$

where n_X is the power-law exponent associated with the variable X. Then the Monin-Yaglom relation (23) forces the relation

$$n_v + 2n_j = 1.$$

Furthermore, we conjecture that $\Delta\rho \to 0$ as $r \to 0$ and more specifically that $n_\rho \geqslant n_v$. Then, using the relation $\Delta(\rho X) = \Delta\rho\{X\}_{\frac{1}{2}} + \{\rho\}_{\frac{1}{2}}\Delta X$, one deduces that $n_j = n_v$. Therefore, with this set of assumptions, one arrives at the conclusion that

$$n_v = n_j = \tfrac{1}{3}.$$

This is the usual Kolmogorov-Obukhov scaling of the velocity increment, corresponding to a $k^{-5/3}$ spectrum. Note that the self-similarity assumption (26) does not allow us to separate the exponents of the different components of the velocity and momentum fields. Given the weak dilatational assumption, which is summed by Eq. (22), the common value of n_j and n_v is that of the principal components of these fields, it is to say, their solenoidal components. The dilatational components can have a different exponent as long as it is larger than $1/3$.

There is also another important conclusion that can be derived from the approximate Monin-Yaglom relation in the weakly dilatational regime. The sole dimensioning parameter that appears in this law is $\overline{\varepsilon_c}$, which is a dissipation rate based on solenoidal fields and on the sole shear viscosity μ. Therefore, one can refine the similarity assumption for the velocity and momentum fields and write that

$$\Delta v_i \overset{\text{law}}{\propto} [(\overline{\varepsilon_c}/\overline{\rho}^2)\,r]^{1/3} \quad \text{and} \quad \Delta j_i \overset{\text{law}}{\propto} [(\overline{\rho}\,\overline{\varepsilon_c})\,r]^{1/3} \tag{27}$$

This prediction is reminiscent of several simulations where turbulent velocity spectra with a Kolmogorov-Obukhov scaling are seen to depend on the solenoidal dissipation rate of energy and not the total one [52,56].

G. Inertial range scalings with shocks

Let us assume that a homogeneous distribution of shocks is present in the flow. In that case, the self-similarity assumption (26) cannot be upheld. In particular, Lindborg [57] showed that when dissipation occurs exclusively in shocks, the velocity field does not verify Eq. (25). Instead, one has

$$\overline{|\Delta\boldsymbol{v}|^p} = \langle|\Delta\boldsymbol{v}|^p\rangle_{\text{sh}}\frac{r}{d_{\text{sh}}}, \tag{28}$$

where d_{sh} is the mean distance between shocks and $\langle|\Delta\boldsymbol{v}|^p\rangle_{\text{sh}}$ is the average jump of $|\Delta\boldsymbol{v}|^p$ across shocks.

If we want to analyze scalings compatible with Eq. (16) we must consequently account for these two distinct behaviours. To this end, we propose to separate the average of two-point correlations in order to make shock contributions explicit. For a given quantity $X(\boldsymbol{x}, \boldsymbol{x}')$ depending on the two positions $\boldsymbol{x}$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$, we split its average as

$$\overline{X}(\boldsymbol{r}) = P_{\text{sh}}(\boldsymbol{r})\overline{X}|_{\text{sh}}(\boldsymbol{r}) + [1 - P_{\text{sh}}(\boldsymbol{r})]\overline{X}|_{\text{ns}}(\boldsymbol{r}), \tag{29}$$

where P_{sh} is the probability of finding a shock between two points separated by $\boldsymbol{r}$ and where $\overline{X}|_{\text{sh}}$ and $\overline{X}|_{\text{ns}}$ are the averages of X conditioned respectively on the event of finding or not a shock between two points separated by $\boldsymbol{r}$.

Then the Monin-Yaglom relation (16) can be written as

$$\partial_{r_j}\big[P_{\text{sh}}\big(\overline{\Delta v_j \Delta(\rho v_i)\Delta(\rho v_i)}|_{\text{sh}} - 2\overline{\Delta v_j\{\rho^2 v_i v_i\}_{\frac{1}{2}}}\big|_{\text{sh}} - 4\overline{\Delta d_i\{\rho v_i v_j\}_{\frac{1}{2}}}\big|_{\text{sh}}\big)\big]$$
$$+ \partial_{r_j}\big[(1 - P_{\text{sh}})\big(\overline{\Delta v_j \Delta(\rho v_i)\Delta(\rho v_i)}|_{\text{ns}} - 2\overline{\Delta v_j\{\rho^2 v_i v_i\}_{\frac{1}{2}}}\big|_{\text{ns}} - 4\overline{\Delta d_i\{\rho v_i v_j\}_{\frac{1}{2}}}\big|_{\text{ns}}\big)\big] = -4\mathscr{E}_{\text{inj}}. \tag{30}$$

This equation only makes explicit the presence of shocks but does not introduce any additional assumption compared to the Monin-Yaglom relation (16) from which it is derived. However, if we want to interpret it further, some simplifications must be made. First, for any given quantity $X(\boldsymbol{x}, \boldsymbol{x}')$ with $\boldsymbol{x}$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$, we assume that $\overline{X}|_{\text{sh}}$ is dominated by the shock contribution. In that case, we

can write that

$$\overline{X}|_{\text{sh}}(\boldsymbol{r}) = \langle X \rangle_{\text{sh}},$$

where $\langle X \rangle_{\text{sh}}$ represents the contribution of the shock jumps to X, averaged over all shocks. This contribution is independent from $\boldsymbol{r}$. Second, we assume that nonshocked contribution to increments obeys a power law. Thus, we reduce Eq. (26) to the nonshocked intervals:

$$\Delta X|_{\text{ns}} \overset{\text{law}}{\propto} r^{n_X}.$$

Finally, we consider that the spatial repartition of shocks follows an isotropic Poisson distribution with a mean distance d. As a result, as in [57], one has $P_{\text{sh}} \approx \frac{r}{d}$ for $r \ll d$ and $1 - P_{\text{sh}} \approx 1$. With these approximations, the Monin-Yaglom relation (30) becomes

$$\partial_{r_j}\left(\overline{\Delta v_j \Delta(\rho v_i)\Delta(\rho v_i)}|_{\text{ns}} - 2\overline{\Delta v_j \{\rho^2 v_i v_i\}_{\frac{1}{2}}}|_{\text{ns}} - 4\overline{\Delta d_i \{\rho v_i v_j\}_{\frac{1}{2}}}|_{\text{ns}}\right) = -4(\mathscr{E}_{\text{inj}} - \mathcal{E}_{\text{sh}})$$

$$\text{with} \quad \mathcal{E}_{\text{sh}} = -\frac{1}{4d_{\text{sh}}}\left(\langle \Delta v_\parallel \Delta(\rho v_i)\Delta(\rho v_i)\rangle_{\text{sh}} - 2\langle \Delta v_\parallel \{\rho^2 v_i v_i\}_{\frac{1}{2}}\rangle_{\text{sh}} - 4\langle \Delta d_i \{\rho v_i v_\parallel\}_{\frac{1}{2}}\rangle_{\text{sh}}\right). \quad (31)$$

If we except the particular case $\mathscr{E}_{\text{inj}} = \mathcal{E}_{\text{sh}}$, this formula can only be fulfilled if at least one of the terms in the divergence on the left-hand side scales as r. Given the scaling laws assumed in the nonshocked region, this condition implies that at least one of this condition must be verified:

$$n_v + 2n_{\rho v} \leqslant 1, \quad n_v + n_{\rho^2 v^2} \leqslant 1, \quad n_d + n_{\rho v^2} \leqslant 1. \quad (32)$$

