Central mean temperature scaling in compressible turbulent channel flows with symmetric isothermal boundaries

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(Received 22 January 2022; accepted 8 April 2022; published 28 April 2022)

As one of the canonical flow problems in compressible wall-bounded turbulence, compressible turbulent channel flow (CTCF) with symmetric isothermal boundaries has been studied a lot in the past. In the present work, an empirical scaling for the central mean temperature in CTCFs is proposed. The scaling originates from the generalized Reynolds analogy (GRA) theory, and it depends on the Mach number, Prandtl number, and ratio of the central mean velocity to the bulk mean velocity. The available direct numerical simulation data with the bulk Reynolds number ranging from 3000 to 34 000 and Mach number ranging from 0.5 to 4.0 are used to assess the proposed scaling. It is found that the empirical scaling is quite accurate and most of the relative errors are below 1.5%. With the scaling of the central mean temperature and the GRA theory, the mean temperature profile can be quantitatively obtained through the mean velocity in CTCFs.

DOI: 10.1103/PhysRevFluids.7.044606

I. INTRODUCTION

Compressible wall-bounded turbulent flows are of great importance in many aerospace applications. In such flows, the heat transfer is also an important quantity of concern in addition to the aerodynamic force [1–4]. Due to the nonlinear coupling between the kinetic and thermal quantities and the influence of various flow conditions, accurate estimations of the velocity and temperature fields are needed in order to ensure reliable and efficient structures. Since the pioneering work by Reynolds [5], plenty of effort has been devoted to establish quantitative relationships between temperature and velocity based on the similarity between the Reynolds-averaged momentum and energy equations—the so-called Reynolds analogy—in compressible wall-bounded flows [6–12]. For the compressible turbulent boundary layer flows (CTBLs) with zero-pressure gradient (ZPG), Van Driest [8] proposed a temperature-velocity relation with the assumption of unity Prandtl number Pr, which reads

$$\frac{\overline{T}}{\overline{T}_\delta} = \frac{\overline{T}_w}{\overline{T}_\delta} - \left(\frac{\overline{T}_w}{\overline{T}_\delta} - 1\right)\frac{\overline{u}}{\overline{u}_\delta} + \frac{\gamma - 1}{2}\mathrm{Ma}_\infty^2 \frac{\overline{u}}{\overline{u}_\delta}\left(1 - \frac{\overline{u}}{\overline{u}_\delta}\right). \tag{1}$$

Here and thereafter, T is temperature, u is the streamwise velocity component, $\overline{(.)}$ stands for the Reynolds-average operator, γ is the ratio of the specific heats, and the subscript w refers to the wall, while the subscript δ denotes the boundary layer edge, or the center of the channel or pipe for compressible channel or pipe flows, respectively. Ma_∞ is the ratio of the free-stream velocity to the

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sound velocity in the free stream. From the above equation (1), the mean temperature profile can be obtained if the mean velocity profiles, $\overline{T}_\delta$ and $\overline{T}_w$, are known.

Walz [10] also obtained a similar quadratic function of the mean velocity for the mean temperature for ZPG-CTBLs,

$$\frac{\overline{T}}{\overline{T}_\delta} = \frac{\overline{T}_w}{\overline{T}_\delta} + \frac{\overline{T}_r - \overline{T}_w}{\overline{T}_\delta}\left(\frac{\overline{u}}{\overline{u}_\delta}\right) + \frac{\overline{T}_\delta - \overline{T}_r}{\overline{T}_\delta}\left(\frac{\overline{u}}{\overline{u}_\delta}\right)^2, \tag{2}$$

where $T_r = T_\delta\{1 + r[(\gamma - 1)/2]\mathrm{Ma}_\infty^2\}$ is the adiabatic (recovery) temperature, with $r \sim 0.9$ being the recovery factor, which was introduced to account for the deviation of Pr from unity. Although the commonly used Walz's equation (2) was verified to match very well with the direct numerical simulation (DNS) data in adiabatic CTBLs [12–14], its performances in diabatic CTBLs are rather poor, where it apparently deviated from the DNS data [12]. By introducing a general recovery factor r_g, Zhang *et al.* [14] proposed a general Reynolds analogy (GRA) for compressible wall-bounded turbulent flows, which is independent of Pr, wall temperature, Mach number, Reynolds number, and pressure gradient. The mean temperature-velocity relation coming from the GRA theory, which reads

