Activity induced turbulence in driven active matter

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Turbulence in driven stratified active matter is considered. The relevant parameters characterizing the problem are the Reynolds number $\mathrm{Re} = \bar{u}L/\nu$ and an active matter Rayleigh number, $R = |\zeta|S^2L/D\bar{u}$. Here $\bar{u}$ is the mean velocity of flow, L is the system size, ν is the kinematic viscosity, ζ is the activity coefficient, S is the concentration gradient, and D is the active matter diffusion coefficient. In the *mixing* limit, $\mathrm{Re} \gg 1, R \ll 1$, we show that the standard Kolmogorov energy spectrum law, $E(k) \propto k^{-5/3}$, is realized. On the other hand, in the *stratified* limit, $\mathrm{Re} \gg 1, R \ll 1$, there is a new turbulence universality class with $E(k) \propto k^{-7/5}$. The crossover from one regime to the other is discussed in detail. Experimental predictions and probes are also discussed.

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I. INTRODUCTION

Over the last couple of decades, a new variety of hydrodynamic instabilities and turbulence has been extensively studied [1–5] in the context of "active matter" hydrodynamics (e.g., bacteria swimming in a fluid). The earliest such studies involved "active nematics" [6–8] where a transition from a quiescent state to a spontaneously flowing state was predicted by Simha and Ramaswamy [9] and observed by Voituriez *et al.* [10]. A large number of interesting results have also been obtained on turbulence in living fluids (e.g., Wensink *et al.* [11]). The special feature of active matter hydrodynamics is that, because of its own energy source, active matter can introduce large additional stress terms in the Navier Stokes equation. The usual stress term for the velocity ($\vec{u}$) dynamics of an incompressible fluid is $T_{\alpha\beta} = -p\delta_{\alpha\beta} + \eta(u_{\alpha,\beta} + u_{\beta,\alpha})$ where "p" is the pressure, $u_{\alpha,\beta} = \partial_\beta u_\alpha$, and η is the shear viscosity. The additional contribution to $T_{\alpha\beta}$ because of the active matter can take different forms depending on what is being studied. A particular form [12] of this extra term is reminiscent of the model "H" among the different universality classes of dynamic critical phenomena [3–16]. The form of the contribution in the lowest nontrivial order allowed by symmetry considerations is a nonlinear Burnett term [17,18] and can be written as

$$\Sigma_{\alpha\beta} = -\zeta\left(\partial_\alpha\phi\partial_\beta\phi - \frac{\delta_{\alpha\beta}}{3}(\nabla\phi)^2\right), \tag{1}$$

where $\phi(\vec{r}, t)$ is the concentration of the active matter and ζ is a constant which can be termed the activity coefficient. It should be noted that unlike inactive matter, the activity coefficient in dimensionless units need not be small. The dynamics of the system in the presence of statistical forcing for both velocity and concentration fields at large distance scales can be described by

$$\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p + \nu \nabla^2 u_\alpha + \partial_\beta \Sigma_{\alpha\beta} + f_\alpha, \tag{2a}$$

$$\partial_\alpha u_\alpha = 0, \tag{2b}$$

$$\partial_t \phi + u_\alpha \partial_\alpha \phi = D \nabla^2 \phi + g. \tag{2c}$$

The incompressibility condition is expressed by Eq. (2b). The statistical forces $\vec{f}(\vec{r}, t)$ and $g(\vec{r}, t)$ are Gaussian noise terms specified by zero mean and nonzero two-point correlation function. This is precisely as in the pioneering work of De Dominicis and Martin [19] and followed up extensively by Yakhot and Orszag [20,21] and by Smith and Reynolds [22]. We focus on this particular model as it allows for the existence of an inertial range of wave numbers which do not contribute to the total energy input or dissipation as opposed to the active nematics [23] or systems with more complicated couplings in the fluid flow dynamics [24,25].

A driven version of the above model was introduced in Ref. [26] where a fixed concentration gradient of the active matter was maintained across two parallel surfaces separated by a distance L. It was shown that for negative values of ζ, an instability sets in when the dimensionless quantity $N = |\zeta| S^2 L^2 / D\nu$, which we will call the active matter Rayleigh number, exceeds a critical value $N_c = 4\pi^2$. This instability is analogous to the convective instability in a fluid heated from below. It was shown by Das *et al.* [27] that if the active matter Rayleigh number is increased beyond N_c, then a cascade of period-doubling bifurcations occur, culminating in a chaotic state for $N \approx 50 N_c$.