We note that, since $\overline{\Delta(\rho v_i)\Delta(\rho v_i)} = \overline{\Delta s_i \Delta s_i} + \overline{\Delta d_i \Delta d_i}$, one must have $n_d \geqslant n_{\rho v}$. Furthermore, if we assume that $\Delta \rho \to 0$ as $r \to 0$ and more specifically that $n_\rho \geqslant n_v$, then, using the relation $\Delta(\rho X) = \Delta \rho \{X\}_{\frac{1}{2}} + \{\rho\}_{\frac{1}{2}}\Delta X$, one can show that $n_{\rho v} = n_v$, $n_{\rho^2 v^2} = n_{v^2}$ and $n_{\rho v^2} = n_{v^2}$. Finally, the Schwartz relation $\overline{\Delta(v_i v_i)^2} \leqslant 2\sqrt{(\{v_j\}_{\frac{1}{2}}\{v_j\}_{\frac{1}{2}})^2}\sqrt{(\Delta v_i \Delta v_i)^2}$ implies that $n_{v^2} \geqslant n_v$.

From there, we finally arrive at the conclusion that the first inequality of Eq. (32) can be verified if $n_v \leqslant 1/3$, while the two others require $n_v \leqslant 1/2$. This can be summed up as

$$\text{Nonvanishing } \overline{\Delta v_j \Delta(\rho v_i)\Delta(\rho v_i)}|_{\text{ns}}: \quad n_v \leqslant \tfrac{1}{3}, \quad (33a)$$

$$\text{Nonvanishing } -2\overline{\Delta v_j \{\rho^2 v_i v_i\}_{\frac{1}{2}}}|_{\text{ns}} + 4\overline{\Delta d_i \{\rho v_i v_j\}_{\frac{1}{2}}}|_{\text{ns}}: \quad n_v \leqslant \tfrac{1}{2}. \quad (33b)$$

The upper bound of the first constraint corresponds to the usual Kolmogorov-Obukhov scaling of the velocity increment, corresponding to a $k^{-5/3}$ spectrum. The upper bound of the second corresponds to a velocity spectrum scaling as k^{-2}. The first scaling is traditionally observed in simulations with moderate Mach numbers and with solenoidal forcing [52]. By contrast, in simulations of supersonic turbulence, a k^{-2} spectrum is obtained for scales larger than the sonic scale. Below this scale, a shallower spectrum with a scaling close to $k^{-5/3}$ is observed [29,55]. We recall that these scalings are derived for the nonshocked regions of the flow. Given the decomposition $\overline{\Delta v_i \Delta v_i} = P_{\text{sh}}\overline{\Delta v_i \Delta v_i}|_{\text{sh}} + (1 - P_{\text{sh}})\overline{\Delta v_i \Delta v_i}|_{\text{ns}}$, we note that the k^{-2} spectrum observed in simulations of supersonic turbulence can have two contributions. The first comes from the presence of shocks, as in Burgers turbulence. The other comes from the dilatational components of the velocity and momentum, shock excluded. The role of this second contribution was also highlighted in [29].

To conclude on this topic, we would like to recall that without solenoidal forcing, when $\mathscr{E}_{\text{inj}} = \overline{j_i f_i^s} = 0$, an equilibrium takes place in the inertial range. Equation (31) shows that this equilibrium can be understood as a balance between the shocked and nonshocked regions of the flow. In particular, if $\mathcal{E}_{\text{sh}} > 0$, shocks lead to a forward transfer of circulicity from large to small scales, while a backward transfer occurs for nonshocked regions.

IV. COARSE-GRAINING APPROACH

In this section, we return to discussing the existence of a circulicity cascade. However, instead of the point-splitting methodology which led us to derive the Monin-Yaglom equation (16), we now use the coarse-graining approach which was initially propelled by Onsager's work [6,7]. This approach does not require the flow to be homogeneous, and unless said otherwise, we drop this hypothesis. As a whole, we closely follow the works of Eyink and Drivas and Aluie [14–18].

A. Resolved and subscale circulicities

The basic tool of the coarse-graining approach is spatial filtering. For a given variable $X(\boldsymbol{x}, t)$, one defines its filtered value by

$$\langle X \rangle_\ell(\boldsymbol{x}, t) = \int G_\ell(\boldsymbol{r}) X(\boldsymbol{x} + \boldsymbol{r}, t) \, d\boldsymbol{r}, \tag{34}$$

where $G_\ell(\boldsymbol{r}) = G(\boldsymbol{r}/\ell)\ell^{-3}$ is a filter function with a kernel G rapidly decreasing at infinity and verifying the normalization condition $\int G(\boldsymbol{r}) \, d\boldsymbol{r} = 1$. The idea is then to look at the properties of the filtered circulicity $\langle \mathcal{C} \rangle_\ell$. More precisely, $\langle \mathcal{C} \rangle_\ell$ can be split into a resolved and subscale circulicity:

$$\langle \mathcal{C} \rangle_\ell = \mathcal{C}\big|_\ell^{\text{res}} + \mathcal{C}\big|_\ell^{\text{sub}} \quad \text{with} \quad \mathcal{C}\big|_\ell^{\text{res}} = \tfrac{1}{2}\langle s_i \rangle_\ell \langle s_i \rangle_\ell \quad \text{and} \quad \mathcal{C}\big|_\ell^{\text{sub}} = \tfrac{1}{2}\tau(s_i, s_i),$$

where we used the notation $\tau(X, Y)$ to denote the subscale moment of given quantities X and Y:

$$\tau(X, Y) = \langle XY \rangle_\ell - \langle X \rangle_\ell \langle Y \rangle_\ell.$$

The evolution of the resolved circulicity $\mathcal{C}|_\ell^{\text{res}}$ can be deduced by filtering the evolution equation (9a) of $\boldsymbol{s}$ and multiplying it by $\langle \boldsymbol{s} \rangle_\ell$. As for the subscale circulicity, its evolution can be derived by filtering the circulicity equation (10) and substracting the resolved part. By doing so, we obtain that

$$\partial_t \mathcal{C}\big|_\ell^{\text{res}} + \partial_j \mathcal{F}_j^c\big|_\ell^{\text{res}} = -\Pi_\ell - \pi_\ell^{f\mu} + \mathcal{P}_c\big|_\ell^{\text{res}} + \langle s_i f_i^s \rangle_\ell, \tag{35a}$$

$$\partial_t \mathcal{C}\big|_\ell^{\text{sub}} + \partial_j \mathcal{F}_j^c\big|_\ell^{\text{sub}} = \Pi_\ell + \pi_\ell^{f\mu} + \mathcal{P}_c\big|_\ell^{\text{sub}} - \langle \varepsilon_c \rangle_\ell. \tag{35b}$$