$$\frac{\overline{T}}{\overline{T}_\delta} = \frac{\overline{T}_w}{\overline{T}_\delta} + \frac{\overline{T}_{rg} - \overline{T}_w}{\overline{T}_\delta}\left(\frac{\overline{u}}{\overline{u}_\delta}\right) + \frac{\overline{T}_\delta - \overline{T}_{rg}}{\overline{T}_\delta}\left(\frac{\overline{u}}{\overline{u}_\delta}\right)^2, \tag{3}$$

was verified in CTBLs, compressible turbulent channel and pipe flows. Here, $\overline{T}_{rg} = \overline{T}_\delta + r_g\overline{u}_\delta^2/(2C_p)$, where C_p is the specific heat at constant pressure, and the general recovery factor r_g is estimated as

$$r_g = \frac{\overline{T}_w - \overline{T}_\delta}{\overline{u}_\delta^2/(2C_p)} - \frac{2\mathrm{Pr}}{\overline{u}_\delta}\frac{\overline{q}_w}{\overline{\tau}_w}, \tag{4}$$

with $\overline{q}_w$ and $\overline{\tau}_w$ being the mean heat flux and skin friction at the wall, respectively. If the well-known Reynolds analogy factor $s = 2C_h/C_f$ is introduced, where C_f and C_h are the skin-friction coefficient and the heat transfer coefficient, respectively, r_g could also be calculated as

$$r_g = r\left[s\mathrm{Pr} + (1 - s\mathrm{Pr})\frac{\overline{T}_w - \overline{T}_\delta}{\overline{T}_r - \overline{T}_\delta}\right]. \tag{5}$$

For general compressible wall-bounded turbulence, it is seen from the above mean velocity-temperature relation (3) that the mean temperature depends on the mean velocity profile, $\overline{T}_\delta$, $\overline{T}_w$, and s. In CTBLs, $\overline{T}_\delta$ and $\overline{T}_w$ are usually the input parameters, making s the key parameter in the temperature-velocity relation. However, in compressible turbulent channel flows with symmetric isothermal boundaries (CTCFs), the mean temperature in the channel center is no longer the input parameter, while the Reynolds analogy factor s can be easily obtained through the overall energy balance argument, i.e., $q_w = -u_b\tau_w$ [15,16], where $u_b = \int_0^h \overline{\rho u} dy / \int_0^h \overline{\rho} dy$ is the bulk mean velocity with h being the channel half width. Therefore, the mean temperature in the channel center becomes the key parameter in the temperature-velocity relation in CTCFs since the mean velocity profile is assumed to be prescribed. Furthermore, the central mean temperature is the highest mean temperature in the channel flow with symmetric isothermal boundaries. Its value can also be used to guide the numerical setups for compressible turbulent channel flow with symmetric isothermal boundaries, since at least the central mean temperature could not exceed the upper limit of the Sutherland's law. A similar conclusion can be drawn for compressible pipe flows.

The central mean temperature in CTCFs has rarely been studied in the past, although a lot of DNSs have been carried out. Brun *et al.* [17] derived the mean temperature-velocity relation for laminar channel flows with symmetric isothermal walls by assuming constant viscosity, and reported an increase in the central mean temperature with Mach number according to their large eddy simulation data, which was attributed to the direct heating effect of the compressibility.

Gerolymos and Vallet [18,19] argued that the viscous heating increases with Mach number, and thus inferring the increase in the central mean temperature, scaled with the temperature at the wall, with the centerline Mach number. They reported that the increase is nonlinear, and the ratio of the centerline to wall mean temperature rises sharply when the centerline Mach number exceeds 2. Nevertheless, the results are rather qualitative, and no quantitative scaling has been reported.

In this paper, we propose an empirical scaling for the central mean temperature in compressible turbulent channel flows based on the GRA theory [14]. DNS databases from different groups are used to determine the parameter in the scaling and confirm its accuracy. Combining this scaling and the temperature-velocity relation deduced from the GRA theory, the mean temperature profile can be estimated directly from the mean velocity without any additional information.

