Approach and separation of bundles of quantized vorticity

George S. E. Grimes* and Andrew W. Baggaley†
Joint Quantum Centre Durham–Newcastle, School of Mathematics and Statistics, University of Newcastle, Newcastle upon Tyne NE1 7RU, United Kingdom

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In the quasiclassical regime of quantum turbulence, it has long been hypothesized that there exist coherent vortical structures, made up of bundles of quantized vortices. More recently, there has been significant experimental evidence that points to their presence. Here, we perform a quantitative study of the reconnection of bundles of quantized vorticity and show that the approach and separation of bundles during a reconnection is consistent with the symmetric $\delta \sim t^{1/2}$ scaling which is consistent with studies of individual quantized vortex reconnection and classical vortex reconnections. We also examined the phenomena of "bridge" structures that form between the vortex bundles during the reconnection process and have also been observed during the reconnection of classical vortices. We study their persistence and suggest that their dissipation is driven by vortex-vortex interactions within the bridge itself.

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I. INTRODUCTION

Theoretical and experimental work exploring turbulence in the quantum phases of ^{4}He, ^{3}He, and atomic Bose-Einstein condensates (BECs), so-called quantum or superfluid turbulence, has attracted a wealth of attention in recent years [1]. Early studies [2–4], focused on finite (nonzero) temperature turbulence in superfluid ^{4}He. This is a complex system in which a viscous normal fluid interacts with an inviscid superfluid, via mutual friction. In the former vorticity is unconstrained; the building blocks of the turbulence are "swirling" motions, eddies, which can take any size and strength. In the superfluid component vorticity can only exist in the form of topological defects, quantized vortex lines, with atomic thickness and fixed circulation Γ, whose value is specific to the atomic properties of the fluid.

As the temperature is decreased towards zero the relative density of the normal fluid vanishes and we are left with a state of pure superfluid turbulence, with vorticity constrained to these line defects. Remarkably, experimental evidence has repeatedly shown that the statistical properties of this turbulence at absolute zero shares many features observed in classical turbulence [5–7]. At first glance this may seem surprising, that vortex stretching, believed to be an important mechanism in the transfer of energy across length scales, is absent as the radius of the superfluid vortex core is fixed [8]. Subsequent theory suggested the appearance of this quasiclassical behavior could plausibly be due to the formation of coherent structures, bundles of quantized vorticity, which could mimic classical vortex stretching [9]. More recently there has been experimental evidence [10] and subsequent analysis [11] that seems to validate this theory. Prior work [12] has also shown that these structures are robust, and are not destroyed due to vortex reconnections.

Reconnections, in this case topological rearrangements of the vorticity field, are another fundamental process in hydrodynamic systems. In a turbulent system they are responsible for randomizing

*g.grimes2@newcastle.ac.uk
†andrew.baggaley@newcastle.ac.uk

the velocity field, play a role in the energy cascade, contribute to the fine-scale mixing, and enhance diffusion [13–15]. In superfluid turbulence they also act to transfer kinetic energy from a three-dimensional hydrodynamic cascade to a one-dimensional Kelvin wave cascade [16]. This ultimately mediates energy transfer to scales where it is dissipated acoustically [17]. Quantum fluids such as superfluid helium and BECs are particularly conducive to the study of reconnections as the interacting filaments are isolated effectively one-dimensional vortex lines which are topological defects of the governing order parameter. Thus, unlike in the continuous fields of classical physics, reconnections in quantum fluids are isolated dramatic events, strongly localized in space and time and hence conceptually and practically easier to study.

Separately, bundles of quantized vorticity and vortex reconnection have received a wealth of attention in the literature [18–25]. However, the study of the reconnection of bundles has not received attention since the work of Alamri *et al.* [12]. Given the recent detection of these structures experimentally [10] we believe it is timely to revisit the problem and extract quantitative information. Indeed, a natural question to ask is how the approach and separation of a vortex bundle during a reconnection compares to a single vortex, by analyzing how the minimum separation between vortices (or vortex bundles) $\delta(t)$ scales in time.