In these equations, $\mathcal{F}^c|_\ell^{\text{res}}$ and $\mathcal{F}^c|_\ell^{\text{sub}}$ are the resolved and subscale circulicity fluxes. We recall that the local circulicity flux appearing in Eq. (10) is noted $\mathcal{F}^c$. If we mark symbolically the dependency of this flux upon the flow variables as $\mathcal{F}^c \equiv \mathcal{F}^c[X]$, then we have $\mathcal{F}_j^c|_\ell^{\text{res}} = \mathcal{F}_j^c[\langle X \rangle_\ell] + \langle j_i \rangle_\ell \tau(j_i, v_j)$ and $\mathcal{F}_j^c|_\ell^{\text{sub}} = \langle \mathcal{F}_j^c[X] \rangle_\ell - \mathcal{F}_j^c[\langle X \rangle_\ell] - \langle j_i \rangle_\ell \tau(j_i, v_j)$. These two fluxes do not play any role in the forthcoming analysis. The term $\pi_\ell^{f\mu}$ combines a filtered scale dissipation and the subscale contribution of the force $\boldsymbol{f}^s$. It is defined as $\pi_\ell^{f\mu} = \tau(s_i, fi^s) + \mu \langle \Omega_i \rangle_\ell \langle \omega_i \rangle_\ell$. Like the fluxes, the role of $\pi_\ell^{f\mu}$ is inessential since it vanishes in the high Reynolds limit and with the assumption that the forcing is at large scale. The last three terms that need to be defined, $\mathcal{P}_c|_\ell^{\text{res}}$, $\mathcal{P}_c|_\ell^{\text{sub}}$, and Π_ℓ, are the ones playing a central role in the present analysis. The first two correspond to the resolved and subscale contributions of the circulicity production term $\mathcal{P}_c$. More precisely, we have

$$\langle \mathcal{P}_c \rangle_\ell = \mathcal{P}_c\big|_\ell^{\text{res}} + \mathcal{P}_c\big|_\ell^{\text{sub}} \tag{36a}$$

$$\text{with } \mathcal{P}_c\big|_\ell^{\text{res}} = -\tfrac{1}{2}|\langle j_i \rangle_\ell \langle j_i \rangle_\ell|^2 \langle \Theta \rangle_\ell + \langle \pi^s \rangle_\ell \langle \Xi \rangle_\ell \tag{36b}$$

$$\text{and } \mathcal{P}_c\big|_\ell^{\text{sub}} = -\tfrac{1}{2}[\tau(j_i j_i, \Theta) + \langle \Theta \rangle_\ell \tau(j_i, j_i)] + \tau(\pi^s, \Xi). \tag{36c}$$

The third term is defined by

$$\Pi_\ell = -\tau(v_j, j_i)\partial_j \langle j_i \rangle_\ell. \tag{37}$$

This term consists in a subscale stress multiplying the gradient of a filtered quantity and marks the interaction between resolved and unresolved scales. It appears as a sink in the resolved circulicity

equation (35a) and as a source in the subscale circulicity equation (35b). Thus, Π_ℓ can be interpreted as a nonlinear exchange term between the resolved and subscale circulicities.

Note that if we introduce a density-weighted filter and density-weighted subscale moments as $\widetilde{X}|_\ell = \langle \rho X \rangle_\ell / \langle \rho \rangle_\ell$ and $\widetilde{\tau}(X, Y) = \widetilde{XY}|_\ell - \widetilde{X}|_\ell \widetilde{Y}|_\ell$, then the exchange term Π_ℓ can also be written as

$$\Pi_\ell = -\langle \rho \rangle_\ell^2 \widetilde{\tau}(v_i, v_j)\, \partial_j \widetilde{v}_i|_\ell - \tau(\rho, v_j) \partial_j(\langle \rho \rangle_\ell \, \widetilde{v}_i|_\ell \widetilde{v}_i|_\ell) - \tau(v_j, j_i) \widetilde{v}_i|_\ell \partial_j \langle \rho \rangle_\ell. \tag{38}$$

Up to a density factor, the first term on the right-hand side is the deformation work that plays a central role in the transfer of kinetic energy between scales [14–17]. Besides, the second term is similar to the "baropycnal work" which also appears as a means to transfer kinetic energy. However, the actual pressure p which enters the definition of the "baropycnal work" is here replaced with a dynamic one: $\langle \rho \rangle_\ell \widetilde{v}_i|_\ell \widetilde{v}_i|_\ell$. Finally, the last term has no direct equivalent in the transfer of kinetic energy. This comparison between Π_ℓ and the transfer of kinetic energy reinforces the idea that Π_ℓ is an exchange term between resolved and unresolved scales. Furthermore, it indicates that some of the mechanisms at work in the transfer of energy and circulicity are similar.

To follow on this comparison, we note that the exchange term between the resolved and subscale kinetic energies has been a staple of the analysis of subgrid-scale models for large-eddy simulations (LES) for more than five decades (see [58,59] and references therein). Its link with the Kolmogorov-Obukhov spectrum and with the Monin-Yaglom relation has long been discussed [58,59]. However, its use as a proper theoretical tool able to demonstrate the existence of a cascading process seems to have been only recognised more recently, most notably with Refs. [14–17].

B. Circulicity cascade

We now consider the high Reynolds limit of the resolved and subscale circulicities, with the added assumption that the force only acts at large scales. Taking the limit $\mu \to 0$ in Eqs. (35a) and (35b), and discarding the subscale force contribution, we obtain that

$$\partial_t \mathcal{C}\big|_\ell^{\mathrm{res}} + \partial_j \mathcal{F}_j\big|_\ell^{\mathrm{res}} = -\Pi_\ell^* - \mathcal{P}_c\big|_\ell^{\mathrm{sub}*} + \langle \mathcal{P}_c^* \rangle_\ell + \langle s_i f_i^s \rangle_\ell, \tag{39a}$$

$$\partial_t \mathcal{C}\big|_\ell^{\mathrm{sub}} + \partial_j \mathcal{F}_j\big|_\ell^{\mathrm{sub}} = \Pi_\ell^* + \mathcal{P}_c\big|_\ell^{\mathrm{sub}*} - \langle \varepsilon_c^* \rangle_\ell, \tag{39b}$$

where we substituted $\mathcal{P}_c|_\ell^{\mathrm{res}}$ with $\langle \mathcal{P}_c \rangle_\ell - \mathcal{P}_c|_\ell^{\mathrm{sub}}$ and where

$$\Pi_\ell^* = \lim_{\mu \to 0} \Pi_\ell, \quad \varepsilon_c^* = \lim_{\mu \to 0} \varepsilon_c, \quad \mathcal{P}_c\big|_\ell^{\mathrm{sub}*} = \lim_{\mu \to 0} \mathcal{P}_c\big|_\ell^{\mathrm{sub}} \quad \text{and} \quad \mathcal{P}_c^* = \lim_{\mu \to 0} \mathcal{P}_c.$$

These limits are taken in a weak distributional sense. For instance, one has

$$\mathcal{P}_c^* = -\tfrac{1}{2}|\boldsymbol{j}|^2 * \Theta + \pi^s * \Xi,$$

where the operator $*$ stands for the distributional product in the limit $\mu \to 0$. The reason for introducing this distributional meaning is explained in [17]. It comes from the fact that ρ and $\boldsymbol{v}$ become nonsmooth when $\mu \to 0$. Hence, $\Theta = \partial_j u_j$ and $\Xi = \partial_j(\rho u_j)$ exhibit divergences that can only be described by distributions. As a result, products such as $|\boldsymbol{j}|^2 \Theta$ and $\pi^s \Xi$ behave as the products of Heaviside functions with Dirac distributions. Their values are ill-posed in the sense that they depend on the particular regularizing path chosen to describe them [60,61]. This regularizing path does not need to be made explicit for our purpose. It is sufficient to know that it depends on the physics of the problem and on the particular way the limit $\mu \to 0$ is approached.