II. SCALING OF CENTRAL MEAN TEMPERATURE IN CTCFs

In the GRA theory proposed by Zhang *et al.* [14], the mean temperature in compressible wall-bounded turbulence under the quasiparallel flow approximation and the small turbulence intensity assumption ($u_i u_i \approx u^2 \approx \overline{u}^2 + 2\overline{u}u'$) satisfies a differential equation as

$$\overline{T} - \frac{\overline{u}}{2}\left[\left.\frac{\partial \overline{T}}{\partial \overline{u}}\right|_w + \frac{1}{\overline{\mathrm{Pr}_e}}\frac{\partial \overline{T}}{\partial \overline{u}}\right] = \overline{T}_w, \tag{6}$$

and the authors further assumed that the effective turbulent Prandtl number $\overline{\mathrm{Pr}_e} = 1$. By integrating Eq. (6), the analytical relation between the mean temperature and the mean velocity was deduced, and the relation was well verified *a priori* by Zhang *et al.* [14] in CTBLs, CTCFs, and compressible pipe flows, where $\overline{T}_\delta$, $\overline{T}_w$, and s were obtained from the DNS data.

As mentioned above, Huang *et al.* [15] reported in CTCFs that $q_w = -u_b \tau_w$ based on the overall energy balance argument, i.e., the total heat generation across the channel and the heat transfer into the walls are the same. Therefore, we may have

$$\left.\frac{\partial \overline{T}}{\partial \overline{u}}\right|_w = \frac{[\partial \overline{T}/\partial y]|_w}{[\partial \overline{u}/\partial y]|_w} = u_b \frac{\mathrm{Pr}}{C_p}. \tag{7}$$

By taking Eq. (6) at the center of CTCF and using the above Eq. (7), one may have

$$\overline{T}_c - \frac{\overline{u}_c}{2}\left[u_b \frac{\mathrm{Pr}}{C_p} + \frac{1}{\overline{\mathrm{Pr}_e}}\left.\frac{\partial \overline{T}}{\partial \overline{u}}\right|_c\right] = \overline{T}_w, \tag{8}$$

which may provide an estimation of the central mean temperature. However, it should be noted that due to the symmetry of the channel with the thermal and velocity boundary conditions, $\partial \overline{T}/\partial \overline{u}|_c$ cannot be determined analytically. Nevertheless, considering the fact that $\overline{T}_c$, $\overline{u}_c$, $\partial \overline{T}/\partial \overline{u}|_w$, and $\overline{T}_w$ are all finite for certain cases, we can assume

$$\frac{1}{\overline{\mathrm{Pr}_e}}\left.\frac{\partial \overline{T}}{\partial \overline{u}}\right|_c = \frac{\overline{T}_w}{u_b} C, \tag{9}$$

with C being a dimensionless parameter to be determined. It should be noted that the above Eq. (9) does not imply anything on the functional form of C, but introduces a dimensionless parameter to denote $\frac{1}{\overline{\mathrm{Pr}_e}}\frac{\partial \overline{T}}{\partial \overline{u}}|_c$ with the help of $\overline{T}_w$ and u_b. C will be determined by using the available DNS data later.

By using Eq. (9), Eq. (8) can be rearranged as

$$\frac{\overline{T}_c}{\overline{T}_w} - \frac{\overline{u}_c}{2}\left[u_b \frac{\mathrm{Pr}}{C_p \overline{T}_w} + \frac{1}{u_b} C\right] = 1. \tag{10}$$

With the relation $u_b^2/(C_p \overline{T}_w) = (\gamma - 1)u_b^2/c_w^2 = (\gamma - 1)\mathrm{Ma}^2$, where Mach number $\mathrm{Ma} = u_b/c_w$ is the ratio of the bulk mean velocity to the velocity of sound at the wall, Eq. (10) can then be

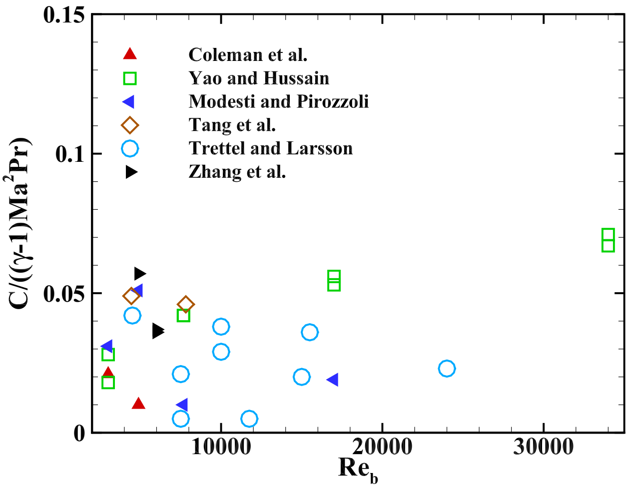

FIG. 1. The distribution of parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ with different Re_b from Table I.