In this work we show that a new universality class for turbulence can be induced in this system by tuning the active matter Rayleigh number to values such that the parameter $R = N/\mathrm{Re} = |\zeta| S^2 L / D\bar{u}$ becomes much larger than unity. The quantity Re is the usual Reynolds number defined by $\mathrm{Re} = \bar{u} L / \nu$ where $\bar{u}$ is mean flow velocity and ν is the kinematic viscosity. Conventional Kolmogorov turbulence occurs for $\mathrm{Re} \gg 1$, $R \ll 1$. In this work we concentrate on the new regime characterized by $R \gg 1$, i.e., $N \gg \mathrm{Re}$ (the Reynolds number Re is still greater than unity) and find that.

(i) For $R \gg 1$, the Kolmogorov 5/3 law changes to a 7/5 law for the energy spectrum, i.e., $E(k) \propto k^{-7/5}$ in the inertial range of wave numbers, setting up a new universality class of turbulence. This result is obtained in two ways: first via a scaling argument and then from the dynamical equations themselves.

(ii) For a given value of R, the new spectrum is seen for wave numbers $k \ll k_1$ and the Kolmogorov spectrum for $k \gg k_2$. Denoting the Prandtl number (ν/D) by σ, the wave number k_1 is found to be proportional to $(\zeta S^2/\sigma)^{5/4}$ and the wave number k_2 to $(\zeta S^2/\sigma)^{3/2}$. Clearly for very large values of $\zeta S^2/\sigma$ the spectrum is almost entirely the 7/5 variety, and for very small values of $\zeta S^2/\sigma$, the spectrum is almost entirely Kolmogorov like. This result is obtained from the dynamical equations. We emphasize that the new scaling law for the turbulent energy spectrum is relevant at large length scales (small wave number) compared to length scales where the Kolmogorov scaling holds and that is what makes it different from normal stratified fluid turbulence.

It is interesting to note a special feature of this driven active model H by contrasting it with the convective fluid system where a fluid layer is subjected to an adverse temperature gradient. For the latter case, using a three-mode model (a three-dimensional dynamical system for the convecting fluid layer), Lorenz [28] found that as the gradient reaches a critical value the dynamics becomes extremely sensitive to initial conditions (chaotic dynamics). In real experiments, however, the dynamics evolves from stationary-in-time states to time periodic states (Hopf bifurcation) followed by more complicated time dependences before making a transition to a chaotic state. The subsequent discovery of dynamical systems showing period-doubling [29], intermittency [30], inherent instability [31] of more than two incommensurate frequency states had a very strong impact on hydrodynamic turbulence. These chaotic systems exhibited a complicated time dependence with the Fourier spectrum showing a continuous distribution of frequencies [32] but the spatial dynamics was ordered and characterized by only a few length scales and therefore not turbulent which requires a very large or infinite number of length scales.

Turbulence had long been a difficult problem to handle as it involved not only the complicated time dependence characteristic of chaotic systems but also involved an infinite number of length scales from the smallest (scale of viscous dissipation) to the largest (scale over which energy was supplied to the system, e.g., the length L introduced above). It has always been a challenge to find a physical system which, by changing a few parameters, can be taken from a zero velocity nonequilibrium steady state (NESS) to a nontrivial steady state followed by a passage to chaos through a sequence of instabilities modeled by an appropriate low-dimensional dynamical system and then to a fully turbulent state by a further manipulation of the parameters.

To the best of our knowledge the driven active model H is the first example where one can study the passage to chaos (increase the value of N at a low Reynolds number) and then study the passage to turbulence by either increasing the Reynolds number way beyond the Rayleigh number or by increasing the Rayleigh number way beyond the Reynolds number, which for us is still high enough to ignore the effects of the viscous drag. In Sec. II, we obtain the scaling laws in the Kolmogorov limit (small values of R) and a new scaling regime in the large R limit. In Sec. III, we discuss the rather unexpected nature of crossover between the two scaling regimes. We conclude with a short summary and the possibility of experimental investigation in Sec. IV.