To discuss the existence of a circulicity cascade, we make two crucial hypotheses, similar to those made in [17]. The first one is that dissipative anomalies exist and that the dissipation of circulicity ε_c has a nonzero finite value in the infinite Reynolds limit:

$$\varepsilon_c^* \neq 0. \tag{40}$$

The second is that in the limit $\ell \to 0$, the evolution equation (39a) of $\mathcal{C}|_\ell^{\mathrm{res}}$ tends, in the sense of distributions, to the evolution equation of $\mathcal{C}$ in the ideal limit $\mu \to 0$. The limit of Eq. (39a) can be

written as follows:

$$\partial_t \mathcal{C} + \partial_j \mathcal{F}_j^c = \mathcal{P}_c^* + s_i f_i^s - \Pi_0^* - \mathcal{P}_c \big|_0^{\text{sub}*}, \tag{41}$$

with $\Pi_0^* = \lim_{\ell \to 0} \Pi_\ell^*$ and $\mathcal{P}_c|_0^{\text{sub}*} = \lim_{\ell \to 0} \mathcal{P}_c|_\ell^{\text{sub}*}$. The evolution of $\mathcal{C}$ in the ideal limit is deduced from Eq. (10) and can be written as

$$\partial_t \mathcal{C} + \partial_j \mathcal{F}_j^c = \mathcal{P}_c^* + s_i f_i^s - \varepsilon_c^*. \tag{42}$$

The reason for the identity between these two limit equations is discussed in [17] (p. 10, second paragraph) and comes for the principle that "objective physical facts such as the rate of decay of energy . . . cannot depend upon an arbitrary scale ℓ." One may expect that this "objectivity principle" also applies to circulicity, its dissipation and injection rates. Comparing Eqs. (42) and (41), one deduces that

$$\Pi_0^* + \mathcal{P}_c \big|_0^{\text{sub}*} = \lim_{\ell \to 0} \lim_{\mu \to 0} \Pi_\ell + \mathcal{P}_c \big|_\ell^{\text{sub}} = \varepsilon_c^*. \tag{43}$$

This equality is the main result of this section. It states that the circulicity transferred nonlinearly from resolved to subscales (Π_0^*) summed to the one produced at subscales ($\mathcal{P}_c|_0^{\text{sub}*}$) does not vanish in the limit $\mu \to 0$, $\ell \to 0$: it tends to a nonzero finite value independent from ℓ and equal, in the sense of distributions, to the dissipation ε_c^*. Equation (43) does not assert the existence of a circulicity cascade. Instead, it expresses a balance between transfer, production and dissipation of circulicity in the subscale range. Since $\varepsilon_c^* \neq 0$, this balance is anomalous and both the transfer and subscale production of circulicity may be the source of the anomalous dissipation ε_c^*. A similar result has already been obtained for the kinetic energy: in [17], it was shown that the anomalous dissipation of energy could have two origins, an energy cascade and a pressure-dilatation defect akin to a subscale energy production. Both elements are similar to the mechanisms identified in this work for circulicity.

Equation (43) has been cast in the form of a subscale balance. It can also be used to express a balance for resolved scales. Indeed, using Eq. (36), one can show that

$$\mathcal{P}_c \big|_0^{\text{res}*} - \Pi_0^* = \lim_{\ell \to 0} \lim_{\mu \to 0} \mathcal{P}_c \big|_\ell^{\text{res}} - \Pi_\ell = \mathcal{P}_c^* - \varepsilon_c^*. \tag{44}$$

This equation shows that, in the limit $\mu \to 0$, $\ell \to 0$, the circulicity transferred to small scales and produced at resolved scaled tends to a finite value equal to the difference between the production and dissipation of circulicity.

Equations (43) and (44) can be understood as the coarse-grained equivalents of the Monin-Yaglom relations (18) and (16). However, as opposed to a Monin-Yaglom relation, Eqs. (43) and (44) are not statistical and do not involve an ensemble mean or, if ergodicity applies, an integration over a spatial domain. Instead, they are written in a weak distributional sense. In this regard, it is worth stressing that Eq. (43) does not imply that $\Pi_0^* + \mathcal{P}_c|_0^{\text{sub}*} = \varepsilon_c^*$ locally and instantly. As a weak equality, it involves spatial integrations with test functions. When seen under this light, equality (43) has a meaning which is indeed stronger that the one given to a Monin-Yaglom relation, but, at the same time, which is maybe not so far removed from it. In the same line of thought, it should also be mentioned that in a real flow, with finite μ and ℓ, the measurable quantities which give rise to equality (43) are Π_ℓ, $\mathcal{P}_c|_\ell^{\text{sub}}$, and $\langle \varepsilon_c \rangle_\ell$ which appear in the right-hand side of Eq. (35b). The left-hand side of this equation has no reason to be equal to 0 so that the values of $\Pi_\ell + \mathcal{P}_c|_\ell^{\text{sub}}$ and $\langle \varepsilon_c \rangle_\ell$ have no reason to be equal locally and instantaneously. Again, this is not what Eq. (43) is meant to say. The weak distributional sense of this equation should always be kept in mind in order to avoid overinterpreting it. The same also applies to Eq. (44).

Another remark worth making about equalities (43) and (44) comes from the fact that a double limit is taken, first $\mu \to 0$ and then $\ell \to 0$. In doing so, Eyink and Drivas [17,18] noted that several distributional products appear. For instance, the contribution $\tau(\pi^s, \Xi)$ of $\mathcal{P}_c|_\ell^{\text{sub}}$ is equal to $\langle \pi \Xi \rangle_\ell -$

$\langle \pi \rangle_\ell \langle \Xi \rangle_\ell$. Taking the double limit, then leads to write

$$\lim_{\ell \to 0} \lim_{\mu \to 0} \tau(\pi^s, \Xi) = \pi^s * \Xi - \pi^s \circ \Xi.$$

In this equality, the symbol $\circ$ refers to the regularization of $\langle \pi \rangle_\ell \langle \Xi \rangle_\ell$ which is different from the one associated with the symbol $*$ and the regularization of $\langle \pi \, \Xi \rangle_\ell$. This double regularization arises because circulicity is not an inviscid invariant and its coarse-grained dynamics cannot be closed as fluxes. This is similar to the case of kinetic energy for which work done by fields which become distributions also lead to a double regularization [17,18].

To conclude this discussion of Eqs. (43) and (44), we would like to mention that a purely statistical version of these equalities can also be derived, without invoking the finiteness of ε_c^* and an "objectivity principle." Instead of these assumptions, one may assume that the flow is homogeneous and statistically stationary, and that the mean of ε_c^* is finite, i.e., that $\lim_{\mu \to 0} \overline{\varepsilon_c} \neq 0$, which is a weaker statement than $\varepsilon_c^* \neq 0$. Then, by averaging Eqs. (35b) and (35a), one deduces that, in the limit $\mu \to 0$ and $\ell \to 0$, the following equality stands:

$$\lim_{\ell \to 0} \lim_{\mu \to 0} \overline{\Pi_\ell} + \overline{\mathcal{P}_c\big|_\ell^{\text{sub}}} = \overline{\varepsilon_c} \quad \text{and} \quad \lim_{\ell \to 0} \lim_{\mu \to 0} \overline{\Pi_\ell} - \overline{\mathcal{P}_c\big|_\ell^{\text{res}}} = \overline{\varepsilon_c} - \overline{\mathcal{P}_c} = \mathscr{E}_{\text{inj}}. \tag{45}$$

These two relations can be obtained by averaging Eqs. (43) and (44). They can also be derived by filtering the Monin-Yaglom relations (18) and (16). Conditions under which $\overline{\mathcal{P}_c|_\ell^{\text{sub}}}$ and $\overline{\mathcal{P}_c|_\ell^{\text{res}}}$ could be neglected compared to $\overline{\Pi_\ell}$ were given in Sec. III E. They are met in a so-called weakly dilatational regime. In that case, the mean transfer of circulicity $\overline{\Pi_\ell}$ becomes independent of ℓ and tends to a finite nonzero value: a circulicity cascade takes place. Otherwise, a mean transfer-production-dissipation equilibrium occurs in the inertial range in lieu of this sole cascade. Still, in Sec. III C it was shown that this mean equilibrium could be interpreted as an effective cascade by recasting the source term in the form of a flux-like transfer term.