rewritten as

$$\frac{\overline{T}_c}{\overline{T}_w} = 1 + \mathrm{Pr}\frac{\gamma-1}{2}\mathrm{Ma}^2\frac{\overline{u}_c}{u_b}\left(1+\frac{C}{(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}}\right), \tag{11}$$

which can be used to calculate the central mean temperature using Mach number Ma, Prandtl number Pr, central velocity ratio $\overline{u}_c/u_b$, and the parameter C. Equivalently, Eq. (11) can also be rewritten as

$$\frac{\overline{T}_c}{\overline{T}_w} = 1 + r_c\frac{\gamma-1}{2}\mathrm{Ma}^2, \tag{12}$$

where

$$r_c = \mathrm{Pr}\frac{\overline{u}_c}{u_b}\left(1+\frac{C}{(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}}\right) \tag{13}$$

can be named as a recovery factor for the channel's central mean temperature, where $\partial\overline{T}/\partial y = 0$. If r_c is known, then the central mean temperature $\overline{T}_c$ can be easily estimated through Eq. (12).

Now, we would like to use the DNS data to give an estimation of the r_c. First, the parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ was calculated from the available DNS data by utilizing Eq. (11) and the related results are listed in Table I. Here, Re_b, Ma, Pr, and $\overline{T}_c/\overline{T}_w$ are obtained directly from the references (which will result in different precision of the data, as shown in Table I), where $\mathrm{Re}_b = \rho_b u_b h/\mu_{\mathrm{ref}}$ is the bulk Reynolds number with $\rho_b = \int_0^h \overline{\rho} dy/h$ being the bulk-averaged density, and $\overline{u}_c/u_b$ is calculated using the parameters in the references. The DNS database has a bulk Reynolds number Re_b ranging from 3000 to 34 000 and a Mach number Ma ranging from 0.5 to 4.0 [20–25,27]. The distribution of the parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ with different Re_b and Ma is shown in Figs. 1 and 2, respectively. At least two observations could be made from Fig. 1 and Fig. 2. On the one hand, the parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ varies in the range of ~0.005–0.071 for the present available DNS data, and it is small as compared to 1 in Eq. (13). Therefore, $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ could be viewed as a small correction for r_c. On the other hand, the parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ has no obvious dependence on Re_b or Ma, which inspires us to use a constant to approximate it at different Re_b and Ma. This is equivalent to the simplest assumption that the parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ is insensitive to Re_b and Ma. An average of all the data listed in Table I gives a value 0.034 for $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$, resulting in an approximation of $r_c = 1.034\mathrm{Pr}\overline{u}_c/u_b$ and the central mean temperature

TABLE I. Parameters of different DNS data, calculated $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ utilizing Eq. (11), estimated $\overline{T}_c/\overline{T}_w$ utilizing Eq. (14), and the corresponding relative errors.

Case	Re_b	Ma	Pr	$\overline{u}_c/u_b$	$\overline{T}_c/\overline{T}_w$	$C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$	$\overline{T}_{c,\mathrm{cal}}/\overline{T}_w$	Error(%)
Coleman *et al.* [20]	3000	1.5	0.70	1.175	1.378	0.021	1.383	0.36
	4880	3.0	0.70	1.171	2.490	0.010	2.526	1.45
Yao and Hussain [21]	3000	0.8	0.72	1.172	1.11	0.018	1.112	0.18
	7667	0.8	0.72	1.146	1.11	0.042	1.109	0.09
	17000	0.8	0.72	1.133	1.11	0.053	1.108	0.18
	34000	0.8	0.72	1.119	1.11	0.067	1.107	0.27
	3000	1.5	0.72	1.171	1.39	0.028	1.392	0.14
	7667	1.5	0.72	1.155	1.39	0.042	1.387	0.22
	17000	1.5	0.72	1.140	1.39	0.056	1.382	0.58
	34000	1.5	0.72	1.124	1.39	0.071	1.377	0.94
Modesti and Pirozzoli [22]	4880	3.0	0.72	1.153	2.570	0.051	2.545	0.97
	3000	1.5	0.72	1.158	1.387	0.031	1.388	0.07
	7667	1.5	0.72	1.137	1.372	0.010	1.381	0.66
	17000	1.5	0.72	1.115	1.368	0.019	1.374	0.44
Trettel and Larsson [23],Trettel [24]	7500	0.7	0.70	1.189	1.082	0.005	1.084	0.18
	11750	0.7	0.70	1.189	1.082	0.005	1.084	0.18
	4500	1.7	0.70	1.146	1.483	0.042	1.479	0.27
	10000	1.7	0.70	1.145	1.481	0.038	1.479	0.14
	15500	1.7	0.70	1.145	1.480	0.036	1.479	0.07
	7500	3.0	0.70	1.156	2.487	0.021	2.506	0.76
	15000	3.0	0.70	1.156	2.486	0.020	2.506	0.80
	24000	3.0	0.70	1.157	2.491	0.023	2.507	0.64
	10000	4.0	0.70	1.144	3.637	0.029	3.650	0.36
Tang *et al.* [25]	7813	1.56	0.72	1.146	1.42	0.046	1.415	0.35
	4438	3.83	0.72	1.191	3.64	0.049	3.601	1.07
Zhang *et al.* [26,27]	6000	0.5	0.72	1.153	1.043	0.036	1.043	0.00
	6000	1.5	0.72	1.158	1.389	0.037	1.388	0.07
	4880	3.0	0.72	1.186	2.624	0.057	2.589	1.33