II. THE SCALING LAWS

A turbulent state [33–35] in a homogeneous isotropic fluid is generally observed at very high values of the Reynolds number Re. In steady state turbulence, the simplest and one of the most well-known results is the 5/3 law of Kolmogorov [36,37]. The steady state means that the amount of energy (ε) introduced in unit time at large length scales is dissipated in unit time at the short viscous scales. In the intermediate length scales (smaller than the typical system size and larger than the viscous boundary layer thickness) the inducted energy cascades from the large length scales to small length scales at a constant rate ε independent of the scale. This defines the inertial range. The total kinetic energy per unit mass of the system defines the energy spectrum $E(k)$ by the relation

$$E = \frac{1}{2V}\int d^3r\langle u_\alpha(\vec{r})u_\alpha(\vec{r})\rangle = \int \frac{d^3k}{(2\pi)^3}\langle u_\alpha(\vec{k})u_\alpha(-\vec{k})\rangle = \int E(k)dk. \tag{3}$$

The assumption made by Kolmogorov was that the energy spectrum $E(k)$ is determined by ε and k. A dimensional analysis leads to $E(k) \propto k^{-5/3}$, the 5/3 law [36]. For the case of the stratified fluid when the anisotropy is still not too big, the 5/3 changes to 11/5 as first predicted by Bolgiano [38] and Obukhov [39].

For the active stratified fluid, we begin by rewriting Eqs. (2a) and (2c) in terms of variables which are centered around the NESS, characterized by $\vec{u} = 0$, constant pressure, and a concentration distribution $\phi_0(\vec{r})$ which is written as $\phi_0(\vec{r}) = \phi_{00} + Sz$, where ϕ_{00} is a constant. We use the variable $\psi(\vec{r}, t) = \phi(\vec{r}, t) - \phi_0(\vec{r})$ and introduce the curl-free vector field $\vec{B} = \vec{\nabla}\psi$. Our interest being in the inertial range where the distance scale is always much larger than the viscous scale, the $\vec{B}$ field can be considered small (small means small compared to S and hence in what follows the B field that will be written is actually B/S and we can linearize in it to arrive at the system (we write the fields in wave-number space to facilitate calculations later)

$$\partial_t u_\alpha(k) + M_{\alpha\beta\gamma}(k)\int \frac{d^3p}{(2\pi)^3}u_\beta(p)u_\gamma(k-p) = -iS^2\zeta[\delta_{\alpha 3}k_\beta B_\beta(k) - k_3 B_\alpha(k)] - \eta k^2 u_\alpha(k) + f_\alpha(k), \tag{4a}$$

$$M_{\alpha\beta\gamma}(k) = \frac{i}{2}[k_\beta P_{\alpha\gamma}(k) + k_\gamma P_{\alpha\beta}(k)], \quad k_\alpha u_\alpha(k) = 0, \tag{4b}$$

$$\partial_t B_\alpha + ik_\alpha \int \frac{d^3p}{(2\pi)^3}u_\beta(p)B_\beta(k-p) = -Dk^2 B_\alpha(k) + iS^2 u_3 k_\alpha + ik_\alpha g(k). \tag{4c}$$

The tensor $P_{\alpha\beta}(k) = \delta_{\alpha\beta} - (\frac{k_\alpha k_\beta}{k^2})$ in Eq. (4b) above stands for the projection operator.

We need to focus on the conserved quantity in the inviscid, unforced limit ($\eta = D = \vec{f} = g = 0$). We define the total energy per unit mass for the active fluid as

$$E = \int \frac{d^3p}{(2\pi)^3}[u_\alpha(p)u_\alpha(-p) + \zeta S^2 B_\alpha(p)B_\alpha(-p)]. \tag{5}$$

The difference from the Kolmogorov situation of Eq. (3) is that the energy, in addition to the usual kinetic energy term, has a term which we label as the potential energy. It is the total energy given above which is dissipated at large values of the wave vector by the viscosity η and the concentration diffusion D. To keep the total energy constant in the presence of the dissipative terms, we have added the small (in wave-vector) scale noise terms $\vec{f}(k,t)$ and $g(k,t)$ in Eqs. (2a)–(2c). The time derivative of the total energy is seen to be [using Eqs. (2a)–(2c) in Fourier space]

$$\dot{E} = -\int \frac{d^3p}{(2\pi)^3} p^2[2\nu u_\alpha(p)u_\alpha(-p) + 2DB_\alpha(p)B(-p) - \{u_\alpha(p)f_\alpha(-p) + \phi(p)g(-p) + \text{c.c.}]. \tag{6}$$

In the inviscid and unforced limit, the energy E is a conserved quantity. From the right-hand side of Eq. (6), it is seen that the energy input is at small p (large length scales) and the dissipation is at large p (small length scales). What is very important to note is that even if there is no external stirring force in the velocity equation (i.e., $f = 0$), it is possible to have a steady state due to the "stirring action" in the dynamics of the B field alone. This is the signature that the B field can generate and sustain the velocity field under the appropriate circumstances. We take this as the signature for active matter.