Building on the pioneering work of de Waele and Aarts [20], it is now well established [21,25] that in a system such as turbulent superfluid helium (away from the boundaries), the quantum of circulation dictates the dynamics and a symmetric power-law scaling $\delta(t) \sim t^{1/2}$ is observed as vortices approach, and separate from, a reconnection. We note the recent study of Galantucci *et al.* highlighted a linear scaling regime when extrinsic factors, for example, image vortices arising from external potentials in BEC systems, drive the vortices. Our work here will focus on the regime where a mutual interaction between the vortex bundles drives the reconnection process.

The most pertinent question is, do we see the same separation scaling during the reconnection of a bundle of quantized vortices? Since the formation of bundles allows stretching to occur, as the relative position of vortex strands within a bundle changes during its evolution, it is natural to consider how it compares to the reconnection of classical vortices [26,27]. Thus, the aim of this paper is to revisit the problem of reconnecting bundles of quantized vorticity and track their approach to, and separation from, reconnection.

II. METHODOLOGY

Hereafter we use parameters which refer to superfluid ^{4}He, circulation $\Gamma = 9.97 \times 10^{-4}$ cm^2/s, and vortex core radius $a_0 \approx 10^{-8}$ cm, however, our results can be generalized to turbulence in low-temperature ^{3}He-B, or atomic BECs. Following the seminal work of Schwarz [28], we describe vortex filaments as space curves $\mathbf{s} = \mathbf{s}(\xi, t)$, where ξ is the arclength and t is time. In the absence of mutual friction and of any externally applied superflow, the self-induced velocity of a superfluid filament at the point $\mathbf{s}$ is given by the Biot-Savart law

$$\frac{d\mathbf{s}}{dt} = -\frac{\Gamma}{4\pi} \oint_{\mathcal{L}} \frac{(\mathbf{s} - \mathbf{r})}{|\mathbf{s} - \mathbf{r}|^3} \times \mathbf{dr}. \tag{1}$$

The line integral extends over the entire vortex configuration $\mathcal{L}$, which is discretized into a large number of points $\mathbf{s}_i$ $(i = 1, \dots N)$. The singularity at $\mathbf{s} = \mathbf{r}$ is removed in a standard way by considering local and nonlocal contributions to the integral. If $\mathbf{s}_i$ is the position of the ith discretization point along the vortex line, Eq. (1) becomes [29]

$$\frac{d\mathbf{s}_i}{dt} = \frac{\Gamma}{4\pi} \ln\left(\frac{\sqrt{\ell_i \ell_{i+1}}}{a}\right) \mathbf{s}_i' \times \mathbf{s}_i'' - \frac{\Gamma}{4\pi} \oint_{\mathcal{L}'} \frac{(\mathbf{s}_i - \mathbf{r})}{|\mathbf{s}_i - \mathbf{r}|^3} \times \mathbf{dr}. \tag{2}$$

Here, ℓ_i and ℓ_{i+1} are the arclengths of the curve t between points $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ and between $\mathbf{s}_i$ and $\mathbf{s}_{i+1}$, and $\mathcal{L}'$ is the original vortex line without the section between $\mathbf{s}_{i-1}$ and $\mathbf{s}_{i+1}$.

Vortex lines reconnect [30] when they become sufficiently close to each other, provided that the total length (as a proxy for energy) is reduced [31]. Within the framework of the vortex filament