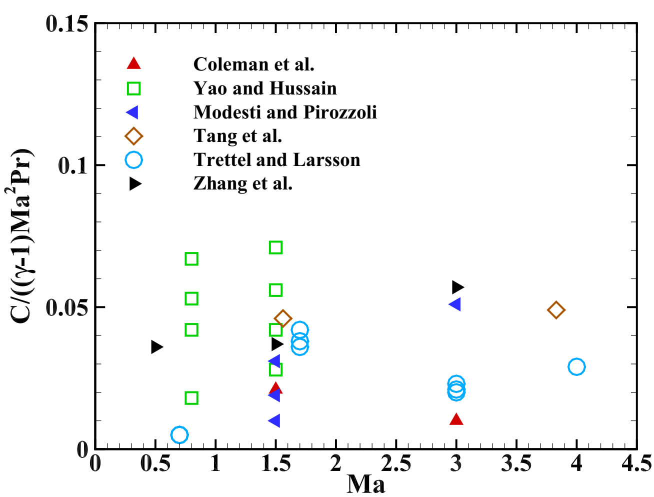

FIG. 2. The distribution of parameter $C/[(\gamma-1)\mathrm{Ma}^2\mathrm{Pr}]$ with Ma from Table I.

TABLE II. $\overline{T}_{c,\text{cal}}/\overline{T}_w$ with different Pr utilizing Eq. (14) and the corresponding relative errors. The reference data are from Gerolymos and Vallet [18].

Ma	$\overline{u}_c/u_b$	$\overline{T}_c/\overline{T}_w$	Pr_1	$\overline{T}_{c,\text{cal}}/\overline{T}_w$	Error (%)	Pr_2	$\overline{T}_{c,\text{cal}}/\overline{T}_w$	Error (%)	Pr_3	$\overline{T}_{c,\text{cal}}/\overline{T}_w$	Error (%)
1.50	1.286	1.417	0.70	1.419	0.14	0.68	1.407	0.71	0.71	1.425	0.56
1.48	1.198	1.379	0.70	1.380	0.07	0.68	1.369	0.73	0.71	1.385	0.44
1.50	1.176	1.383	0.70	1.383	0.00	0.68	1.372	0.80	0.71	1.389	0.43
1.56	1.154	1.403	0.70	1.407	0.29	0.68	1.395	0.57	0.71	1.412	0.64
0.30	1.142	1.015	0.70	1.015	0.00	0.68	1.014	0.10	0.71	1.015	0.00

scaling

$$\frac{\overline{T}_c}{\overline{T}_w} = 1 + 1.034\text{Pr}\left(\frac{\overline{u}_c}{u_b}\right)\frac{\gamma - 1}{2}\text{Ma}^2. \tag{14}$$

III. VALIDATION AND APPLICATION OF THE SCALING

By using Eq. (14), we may estimate the central mean temperature. The estimated $\overline{T}_c/\overline{T}_w$ and the corresponding relative errors to the DNS data are listed in Table I. Here, the relative error is defined as

$$\text{error} = \left|\frac{\frac{\overline{T}_{\text{DNS}}}{\overline{T}_w} - \frac{\overline{T}_{\text{cal}}}{\overline{T}_w}}{\frac{\overline{T}_{\text{DNS}}}{\overline{T}_w}}\right| \times 100\%. \tag{15}$$