The rate of energy flow across a given wave number in the inertial range is given by

$$\begin{aligned}\varepsilon(k) &= \partial_t \int_0^k \frac{d^3p}{(2\pi)^3}[u_\alpha(p)u_\alpha(-p) + \zeta S^2 B_\alpha(p)B_\alpha(-p)] \\ &= 2\int_0^k \frac{d^3p}{(2\pi)^3}[\dot{u}_\alpha(p)u_\alpha(-p) + \zeta S^2 \dot{B}_\alpha(p)B_\alpha(-p)].\end{aligned} \tag{7}$$

When the applied concentration gradient is small, the second term is negligible and the energy spectrum $E(k)$ is the Kolmogorov variety. Our primary interest is in the limit when the second term is dominant. In this case we have the interesting situation where the energy spectrum, which is by definition the kinetic energy, is dominated by the flux of the potential energy. To proceed further, we need to first discuss the different length scales (inverse of the momentum or wave-number scales) in the problem.

The largest length scale is the extension L of the system. The shortest length scale is the Kolmogorov scale l_K beyond which the viscous dissipation controls the dynamics. This is known to be $l_K = (\nu^{3/4}/\varepsilon^{1/4})$. In the situation considered, because of the diffusion of the concentration field, there will be an analogous length scale $D^{3/4}/\varepsilon^{1/4}$. However, the turbulent Prandtl number ν/D is close to unity and hence we will not treat this scale separately. There is the larger length scale l_R, determined by the Reynolds number. This is known to be $l_R = (\text{Re})^{3/4} l_K$. For Kolmogorov turbulence, the range of length scales where the energy cascades to lower and lower length scales without loss is set by the condition $l_R \gg k^{-1} \gg l_K$. A significant inertial range is obtained for $\text{Re} \gg 1$. In our case, we have a new length scale which is independent of those discussed so far. This can be constructed from the activity coefficient ζ, which from Eqs. (1) and (2a) has the dimension L^{10}/T^2, and the energy transfer rate ε. We denote this scale by l_ζ and find by dimensional analysis that $l_\zeta = (\zeta/\varepsilon^{2/3})^{3/26}$. The new regime of turbulence that we are putting forward is relevant at large length scales provided $l_\zeta > (\text{Re})^{3/4} l_K$. Thus we have a sequence of scales $L > l_\zeta > l_t > l_K$. The new scaling behavior $E(k) \propto k^{-7/5}$ is dominant for $l_\zeta > k^{-1} > l_t$ and will cross over to the $E(k) \propto k^{-5/3}$ spectrum for $l_t > k^{-1} > l_K$. We are now in a position to redo the Kolmogorov scaling argument.

To do this we need to find the scaling dimension of the B field. It is simplest to do so by considering Eq. (4a) in real space and looking at the linear terms where the acceleration is driven by the gradient of the B field. The scaling dimension is the dimension that leaves the real-space version unchanged when the length scale changes by a factor α, i.e., when we consider the transformation $l \to \alpha l$. If time scales under this transformation as α^z, then the acceleration scales as α^{1-2z} and hence B as α^{2-2z}. The quantity $\varepsilon(k)$ consequently scales as α^{4-5z}. For the flux to be k independent we need $z = 4/5$. The energy spectrum $E(k)$ has the dimension L^3/T^2 and thus scales as l^{3-2z}, which is k^{2z-3}. Using $z = 4/5$, we have the spectrum $E(k) \propto k^{-7/5}$ for $\zeta \gg 1$. Denoting the B field flux by ε_B, we have the kinetic energy spectrum given by

$$E(k) = K' \varepsilon_B^{2/5} k^{-7/5}. \tag{8}$$

In Eq. (8) above the universal numerical constant K′ is the analog of the Kolmogorov constant for the usual turbulence. Note that if $S = 0$, the same sort of arguments lead to the Kolmogorov result $E(k) \propto k^{-5/3}$.