C. Increment scalings and locality

There are two ways by which the anomalous dissipation ε_c^* can be sustained: one is the transfer term Π_ℓ, the other is the subscale production $\mathcal{P}_c|_\ell^{\text{sub}}$. The orders of magnitude of these terms are linked to the space increments of the velocity, density, and momentum fields. The order of magnitude of the increment $\Delta X(\boldsymbol{x}, \boldsymbol{r})$ of a quantity X is noted $\delta X(r)$. As discussed in [14,17,54], one has

$$\tau(X, Y) = O(\delta X \delta Y), \quad \partial_i \langle X \rangle_\ell = O\left(\frac{\delta X}{\ell}\right), \quad X - \langle X \rangle_\ell = O(\delta X).$$

Using these relations, it is straightforward to show from Eq. (37) that

$$\Pi_\ell = O\left(\frac{\delta v (\delta j)^2}{\ell}\right).$$

One may also use Eq. (38) and the developments of Aluie [14] to show that this expression can be simplified as

$$\Pi_\ell = O\left(\frac{(\delta v)^3}{\ell}\right) \rho^2 \left[1 + O\left(\frac{\delta \rho}{\rho}\right) + O\left(\frac{\delta \rho^2}{\rho^2}\right)\right]. \tag{46}$$

From this estimate, one deduces that the transfer term does not vanish in the limit $\ell \to 0$ provided the velocity field is rough enough. More precisely, let us assume that the order of magnitude δX of the increment a quantity X obeys a power law:

$$|\delta X(\ell)^n| \propto \ell^{\zeta_n^X}.$$

Then, if $\zeta_n^\rho > 0$, i.e., if $\delta\rho/\rho \to 0$ when $\ell \to 0$, Eq. (46) implies that Π_ℓ is nonvanishing provided

$$\zeta_n^v \leqslant \frac{n}{3} \quad \text{for } n \geqslant 3. \tag{47}$$

This is a necessary condition for Π_ℓ to contribute to the anomalous dissipation ε_c^*. This condition is identical to the one derived in [17] for the energy cascade. This is not entirely unexpected: the resolved to unresolved exchange terms of circulicity and energy both involve the deformation work as a common mechanism.

We now turn our attention to the subscale circulicity production $\mathcal{P}_c|_\ell^{\text{sub}}$ defined in Eq. (36). This term has three components involving $\Theta = \partial_j v_j = \partial_j v_j^d$ and $\Xi = \partial_j j_j = \partial_j d_j$. By comparing the right- and left-hand sides of the relation $\partial_i \langle \Theta \rangle_\ell = \partial_{ij}^2 \langle v_j^d \rangle_\ell$, one deduces that $\delta\Theta/\ell \sim \delta v^d/\ell^2$ so that $\delta\Theta \sim \delta v^d/\ell$. Similarly, one derives that $\delta\Xi \sim \delta d/\ell$. Therefore, the three contributions of $\mathcal{P}_c|_\ell^{\text{sub}}$ lead to the following order of magnitude:

$$\mathcal{P}_c\big|_\ell^{\text{sub}} = O\left(\frac{\delta v^d \delta(j^2)}{\ell}, \quad \frac{\delta v^d (\delta j)^2}{\ell}, \quad \frac{\delta \pi^s \delta d}{\ell}\right).$$

The Poisson equation for the pressure (9c) also leads to $\delta\pi^s \sim \delta(vj)$. Thus, the conditions for the production to be nonvanishing are

$$\zeta_n^{v^d} + \zeta_n^{j^2} \leqslant 1, \quad \zeta_n^{v^d} + 2\zeta_n^j \leqslant 1, \quad \zeta_n^d + \zeta_n^{vj} \leqslant 1 \quad \text{for } n \geqslant 3.$$

If the dilatational fields v^d and d decay sufficiently rapidly then these conditions are not fulfilled. In that case, the transfer term Π_ℓ is the only one contributing to the anomalous dissipation. Thus, sufficient conditions for the existence of a circulicity cascade are

$$\zeta_n^{v^d} > 1 - \min\left(2\zeta_n^j, \zeta_n^{j^2}\right) \quad \text{and} \quad \zeta_n^d > 1 - \zeta_n^{vj}. \tag{48}$$

Other conditions may also lead to the fact that $\mathcal{P}_c|_\ell^{\text{sub}}$ vanishes in the limit $\mu \to 0$, $\ell \to 0$. For instance, d and v^d may simply be quantities belonging to the dissipative range that vanish when $\mu \to 0$. This is the case for v^d in the pseudocompressible limit studied in [42,62,63].

To conclude, the locality of the transfer term Π_ℓ obeys the same constraints as the locality of the energy transfer term studied in [14]. We recall that locality requires the velocity and density exponents to satisfy

$$\zeta_n^v < n \quad \text{and} \quad \zeta_n^\rho > 0.. \tag{49}$$

If equations (47), (48), and (49) are verified then a local conservative cascade of circulicity takes place in the inertial range.

V. CONCLUSION

Eyink and Drivas [17] noted that "it is not hard to find infinitely many anomalous balance relations in the ideal limit of turbulence, but most of them are not physically relevant and have no significant consequences." In this work, we started by raising a physical question: we asked whether a cascade of rotational motions (whirls) exist in the inertial range of compressible turbulence. This question was formulated in terms of a quantity called circulicity which carries information about the large scale angular momentum of the flow. We then used a point-splitting and a coarse-graining approach to analyze the properties of the circulicity. The two approaches gave similar results but allowed us to highlight different aspects. Concerning the point-splitting approach, we derived a Monin-Yaglom equation that differs from previously derived expressions [19–30]. In particular, it only involves quantities which can be reconstructed from the velocity and density fields, ρ and v. It does not require any knowledge about the energy or the pressure and is independent of the equation of state of the flow. It is also valid whether the flow is fully compressible or is in a quasi-incompressible limit. This Monin-Yaglom relation allowed us to show that an "effective"

cascade of circulicity exists when the flow is stirred with a solenoidal force, but not when the force is purely dilatational. Besides, it also allowed us to identify a weakly dilatational regime characterized by a Kolmogorov-Obukhov scaling and a scaling parameter given by a solenoidal dissipation. In the opposite regime, it was shown how this Monin-Yaglom relation could be interpreted when strong shocks are present. Finally, the practical relevance of this Monin-Yaglom was discussed.

The coarse-graining point of view brought additional information and allowed us to clarify several points. In particular, we were able to disentangle the role played by the actual transfer of circulicity and its subscale production. This allowed us to propose conditions under which a conservative and local circulicity cascade could take place.