According to the data listed in Table I, it is apparent that Eq. (14) can provide excellent estimations on the central mean temperature at different Re_b and Ma, and the relative errors are all within 1.5%. The DNS data from Gerolymos and Vallet [18] are further used to validate our central mean temperature scaling, and the results are shown in Table II. It should be noted that the Prandtl number in Gerolymos and Vallet [18] is not constant, but varying with temperature. For the cases with $0.30 \leqslant \text{Ma} \leqslant 1.56$, Pr varies within [0.68, 0.71]. With a roughly estimated $\text{Pr} \approx 0.7$, the estimated central mean temperature matches with their DNS data very well, with relative errors less than 0.25%. The relative errors will also be less than 0.80% even when $\text{Pr} = 0.68$ or $\text{Pr} = 0.71$. It is interesting to see that the empirical scaling, given by Eq. (14), can also be used in compressible pipe flows with isothermal walls. Table III shows the results of the DNS data from Modesti and Pirozzoli [28] in pipe flows, and it is seen that the predictions of Eq. (14) are quite well and the largest relative error is around 2.67% at $\text{Ma} = 3.0$. Therefore, we may conclude that Eq. (14) is a good scaling for the central mean temperature, not only for channel flows but also for pipe flows.

As discussed above, in CTCFs, if we have the central mean temperature, then the mean temperature profile can be easily obtained from the mean velocity field from the GRA theory, i.e., Eq. (3).

TABLE III. The results of $\overline{T}_{c,\text{cal}}/\overline{T}_w$ with pipe flows utilizing Eq. (14) and the corresponding relative errors.

Case	Ma	Pr	$\overline{u}_c/u_b$	$\overline{T}_c/\overline{T}_w$	$\overline{T}_{c,\text{cal}}/\overline{T}_w$	Error (%)
Modesti and Pirozzoli [28]	1.5	0.71	1.265	1.434	1.418	1.12
	1.5	0.71	1.255	1.414	1.415	0.07
	1.5	0.71	1.236	1.413	1.408	0.35
	3.0	0.71	1.309	2.805	2.730	2.67

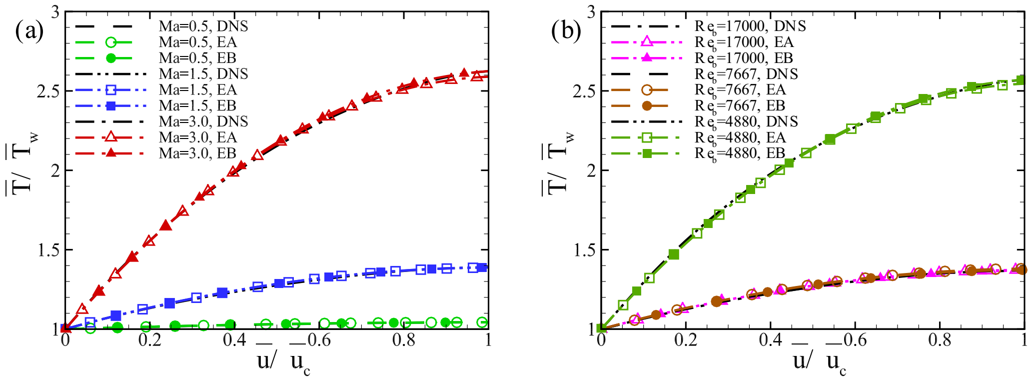

FIG. 3. Mean temperature profiles obtained using the GRA theory at different Ma and Re_b. EA: Eq. (20); EB: Eq. (19), where $\overline{T}_c$ is prescribed using the DNS data. The reference values are the DNS data from (a) Zhang and Xia [26] with Ma = 0.5, 1.5, 3.0 and $\mathrm{Re}_b = 6000, 6000, 4880$; and (b) Modesti and Pirozzoli [22] with Ma = 3.0, 1.5, 1.5 and $\mathrm{Re}_b = 4880, 7667, 17\,000$.