Next we show that the dynamics specified by Eqs. (4a)–(4c) is consistent with the conclusion above. This can be seen by working directly in the approximation $\zeta S^2 \gg 1$, where the nonlinearity in Eq. (4a) is overwhelmed by the ζ-containing term and thus $u_\alpha(k,t) = -i\zeta S^2(\delta_{\alpha 3}k_\beta - \delta_{\alpha\beta}k_3)\int_0^t dt' G_u(k,t-t')B_\beta(k,t')$, where $G_u(k,t)$ is the dressed propagator for the velocity field. To obtain the dynamics of $B_\alpha(k,t)$, we write the solution of Eq. (4c) as

$$B_\alpha(k,t) = i\int_0^t dt' G_B(k,t-t')\int \frac{d^3p}{(2\pi)^3} k_\alpha u_\beta(p,t')B_\beta(k-p,t'). \tag{9}$$

The dressed propagator $G_B(k,t)$ is written in wave-number frequency space by the usual Dyson equation $G_B{}^{-1}(k,\omega) = G_0^{-1}(k,\omega) + \Sigma_B(k,\omega)$ where the dressed one-loop self-energy is given by

$$\Sigma_B(k,\omega) = k^2 \iint \frac{d\omega'}{2\pi}\frac{d^3p}{(2\pi)^3} G_B(p,\omega')C_u(k-p,\omega-\omega'). \tag{10}$$

In the above equation, $C_u(k,\omega)$ is the velocity correlation function. We now match the scaling dimensions of either side. The inverse of the Green's function and the self-energy scale as the frequency and hence behave as k^z. The equal time B-field correlation function $\int d\omega\, C_B(k,\omega)$ is taken to scale as k^{-n}. The relation between the velocity and the B field shown above gives the scaling dimension of $\int d\omega\, C_u(k,\omega)$ as $k^{-n+2-2z}$. The matching of scaling properties of the two sides of Eq. (10) yields $4z = 7-n$.

We need the energy transfer rate $\varepsilon(k)$ of Eq. (7) at the lowest dressed order of perturbation theory in the $\zeta \gg 1$ limit. We drop the first term in the right-hand side of Eq. (7) and use the nonlinear term of Eq. (4c) to obtain

$$\varepsilon(k) = -2i\zeta\left\langle \int_0^k \frac{d^3p}{(2\pi)^3} p_\alpha \int \frac{d^3q}{(2\pi)^3} B_\alpha(-p,t)B_\beta(p-q,t)u_\beta(q,t)\right\rangle, \tag{11}$$

where the angular brackets stand for the average over the random forcing term in Eq. (4c). Because of the dynamics involved in the term $B_\beta(p-q,t)$ of Eq. (11) above, we write from Eq. (4c)

$$B_\beta(p-q,t) = i\int_0^t dt' G_B(p-q,t-t')\int \frac{d^3l}{(2\pi)^3}(p-q)_\beta u_\gamma(l,t')B_\gamma(p-q-l,t'). \tag{12}$$

We substitute for $B_\beta(p-q,t)$ in Eq. (11) from Eq. (12) and use the connection between u_α and B_β given above Eq. (9) to arrive at

$$\varepsilon(k) = \int_0^k \frac{d^3p}{(2\pi)^3}\int_0^t dt' \int \frac{d^3q}{(2\pi)^3}\int \frac{d^3l}{(2\pi)^3} p_\alpha (p-q)_\beta G_B(p-q,t-t') \times \langle B_\alpha(-p,t)B_\gamma(p-q-l,t')\rangle\langle u_\beta(q,t)u_\gamma(l,t')\rangle. \tag{13}$$

Using the relation between the velocity field and the B field, and the Kolmogorov condition of $\varepsilon(k)$ being independent of k leads to $3z + 2n = 10$. Combining with the other relation between the two exponents found from the self-energy consistency leads to $z = 4/5$, $n = 19/5$. The relation between the velocity and the B field now gives $E(k)$ with $E(k) \propto k^{-7/5}$ once again. In terms of the wave number and energy flux, we have the new scaling regime where the energy spectrum follows a 7/5 law.