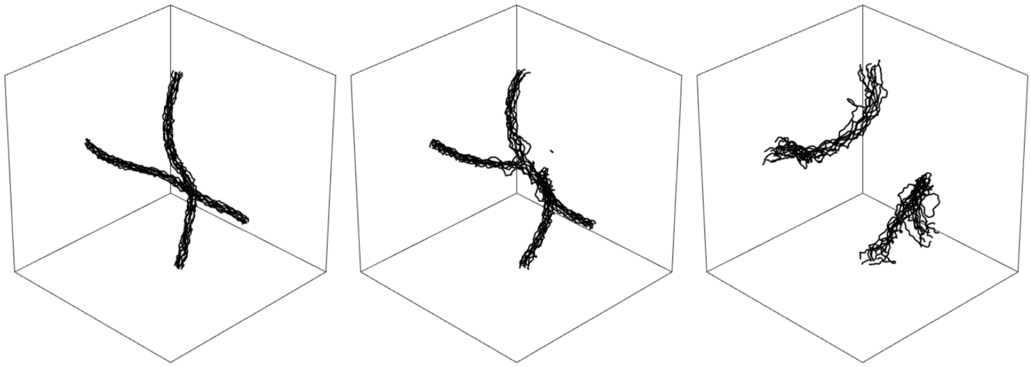

FIG. 1. Snapshots from a numerical simulation of the reconnection of vortex bundles. Here, the initial bundle separation is $\mathcal{D} = 0.1$ cm, the bundle radius is $\mathcal{A} = 0.03$ cm, and the number of vortices in each bundle is $N = 8$.

model (VFM) we use here, an algorithmic approach to incorporate reconnections is required, and its implementation has been described elsewhere in the literature [32]. Likewise for the sake of brevity, other numerical approaches used here including adaptive discretization, adaptive finite-difference schemes, and time stepping can be found in Ref. [33]. We work in a periodic domain of size $D = 1$ cm, the numerical discretization Δs is held between 0.005 and 0.01 cm, and a time step of $\Delta t = 5 \times 10^{-4}$ s is used.

In this work, we study the interaction of two bundles of a given number N of (initially) straight parallel vortex strands, which are initially orthogonal to one another; within a bundle vortices are randomly distributed. This choice of initially orthogonal vortices is perhaps the most well-studied reconnection process [22,24,25]; moreover, numerical results [34] suggest this is the dominant process in quasiclassical quantum turbulence. We denote the initial separation between the axes of the two bundles to be $\mathcal{D}$ and $\mathcal{A}$ be the half width of each bundle. Figure 1 shows the evolution of the reconnection process with $N = 8$, $\mathcal{D} = 0.1$ cm, and $\mathcal{A} = 0.03$ cm, and is qualitatively comparable with the results presented in Ref. [12]. In order to go further and track the dynamics of the vortices we must be able to track the axes of the two bundles as they evolve in time. To do this we follow the approach developed in Ref. [19] to define a coarse-grained vorticity by smoothing the singular vorticity field onto a three-dimensional grid using a cubic spline.

Thus, we define a smoothed vorticity field $\boldsymbol{\omega}(\mathbf{x})$ using a cubic-spline kernel with finite support. In the smoothed particle hydrodynamics (SPH) literature the kernel used here is typically denoted the M_4 kernel [35], and allows us to construct $\boldsymbol{\omega}$ through

$$\boldsymbol{\omega}(\mathbf{x}) = \kappa \sum_{j=1}^{N} \mathbf{s}'_j W(r_{ij}, h) \Delta\xi_j, \tag{3}$$

where $r_{ij} = |\mathbf{x} - \mathbf{s}_j|$, $\Delta\xi_j = |\mathbf{s}_{j+1} - \mathbf{s}_j|$, $W(r, h) = g(r/h)/(\pi h^3)$, h is a characteristic length scale, and

$$g(q) = \begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3, & 0 \leqslant q < 1, \\ \frac{1}{4}(2-q)^3, & 1 \leqslant q < 2, \\ 0, & q \geqslant 2. \end{cases} \tag{4}$$

We discretize $\boldsymbol{\omega}$ on a grid of size M^3 over the entire computational domain. Figure 2 shows a snapshot of the initial configuration of vortices with $N = 6$, $D = 0.2$ cm, and $\mathcal{A} = 0.03$ cm, overlaid with a translucent isosurface of the smoothed vorticity field $\boldsymbol{\omega}$. From this it is straightforward to identify the axis of the vortices as these are associated with the grid points with the largest magnitude