According to the definition of r_g and $\overline{T}_{rg}$, we have

$$r_g = \frac{1 - \frac{\overline{T}_c}{\overline{T}_w}}{\overline{u}_c^2/(2C_p\overline{T}_w)} - 2\mathrm{Pr}\left(\frac{u_b}{\overline{u}_c}\right)\frac{\overline{q}_w}{u_b\overline{\tau}_w} = 0.966\mathrm{Pr}\left(\frac{u_b}{\overline{u}_c}\right) \tag{16}$$

and

$$\frac{\overline{T}_{rg}}{\overline{T}_w} = 1 + \mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2. \tag{17}$$

Therefore, we have the following mean temperature profile:

$$\frac{\overline{T}}{\overline{T}_c} = \frac{\overline{T}_w}{\overline{T}_c} + \frac{\overline{T}_w}{\overline{T}_c}\mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2\left(\frac{\overline{u}}{\overline{u}_c}\right) + \left[1 - \frac{\overline{T}_w}{\overline{T}_c} - \frac{\overline{T}_w}{\overline{T}_c}\mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2\right]\left(\frac{\overline{u}}{\overline{u}_c}\right)^2. \tag{18}$$

Equivalently, we can rewrite the above equation as

$$\frac{\overline{T}}{\overline{T}_w} = 1 + \mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2\left(\frac{\overline{u}}{\overline{u}_c}\right) + \left[\frac{\overline{T}_c}{\overline{T}_w} - 1 - \mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2\right]\left(\frac{\overline{u}}{\overline{u}_c}\right)^2. \tag{19}$$

By using Eq. (14), Eq. (19) can be rearranged as

$$\frac{\overline{T}}{\overline{T}_w} = 1 + \mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2\left(\frac{\overline{u}}{\overline{u}_c}\right) - 0.483\mathrm{Pr}\left(\frac{\overline{u}_c}{u_b}\right)(\gamma - 1)\mathrm{Ma}^2\left(\frac{\overline{u}}{\overline{u}_c}\right)^2, \tag{20}$$

which depends solely on the mean velocity profile (the ratio between the central mean velocity to the bulk mean velocity can be estimated from the mean velocity profile), Pr and Ma. For a CTCF with certain Pr and Ma, if the mean velocity profile is known, which could be obtained through many experimental strategies such as the particle image velocimetry (PIV), the mean temperature profile can then be easily obtained by using Eq. (20).

Figure 3 shows the mean temperature profiles $\overline{T}_c$ obtained using our empirical equation (20) and the GRA equation (19), where $\overline{T}_c$ is obtained from the DNS data directly. The DNS data $\overline{T}_{\mathrm{DNS}}$ from Zhang and Xia [26] with Ma = 0.5, 1.5, 3.0 and $\mathrm{Re}_b = 6000, 6000, 4880$ and Modesti and Pirozzoli [22] with Ma = 3.0, 1.5, 1.5 and $\mathrm{Re}_b = 4880, 7667, 17\,000$ are shown as the references. It is apparent that the mean temperature profiles at different Ma and Re_b obtained using our empirical

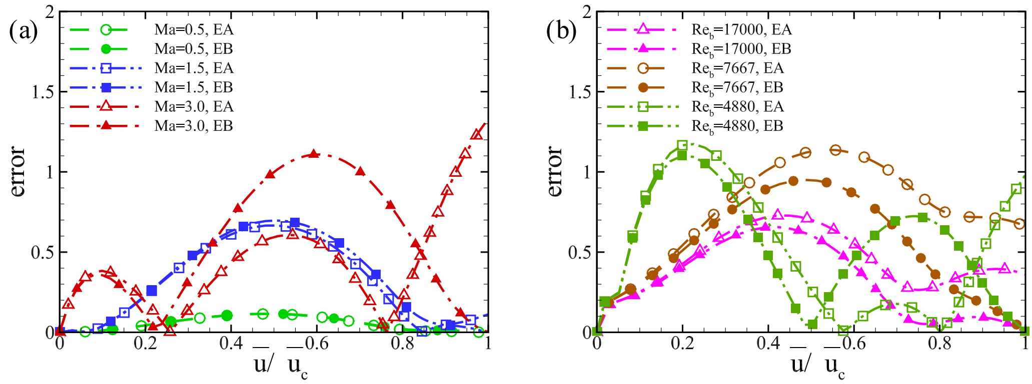

FIG. 4. The corresponding relative errors of the profiles shown in Fig. 3.

equation (20) match very well with those of the reference DNS data and the *a priori* estimations using the GRA equation (19) with the DNS value of $\overline{T}_c$. The corresponding relative errors at different cases are also shown in Fig. 4. It is evident that the relative errors are quite small, and they are all within 1.5% across the channel for the six cases, either for the empirical equation (20) or for the GRA equation (19). This again confirms that the prediction of the mean temperature profile by using the empirical equation (20) in CTCFs is acceptable.