APPENDIX A: ANGULAR MOMENTUM OF A EULERIAN CONTROL VOLUME

We consider a volume of fluid enclosed in sphere of radius R centered at a position $\boldsymbol{x}$. By definition, the angular momentum of this sphere with respect to its center is

$$L_i(\boldsymbol{x}, R, t) = \int_{\mathcal{V}} \epsilon_{ijk}(r_j - x_j)\,[\rho v_k(\boldsymbol{r}, t)]\, d^3\boldsymbol{r}, \tag{A1}$$

where $\int_{\mathcal{V}} \cdot\, d^3\boldsymbol{r}$ represents the integration over the volume of the sphere.

Now, if we let the radius R become infinitely small, we may Taylor-expand the integrand. We obtain

$$\text{for } R \to 0, \quad L_i(\boldsymbol{x}, R, t) = \int_{\mathcal{V}} \epsilon_{ijk}(r_j - x_j)[\rho v_k(\boldsymbol{x}, t) + (r_l - x_l)\partial_l(\rho v_k)(\boldsymbol{x}, t) + O(|\boldsymbol{r} - \boldsymbol{x}|^2)]\, d^3\boldsymbol{r}. \tag{A2}$$

The first term of the expansion vanishes by symmetry so that the main contribution to the angular momentum is given by the second term:

$$\text{for } R \to 0, \quad L_i(\boldsymbol{x}, R, t) = \frac{4\pi}{15} R^5 \epsilon_{ijk}\partial_j(\rho v_k)(\boldsymbol{x}, t) + O(R^7). \tag{A3}$$

Thus, as explained in the introduction, the angular momentum of a Eulerian control volume is defined with respect to the curl of the linear momentum $\rho\boldsymbol{v}$. When density is constant, for instance when $\rho(\boldsymbol{x}, t) = \rho_0$, then one has

$$\text{for } R \to 0 \quad \text{and} \quad \rho(\boldsymbol{x}, t) = \rho_0, \quad L_i(\boldsymbol{x}, R, t) = \frac{4\pi}{15}\rho_0 R^5 \epsilon_{ijk}\partial_j v_k(\boldsymbol{x}, t) + O(R^7). \tag{A4}$$

In that case, the angular momentum of the Eulerian control volume is set by the value of the vorticity field. This is why we wrote in the introduction that vorticity becomes a proxy for measuring angular momentum when density is constant.

Note that the inertial moment of the infinitesimal sphere with respect to an axis passing through its center is

$$I = \frac{8\pi}{15}\rho R^5. \tag{A5}$$

Thus, the main order of the angular momentum can be written as

$$\boldsymbol{L} = I\boldsymbol{\varpi} \quad \text{with} \quad \varpi_i = \frac{1}{2\rho}\epsilon_{ijk}\partial_j(\rho v_k). \tag{A6}$$

The quantity $\boldsymbol{\varpi}$ corresponds to the angular velocity of the sphere.

APPENDIX B: EVOLUTIONS OF THE ANGULAR MOMENTUM AND OF THE DILATATIONAL PART OF THE LINEAR MOMENTUM

The evolution of the angular momentum $\mathbf{\Omega} = \epsilon_{ijk}\partial_j(\rho v_k)$ is deduced from Eq. (1b), knowing that $\partial_j(\rho v_j v_k) = v_j\partial_k(\rho v_j) + \rho v_k\partial_j v_j - \epsilon_{kpq}v_p\Omega_q$:

$$\partial_t\Omega_i + v_j\partial_j\Omega_i = \Omega_j\partial_j v_i - \Omega_i\partial_j v_j + \epsilon_{ijk}\partial_k\left(\frac{\rho v_l v_l}{2}\right)\frac{\partial_j\rho}{\rho} - \epsilon_{ijk}\partial_j(\rho v_k\partial_l v_l) + \mu\partial_{jj}^2\omega_i + \epsilon_{ijk}\partial_j f_k^s.$$
(B1)

This equation can be compared against the vorticity equation

$$\partial_t\omega_i + v_j\partial_j\omega_i = \omega_j\partial_j v_i - \omega_i\partial_j v_j + \epsilon_{ijk}\frac{\partial_k p}{\rho}\frac{\partial_j\rho}{\rho} + \frac{\mu}{\rho}\partial_{jj}^2\omega_i + \epsilon_{ijk}\frac{\partial_l\sigma_{kl}}{\rho}\frac{\partial_j\rho}{\rho} + \epsilon_{ijk}\partial_j(f_k/\rho).$$
(B2)

We note that similar stirring terms are present in both equations, while the standard baroclinic torque acting in the vorticity equation is replaced by a torque involving the dynamic pressure $\rho|\mathbf{v}|^2/2$ in the angular momentum equation.

As for $\mathbf{d}$, its evolution is given by

$$\partial_t d_i = -\partial_i\pi^d - \partial_j\sigma_{ij}^d + f_i^d,$$
(B3)

$$\text{with}\quad \pi^d = p - \pi^s, \quad \sigma_{ij}^d = -\left(\eta + \tfrac{4}{3}\mu\right)\partial_k v_k\delta_{ij}, \quad f_i^d = -\partial_i\phi^f,$$

and $\partial_{jj}\phi^f = -\partial_j f_j$. This equation is readily integrated and shows that the scalar potential ϕ, such that $d_i = -\partial_i\phi$, is directly related to the dilatational pressure π^d in addition to forcing and viscous terms:

$$\phi(\mathbf{x}, t) = \phi_0(t) + \int_0^t \pi^d(\mathbf{x}, t')\,dt' + \int_0^t \phi^f(\mathbf{x}, t') - \left(\eta + \frac{4}{3}\mu\right)\partial_k v_k(\mathbf{x}, t')\,dt',$$
(B4)

with ϕ_0 an arbitrary function of time.

APPENDIX C: USEFUL RELATIONSHIPS FOR TWO-POINT CORRELATIONS

For some given quantities A, B, and C with homogeneous statistics, we have

$$\overline{\Delta A\Delta B} = 2\overline{AB} - \overline{AB'} - \overline{A'B},$$
(C1)

$$\overline{\Delta A\Delta B\Delta C} = \overline{A'(BC)} - \overline{A(BC)'} + \overline{B'(AC)} - \overline{B(AC)'} + \overline{C'(AB)} - \overline{C(AB)'},$$
(C2)

$$\overline{A'B} - \overline{AB'} = 2\overline{\Delta A\,\{B\}_{\frac{1}{2}}} = -2\overline{\Delta B\,\{A\}_{\frac{1}{2}}} = \overline{\Delta A\,\{B\}_{\frac{1}{2}}} - \overline{\Delta B\,\{A\}_{\frac{1}{2}}},$$
(C3)

$$\overline{\partial_i'A'B} - \overline{\partial_i AB'} = \partial_{r_i}(\overline{A'B} + \overline{AB'}) = -\partial_{r_i}\overline{\Delta A\Delta B}.$$
(C4)