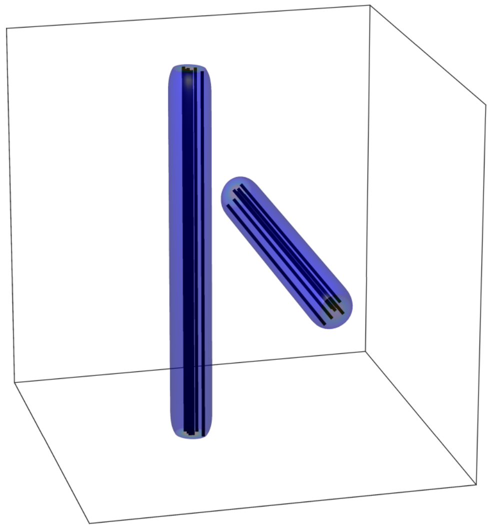

FIG. 2. The initial condition used in our simulations, two straight orthogonal bundles of quantized vortices, where the initial bundle separation is $\mathcal{D} = 0.1$ cm, the bundle radius is $\mathcal{A} = 0.03$ cm, and the number of vortices in each bundle is $N = 6$. Overlaid is an isosurface in blue showing a level set of constant vorticity obtained through our kernel smoothing procedure [see Eq. (3)].

of vorticity. Due to the compact support of the kernel, Eq. (4), it is also feasible to compute $\boldsymbol{\omega}$ on a grid of size $M = 64^3$ at frequent points during the evolution of the bundles' approach and separation during a reconnection. From this we can extract the key quantity of interest, the bundle separation $\delta(t)$.

III. RESULTS

The two quantities of interest in this study are the role the number of vortices (N) within a bundle and the initial width of the bundle ($2\mathcal{A}$) both play in determining the dynamics of the reconnection. In order to make progress we introduce a dimensionless distance $\delta^\star = \delta/\mathcal{D}$ and time $t^\star = t/\tau$, where τ is based on the azimuthal velocity of the bundle,

$$\tau = \frac{\mathcal{D}}{v_\theta} = \frac{2\pi\mathcal{D}\mathcal{A}}{N\Gamma}. \tag{5}$$

Preliminary simulations confirmed our results were essentially independent of the initial separation (assuming this was large enough that the scaling of the approach of the bundles could be studied) and so we fix $\mathcal{D} = 0.1$ cm for all simulations presented here. For a given parameter of interest we perform ten simulations, with all parameters fixed, but with the position of the vortices within the two bundles randomized, the results presented in the paper are then ensemble averaged over these ten simulations.

The first results we shall present have a fixed bundle radius $\mathcal{A} = 0.03$ cm, and varying number of vortices in the bundle N. Figure 3 shows the approach and separation in unscaled (inset) and dimensionless units. We see that the choice of nondimensionalization based on the azimuthal velocity of the bundle leads to the collapse of the data onto a single curve. Note there is a minimum separation between the bundles that can be resolved due to both the reconnection scheme and our approach to identify the center of vorticity.

To further our understanding of the reconnection of bundles of quantized vortices we seek to gain an understanding of the functional form for $\delta^\star(t)$, and in particular if we observe the symmetric power-law scaling $\delta^\star(t) \sim t^{1/2}$ which has been observed in numerous studies of