At last, we would like to make several remarks:

(i) As pointed out by one of the anonymous reviewers, the central mean temperature could be nearly identical to the recovery temperature based on the bulk Mach number of the channel. The present empirical scaling indeed confirms this conjecture [see Eqs. (12) and (14)] and also provides an empirical relation for the recovery factor $r_c = 1.034\mathrm{Pr}\overline{u}_c/u_b$. This recovery factor, which is obtained by following the GRA theory and the data calibration, is different from the commonly used one, i.e., $r = \mathrm{Pr}^{1/3}$, in compressible turbulent boundary layers. Furthermore, the present empirical scaling can predict $\overline{T}_c$ with a higher accuracy, especially at higher Ma cases. The largest error of the present scaling is around 1.45%, while the one with $r = \mathrm{Pr}^{1/3}$ can result in an error as large as 5.61%.

(ii) In $r_c = 1.034\mathrm{Pr}\overline{u}_c/u_b$, it is evident that r_c depends on $\overline{u}_c/u_b$. Therefore, if we want to predict $\overline{T}_c$, $\overline{u}_c/u_b$ should be given besides the information of Pr, Ma, and T_w. From this point, the present empirical scaling is not *a priori* one as compared to the recovery temperature in ZPG-CTBLs, where the recovery factor depends solely on Pr. Nevertheless, in the mean temperature-velocity relation, the mean velocity profile is assumed to be given, and thus $\overline{u}_c/u_b$ could also be estimated in advance.

(iii) The small correction $0.034\mathrm{Pr}\overline{u}_c/u_b$ is quite small as compared to the rest of the empirical scaling in the right-hand side of Eq. (14). If this correction is neglected, the scaling is more concise. However, the prediction error will also increase, especially for the cases with higher Ma, and the largest error is around 3.41%, which is more than twice the largest error with the small correction. Since the present scaling is an empirical one, we prefer to keep the small correction due to its higher accuracy.

(iv) In all the DNS data cited in the present work, Pr varies in the small range $[0.7, 0.72]$. Therefore, the applicability of the present scaling should be restricted to the small vicinity of $\mathrm{Pr} = 0.7$. We have carried out an extra DNS with $\mathrm{Pr} = 1$, $\mathrm{Ma} = 1.5$, and $\mathrm{Re}_b = 6000$, and the relative error of the prediction for this case is 3.63%.

(v) Although the present empirical relation can be applied to CTCFs as well as the compressible pipe flows, it is invalid for compressible channel flows with asymmetric isothermal boundary conditions [29] and the compressible channel flows with fixed bulk temperature, where a cooling term is added in the energy equation, as was done in Yu *et al.* [30]. The underlying reason is that the hypotheses under which the empirical scaling was derived were broken at such flow problems.

IV. CONCLUSION

To summarize, we propose an empirical scaling, given by Eq. (14), for the central mean temperature in compressible turbulent channel flows with symmetric isothermal boundaries by following the GRA theory and the DNS data. Our scaling shows that the central mean temperature depends on Pr, Ma, and the ratio between the central mean velocity to the bulk mean velocity. The scaling is quite accurate, with relative errors less than 1.5% at Reynolds number Re_b ranging from 3000 to 34 000 and Mach number Ma ranging from 0.5 to 4.0. With the empirical scaling of the central mean temperature and the GRA theory, an empirical mean temperature profile, Eq. (18) scaled with the central mean temperature and Eq. (20) scaled with the wall mean temperature, can also be derived. The one scaled with the wall mean temperature depends solely on the mean velocity profile, Pr and Ma, and it is quite accurate according to the comparisons with the DNS data at different Re_b and Ma. Further evaluations at higher Re_b and Ma as well as higher Pr are necessary when data become available.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC Grants No. 11822208, No. 91852205, No. 92152101, and No. 92152301), the National Defense Science and Technology 173 Program of Technical Field Fund (Grant No. 2021-JCJQ-JJ-1050), and the Guangdong Provincial Key Laboratory (Grant No. 2019B121203001). We are deeply grateful to the anonymous referees for their professional comments.

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