If v_j satisfies $\partial_j v_j = \partial_j a_j$, we have

$$\partial_{r_j}(\overline{v_j'B'C} - \overline{v_j BC'}) = -\partial_{r_j}\overline{\Delta v_j\Delta B\Delta C} - \partial_{r_j}(\overline{v_j'BC'} - \overline{v_j B'C}) + \partial_{r_j}(\overline{a_j'BC} - \overline{a_j B'C'}),$$
(C5)

$$\partial_{r_j}(\overline{v_j'BB'} - \overline{v_j BB'}) = -\tfrac{1}{2}\partial_{r_j}\overline{\Delta v_j\Delta B^2} + \tfrac{1}{2}\partial_{r_j}(\overline{a_j'B^2} - \overline{a_j B'^2}).$$
(C6)

We also recall that homogeneity implies that

$$\overline{\partial_j'(\cdot)} = -\overline{\partial_j(\cdot)} = \partial_{r_j}(\overline{\,\cdot\,}).$$
(C7)

APPENDIX D: KÁRMÁN-HOWARTH EQUATION FOR THE MOMENTUM

The Kármán-Howarth equation for the linear momentum $\boldsymbol{j} = \rho \boldsymbol{v}$ can be found in [17]. Its expression, derived from Eq. (1b), is recalled here:

$$\partial_t \overline{\Delta j_i \Delta j_i}(\boldsymbol{r}, t) = -\partial_{r_j} \mathscr{F}_j^j(\boldsymbol{r}, t) - 4\mathscr{E}_{\text{inj}}^j(t) + \partial_{r_j r_k}^2 \mathcal{D}_{jk}^j(\boldsymbol{r}, t) + 2\overline{\Delta j_i \Delta f_i}, \tag{D1a}$$

with

$$\mathscr{F}_j^j(\boldsymbol{r}, t) = \overline{\Delta v_j \Delta j_i \Delta j_i} + 2\partial_{r_j}\left(\overline{\Delta \Phi \Delta p} - \frac{1}{2}\overline{\Delta \varphi^v \Delta(j_i j_i)}\right), \tag{D1b}$$

$$\mathscr{E}_{\text{inj}}^j(t) = -\left(\overline{\sigma_{ij} \partial_j j_i} + \overline{p \partial_j j_j} - \frac{1}{2}\overline{j_i j_i \partial_j v_j}\right), \tag{D1c}$$

$$\mathcal{D}_{jk}^j(\boldsymbol{r}, t) = 2\mu \overline{\Delta v_i \Delta j_i} \delta_{jk} + 2\left(\eta + \frac{\mu}{3}\right)\overline{\Delta v_j \Delta j_k}. \tag{D1d}$$

APPENDIX E: EXPRESSION OF THE SPECIES MASS FLUXES AND HEAT FLUX

Following [44–46], the species mass fluxes $\sigma^{(\alpha)}$ and heat flux $\sigma^{(e)}$ take the following form:

$$\boldsymbol{\sigma}^{(\alpha)} = \rho Y_\alpha \boldsymbol{\mathcal{V}}_\alpha \quad \text{with} \quad \boldsymbol{\mathcal{V}}_\alpha = -\sum_\beta D_{\alpha\beta}(\boldsymbol{d}_\beta + \chi_\beta \nabla \log T), \tag{E1a}$$

$$\boldsymbol{\sigma}^{(e)} = \sum_\beta h_\beta \boldsymbol{\sigma}^{(\beta)} - \lambda \nabla T + p \sum_\beta \chi_\beta \boldsymbol{\mathcal{V}}_\beta, \tag{E1b}$$

with $D_{\alpha\beta}$ the multicomponent diffusion coefficients, $\boldsymbol{d}_\beta$ the diffusion driving force, χ_β the species thermal diffusion ratio, h_β the species enthalpy and λ the thermal conductivity. The symbol ∇ refers to the gradient operator and there is no summation on repeated Greek indices (as opposed to repeated Latin indices in the main text). Note that all the coefficients of Eqs. (E1) are functions of density, mass fractions and temperature. Simplifications of Eqs. (E1) are given in [44–46], in particular for binary mixtures and for diluted species.

---

[1] L. F. Richardson, *Weather Prediction by Numerical Process* (Cambridge University Press, Cambridge, 1922).

[2] P. A. Davidson, Y. Kaneda, K. Moffatt, and K. R. Sreenivasan, *A Voyage through Turbulence* (Cambridge University Press, Cambridge, 2011).

[3] A. Alexakis and L. Biferale, Cascades and transitions in turbulent flows, Phys. Rep. **767–769**, 1 (2018).

[4] A. N. Kolmogorov, Local turbulence structure in incompressible viscous fluid at very high Reynolds numbers, Dokl. Akad. Nauk. SSSR **30**, 299 (1941).

[5] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

[6] L. Onsager, Statistical hydrodynamics, Nuovo Cimento **6**, 279 (1949).

[7] G. Eyink and K. Sreenivasan, Onsager and the theory of hydrodynamic turbulence, Rev. Mod. Phys. **78**, 87 (2006).

[8] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, Mass, 1975).

[9] G. I. Taylor, Statistical theory of turbulence, Proc. R. Soc. London A **151**, 421 (1935).

[10] A. N. Kolmogorov, On the degeneration of isotropic turbulence in an incompressible viscous fluid, Dokl. Akad. Nauk. SSSR **31**, 538 (1941).

[11] J. Panickacheril John, D. A. Donzis, and K. R. Sreenivasan, Does dissipative anomaly hold for compressible turbulence? J. Fluid Mech. **920**, A20 (2021).

[12] E.-W. Saw, D. Kuzzay, D. Faranda, A. Guittonneau, F. Daviaud, C. Wiertel-Gasquet, V. Padilla, and B. Dubrulle, Experimental characterization of extreme events of inertial dissipation in a turbulent swirling flow, Nat. Commun. **7**, 12466 (2016).

[13] B. Dubrulle, Beyond Kolmogorov cascades, J. Fluid Mech. **867**, P1 (2019).

[14] H. Aluie, Compressible Turbulence: The Cascade and Its Locality, Phys. Rev. Lett. **106**, 174502 (2011).

[15] H. Aluie, S. Li, and H. Li, Conservative cascade of kinetic energy in compressible turbulence, Astrophys. J. Lett. **751**, L29 (2012).

[16] H. Aluie, Scale decomposition in compressible turbulence, Physica D **247**, 54 (2013).

[17] G. Eyink and T. Drivas, Cascades and Dissipative Anomalies in Compressible Fluid Turbulence, Phys. Rev. X **8**, 011022 (2018).

[18] T. D. Drivas and G. L. Eyink, An Onsager singularity theorem for turbulent solutions of compressible Euler equations, Commun. Math. Phys. **359**, 733 (2018).

[19] G. Falkovich, I. Fouxon, and Y. Oz, New relations for correlation functions in Navier–Stokes turbulence, J. Fluid Mech. **644**, 465 (2010).

[20] S. Galtier and S. Banerjee, Exact Relation for Correlation Functions in Compressible Isothermal Turbulence, Phys. Rev. Lett. **107**, 134501 (2011).

[21] R. Wagner, G. Falkovich, A. Kritsuk, and M. Norman, Flux correlations in supersonic isothermal turbulence, J. Fluid Mech. **713**, 482 (2012).

[22] S. Banerjee and S. Galtier, Exact relation with two-point correlation functions and phenomenological approach for compressible magnetohydrodynamic turbulence, Phys. Rev. E **87**, 013019 (2013).

[23] A. Kritsuk, R. Wagner, and M. Norman, Energy cascade and scaling in supersonic isothermal turbulence, J. Fluid Mech. **729**, R1 (2013).