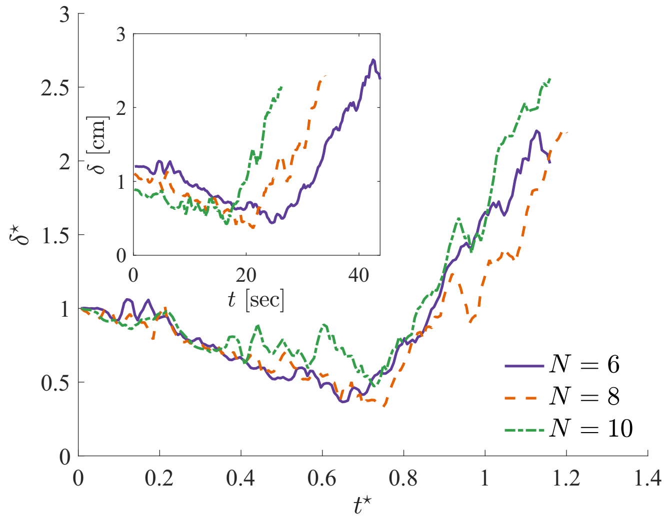

FIG. 3. The main panel shows the nondimensionalized minimum distance between the two bundles $\delta^\star$ as a function of the rescaled time $t^\star$ through the reconnection process; the inset shows the corresponding unscaled plot. In all cases the bundle radius is fixed at $\mathcal{A} = 0.03$ cm with the number of vortices within a bundle N varying.

single-vortex reconnection. Our results are presented in Fig. 4, with the separation of the vortices displayed in the main panel and the approach as an inset. Both appear to be consistent with the $t^{1/2}$ scaling, emphasizing the conclusions of Ref. [12], that bundles of quantized vortices remain coherent structures. Our results mean we can take this statement further and show bundles of quantized vortices approach and separate a reconnection as intense classical vortex structures [27].

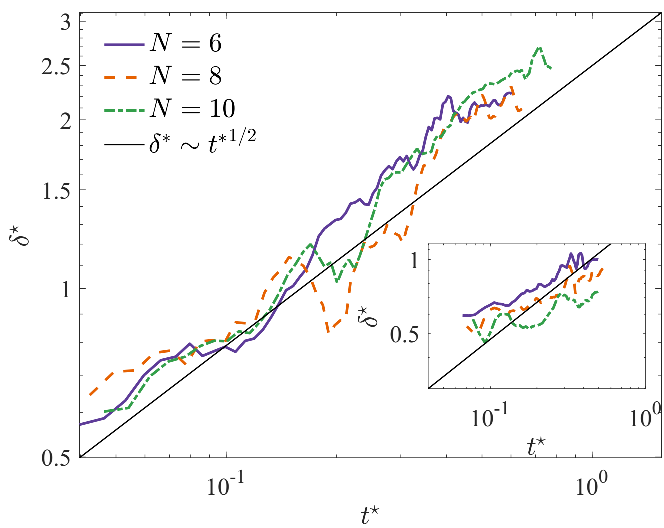

FIG. 4. A log-log plot of the data presented in Fig. 3. The inset shows the approach to reconnection and the main panel shows the subsequent separation of the bundles. Both approach and separation are consistent with the time-symmetric $t^{1/2}$ scaling observed for single quantized vortices [25] and classical vortex tubes [27].

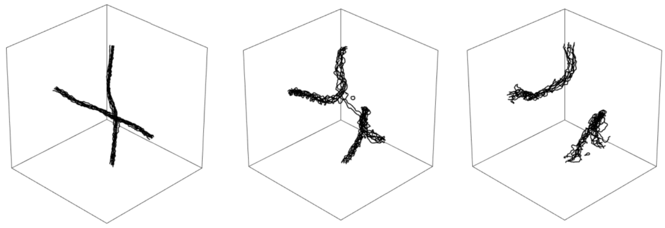

FIG. 5. Snapshots from a numerical simulation of the reconnection of vortex bundles. The left panel shows the vortices before reconnection, and in the central panel a "bridge" structure is clearly visible, which forms during the reconnection process, before dissipating due to internal reconnections (right panel). Here, the initial bundle separation is $\mathcal{D} = 0.1$ cm, the bundle radius is $\mathcal{A} = 0.03$ cm, and the number of vortices in each bundle is $N = 8$.

Interestingly, a further comparison with the reconnection of classical vortices can be drawn. In Ref. [26] an interesting feature, denoted "bridges," is thin strands of vorticity which form during the reconnection and remain attached to the main vortical structures as they move away from one another after the reconnection. We see analog structures which form during the reconnection of bundles of quantized vorticity. Such bridges are clearly visible in Fig. 5, and in a typical simulation they form during the reconnection process, and persist for a period of time before dissipating due to reconnections between the strands which form the bridge, which may generate a number of small vortex rings [36].

We are motivated to understand if their lifetime is dependent on the number of vortices within a bundle, which for a fixed bundle radius $\mathcal{A}$ dictates the timescale for the rotation of the bundle [see Eq. (5)]. Recall for each value of N we have ten simulations of the bundle reconnection, and for each we record the lifetime of the bridge, which we denote t_b. If the breakdown of the bridge is caused by a vortex interaction within this structure, driving a Crow instability [37], we would expect t_b to be independent of N. Instead, if it is driven by the turbulent motions within the bundles

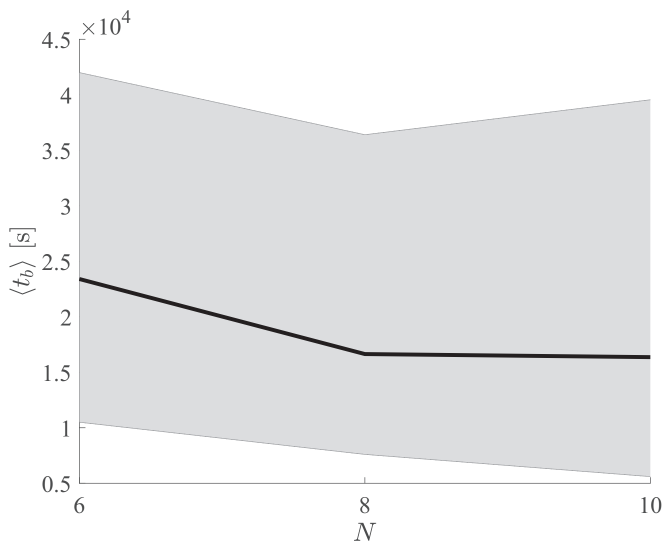

FIG. 6. The average bridge duration $\langle t_b \rangle$, where angle brackets denote ensemble averaging, over ten simulations of bundle reconnections with a varying number of vortices N in each bundle. The shaded area depicts the range of t_b over these simulations.

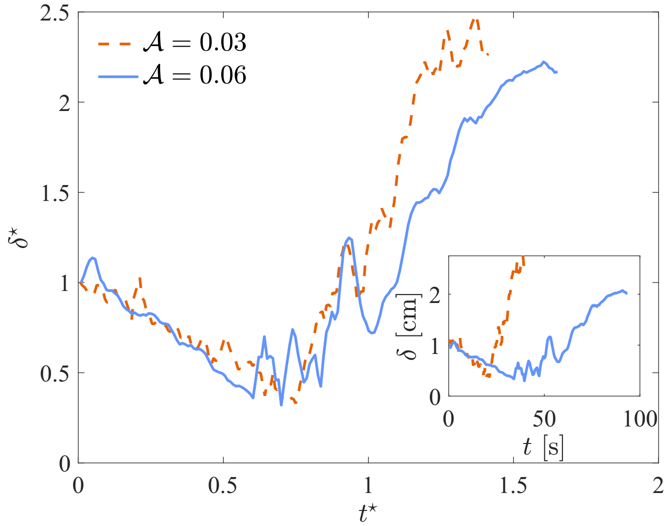

FIG. 7. The main panel shows the nondimensionalized minimum distance between the two bundles $\delta^\star$ as a function of the rescaled time $t^\star$ through the reconnection process; the inset shows the corresponding unscaled plot. In both curves the number of vortices within a bundle is fixed at $N = 8$ with the radius of a bundle $\mathcal{A}$ varying.

we may expect t_b to be related to N. In Fig. 6 we present evidence that the bridge lifetime is plausibly independent of N and thus a vortex-vortex interaction within the bridge is most likely responsible for its breakdown.