[24] S. Banerjee and S. Galtier, A Kolmogorov-like exact relation for compressible polytropic turbulence, J. Fluid Mech. **742**, 230 (2014).

[25] S. Banerjee and A. Kritsuk, Exact relations for energy transfer in self-gravitating isothermal turbulence, Phys. Rev. E **96**, 053116 (2017).

[26] S. Banerjee and A. Kritsuk, Energy transfer in compressible magnetohydrodynamic turbulence for isothermal self-gravitating fluids, Phys. Rev. E **97**, 023107 (2018).

[27] N. Andrés, F. Sahraoui, S. Galtier, L. Hadid, P. Dmitruk, and P. Mininni, Energy cascade rate in isothermal compressible magnetohydrodynamic turbulence, J. Plasma Phys. **84**, 905840404 (2018).

[28] N. Andrés, F. Sahraoui, S. Galtier, L. Hadid, R. Ferrand, and S. Huang, Energy Cascade Rate Measured in a Collisionless Space Plasma with MMS Data and Compressible Hall Magnetohydrodynamic Turbulence Theory, Phys. Rev. Lett. **123**, 245101 (2019).

[29] R. Ferrand, S. Galtier, F. Sahraoui, and C. Federrath, Compressible turbulence in the interstellar medium: New insights from a high-resolution supersonic turbulence simulation, Astrophys. J. **904**, 160 (2020).

[30] P. Simon and F. Sahraoui, General exact law of compressible isentropic magnetohydrodynamic flows: Theory and spacecraft observations in the solar wind, Astrophys. J. **916**, 49 (2021).

[31] W. Schmidt, C. Federrath, and R. Klessen, Is the Scaling of Supersonic Turbulence Universal?, Phys. Rev. Lett. **101**, 194505 (2008).

[32] W. Schmidt, C. Federrath, M. Hupp, S. Kern, and J. C. Niemeyer, Numerical simulations of compressively driven interstellar turbulence—I. Isothermal gas, Astron. Astrophys. **494**, 127 (2009).

[33] W. Schmidt, *Numerical Modelling of Astrophysical Turbulence* (Springer, Heidelberg, NY, 2014).

[34] Y. Zhou, Rayleigh–Taylor and Richtmyer–Meshkov instability induced flow, turbulence, and mixing. I, Phys. Rep. **720–722**, 1 (2017).

[35] Y. Zhou, Rayleigh–Taylor and Richtmyer–Meshkov instability induced flow, turbulence, and mixing. II, Phys. Rep. **723–725**, 1 (2017).

[36] O. Soulard, Implications of the Monin-Yaglom relation for Rayleigh-Taylor Turbulence, Phys. Rev. Lett. **109**, 254501 (2012).

[37] O. Soulard and B.-J. Gréa, Influence of zero-modes on the inertial-range anisotropy of Rayleigh-Taylor and unstably stratified homogeneous turbulence, Phys. Rev. Fluids **2**, 074603 (2017).

[38] P. Sagaut and C. Cambon, *Homogeneous Turbulence Dynamics* (Springer, Heidelberg, NY, 2018).

[39] S. Kassinos, W. Reynolds, and M. Rogers, One-point turbulence structure tensors, J. Fluid Mech. **428**, 213 (2001).

[40] R. McKeown, A. Pumir, S. M. Rubinstein, M. P. Brenner, and R. Ostilla-Mónico, Visualizing the turbulent energy cascade: Fourier and physical space, arXiv:2204.09023 (2022).

[41] T. Tsuruhashi, S. Goto, S. Oka, and T. Yoneda, Self-similar hierarchy of coherent tubular vortices in turbulence, Philos. Trans. R. Soc. A **380**, 20210053 (2022).

[42] O. Soulard, J. Griffond, B.-J. Gréa, and G. Viciconte, Permanence of large eddies in decaying variable-density homogeneous turbulence with small Mach numbers, Phys. Rev. Fluids **5**, 064613 (2020).

[43] O. Soulard and J. Griffond, Permanence of large eddies in Richtmyer–Meshkov turbulence for weak shocks and high Atwood numbers, Phys. Rev. Fluids **7**, 014605 (2022).

[44] V. Giovangigli, *Multicomponent Flow Modeling* (Springer Science, Birkhäuser Boston, 1999).

[45] F. A. Williams, *Combustion Theory*, 2nd ed. (Benjamin/Cummings, Menlo Park, California, 1985).

[46] P. Arnault and S. Guisset, Chapman-Enskog derivation of multicomponent Navier-Stokes equations, Phys. Plasmas **29**, 090901 (2022).

[47] J. J. Podesta, M. A. Forman, and C. W. Smith, Anisotropic form of third-order moments and relationship to the cascade rate in axisymmetric magnetohydrodynamic turbulence, Phys. Plasmas **14**, 092305 (2007).

[48] C. Federrath, J. Roman-Duval, R. S. Klessen, W. Schmidt, and M.-M. Mac Low, Comparing the statistics of interstellar turbulence in simulations and observations-solenoidal versus compressive turbulence forcing, Astron. Astrophys. **512**, A81 (2010).

[49] D. A. Donzis and S. Jagannathan, Fluctuations of thermodynamic variables in stationary compressible turbulence, J. Fluid Mech. **733**, 221 (2013).

[50] C. Federrath, On the universality of supersonic turbulence, MNRAS **436**, 1245 (2013).

[51] S. Jagannathan and D. A. Donzis, Reynolds and Mach number scaling in solenoidally-forced compressible turbulence using high-resolution direct numerical simulations, J. Fluid Mech. **789**, 669 (2016).

[52] J. Wang, M. Wan, S. Chen, C. Xie, and S. Chen, Effect of shock waves on the statistics and scaling in compressible isotropic turbulence, Phys. Rev. E **97**, 043108 (2018).

[53] D. Donzis and J. John, Universality and scaling in homogeneous compressible turbulence, Phys. Rev. Fluids **5**, 084609 (2020).

[54] G. L. Eyink, Turbulence theory, course notes, https://www.ams.jhu.edu/~eyink/Turbulence/notes.html.

[55] C. Federrath, R. S. Klessen, L. Iapichino, and J. Beattie, The sonic scale of interstellar turbulence, Nat. Astron. **5**, 365 (2021).

[56] J. Panickacheril John, D. A. Donzis, and K. R. Sreenivasan, Solenoidal Scaling Laws for Compressible Mixing, Phys. Rev. Lett. **123**, 224501 (2019).

[57] E. Lindborg, A note on acoustic turbulence, J. Fluid Mech. **874**, R2 (2019).

[58] P. Sagaut, *Large Eddy Simulation for Incompressible Flows: An Introduction* (Springer Science & Business Media, Berlin, Heidelberg, 2006).

[59] E. Garnier, N. Adams, and P. Sagaut, *Large Eddy Simulation for Compressible Flows* (Springer Science & Business Media, Dordrecht, 2009).

[60] G. Dal Maso, P. G. Lefloch, and F. Murat, Definition and weak stability of nonconservative products, J. Math. Pures Appl. **74**, 483 (1995).

[61] C. Parés, Numerical methods for nonconservative hyperbolic systems: A theoretical framework, SIAM J. Numer. Anal. **44**, 300 (2006).

[62] D. L. Sandoval, The dynamics of variable density turbulence, Ph.D. thesis, University of Washington, 1995.

[63] D. Livescu and J. R. Ristorcelli, Variable–density mixing in buoyancy–driven turbulence, J. Fluid Mech. **605**, 145 (2008).