The final question we seek to address here is the role of the bundle width. Our motivation is twofold: First, it provides a secondary test of the nondimensionalization presented in Eq. (5). Primarily, however, we seek a comparison with the results of Hussain's two studies of classical vortex reconnection [26,27]. In their first work [26], with relatively diffuse vortex structures they

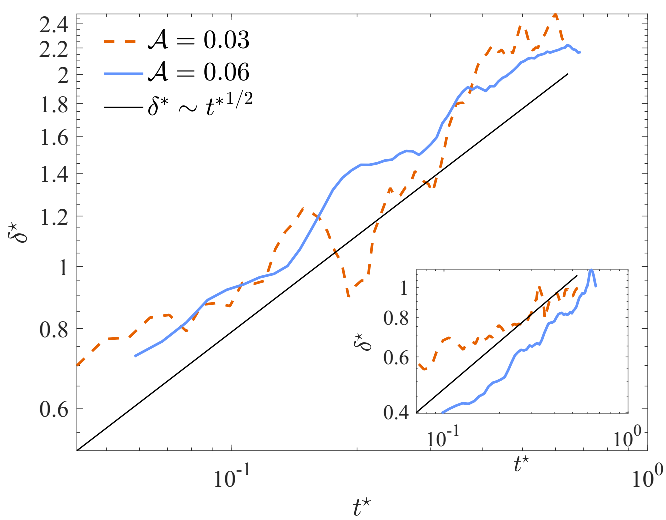

FIG. 8. A log-log plot of the data presented in Fig. 7. The inset shows the approach to reconnection and the main panel shows the subsequent separation of the bundles.

observed asymmetry in the approach and separation of the vortices to reconnection, with scalings of $\delta \sim t^{3/4}$ prior to and $\delta \sim t^2$ after reconnection. In a further study [27] with substantially more concentrated vortical filaments they recovered the symmetric $t^{1/2}$ scaling widely observed in the quantum fluid literature. Hence a natural question is if we increase the core size of the bundles $\mathcal{A}$, do we observe a transition to a different scaling regime? From a suite of ten simulations with $\mathcal{D} = 0.1$ cm and $\mathcal{A} = 0.06$ cm, twice as large as the previous simulations with $N = 8$, we see no evidence of a different scaling.

Indeed, in Fig. 7 we demonstrate that the nondimensionalization presented in this paper leads to a reasonable collapse of the data onto a single curve. Figure 8 shows that for both bundle radii considered the reconnection scaling is consistent with the symmetric $t^{1/2}$ scaling. We attempted simulating the reconnection of bundles with larger widths $\mathcal{A} \approx 0.1$ cm, however, identifying the center of vorticity for these very diffuse bundles proved challenging and the results were dominated by noise.

We also note that Hussain and Duraisamy [26] observed the asymmetric scalings during the reconnection of antiparallel vortex tubes. It would therefore be of interest to study the role of the geometry of the initial condition and its affect on both the scaling laws and bridge formation in a future study.

IV. SUMMARY

To conclude, we have presented a quantitative study of the reconnection of bundles of quantized vorticity. The approach and separation of bundles during a reconnection is consistent with the symmetric $\delta \sim t^{1/2}$ scaling which has been observed in a large number of studies of quantized vortex reconnection [25] and a recent study of classical vortex reconnections [27]. We demonstrate a simple nondimensionalization based on the azimuthal velocity within the bundle captures much of the dynamics of the reconnection process. Finally, we examined the persistence of "bridge" structures which form between the vortex bundles during the reconnection process. Our results suggest that their dissipation is driven by vortex-vortex interactions within the bridge following a Crow-like mechanism.

Given recent experimental evidence for the existence of the coherent vortex structures in quantum turbulence, understanding their interactions is fundamental to our understanding this unique hydrodynamic system; our results present a step in this direction. One important aspect of the reconnection of bundles which we did not address here are topological changes that occur in the reconnection process. Visually the bundles after reconnection appear more complex with the appearance of braided structures. This may prove an opportunity to draw parallels with studies of magnetohydrodynamics (MHD) reconnection [38] in the future.

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