III. THE CROSSOVER

Having established that for $S^2\zeta \gg 1$, the turbulent kinetic energy spectrum follows a new scaling law with the dynamical equations, we now ask the question of how the crossover from Kolmogorov variety to this new variety of turbulence occurs as the control parameter N is varied. We begin by recalling the Heisenberg [40,41] theory of turbulence and the extension of it by Chandrasekhar [42] to account for the crossover from the Kolmogorov regime to the viscosity-dominated regime. Heisenberg began by noting that the energy transfer due to viscosity from small to large wave numbers across a given wave number k in Eq. (4a) is given by $-\eta \int_0^k \frac{d^3p}{(2\pi)^3}\langle u_\alpha(p)u_\alpha(-p)\rangle = -\eta \int_0^k dp\, p^2E(p)$. In analogy with this he decided to write the transfer caused by the nonlinear term in Eq. (4a) as a nonlocal effective viscosity term given by $-\eta_{\rm eff}(k)\int_0^k dp\, p^2E(p)$. Since the transfer occurs to all scales larger than k, Heisenberg used dimensional arguments to write the $\eta_{\rm eff}$ (eddy viscosity) in terms of $E(p)$ and p as

$$\eta_{\rm eff}(k) = \int_k^\infty \frac{dp}{p}\sqrt{\frac{E(p)}{p}}. \tag{14}$$

In our case, in addition to the viscous dissipation there is an additional term from the concentration diffusion. The total dissipation for us is $\int \frac{d^3p}{(2\pi)^3}[\eta\langle u_\alpha(p)u_\alpha(-p)\rangle + D\zeta S^2\langle B_\alpha(p)B_\alpha(-p)\rangle]$. The first term of the dissipation is automatically $-\eta\int E(p)dp$ and is treated as above. The second term will be written in an analogous fashion. We need to write the second term in the dissipation in terms of a $D_{\rm eff}$ analogous to the $\eta_{\rm eff}$ of Eq. (14) and express the B-field correlation function in terms of $E(p)$ and p by a dimensional analysis. This makes its contribution of the form $\nu_{\rm eff}\frac{\zeta S^2}{\sigma}\int p^3E^2(p)dp$ where $\sigma = \nu_{\rm eff}/D_{\rm eff}$ is the turbulent Prandtl number and is assumed to be a simple number. Absorbing all numerical factors of $O(1)$ in $\nu_{\rm eff}$ and σ, we write the energy flux in the inertial range as

$$\varepsilon(k) = \eta_{\rm eff}(k)\left[\int_0^k E(p)p^2dp + \frac{\zeta S^2}{\sigma}\int_0^k E^2(p)p^3dp\right] = \eta_{\rm eff}(k)y(k) \tag{15}$$

It is convenient to define a function $g(k)$ by the relation

$$g(k) = k\frac{dy}{dk} = E(k)k^2 + \frac{\zeta S^2}{\sigma}E^2(k)k^3. \tag{16}$$

This leads to

$$\frac{2\zeta}{\sigma}kE(k) = \sqrt{1 + 4\frac{g(k)}{k}\frac{\zeta S^2}{\sigma}} - 1. \tag{17}$$

Substituting for $\eta_{\rm eff}(k)$ in Eq, (15) from Eq, (14) and imposing the scale-independent energy flux condition, i.e., $\varepsilon(k)$ is independent of k, gives

$$\int_k^\infty \sqrt{\frac{E(p)}{p^3}}dp = y(k)\sqrt{\frac{E(k)}{k^3}}\frac{1}{k^2E(k) + \frac{\zeta}{\sigma}k^3E^2(k)} = \frac{y(k)}{k^2g(k)}\left(\sqrt{1 + \frac{4\zeta S^2}{\sigma}\frac{g(k)}{k^2}} - 1\right)^{1/2}. \tag{18}$$

Further, the left-hand side of Eq, (18) can be rewritten as an integral over $y(p)$ and thus Eq, (18) becomes

$$\int_{y(k)}^{\infty} \frac{dy(p)}{pg(p)} \left(\sqrt{1 + \frac{4\zeta S^2}{\sigma} \frac{g(p)}{p^2}} - 1 \right)^{1/2} = \frac{y(k)}{kg(k)} \left(\sqrt{1 + \frac{4\zeta S^2}{\sigma} \frac{g(k)}{k^2}} - 1 \right)^{1/2}. \tag{19}$$

A derivative with respect to $y(k)$ now yields the exact differential equation satisfied by $g(k)$ and hence by $E(k)$. The most relevant information about the crossover can, however, be extracted from Eq. (19) itself. The vital point about the crossover is that it is not determined by $\zeta S^2/\sigma$ alone as could have been naively expected but by the quantity $\xi = \zeta S^2 g(k)/\sigma k^2$, showing that along with ζ/σ, the wave number plays a very important role in the crossover. For $\xi \ll 1$, the energy spectrum is Kolmogorov, while when it is much greater than 1 it is the new variety $E(k) \propto k^{-7/5}$ established in Eq. (7). For $\xi \ll 1$, Eq. (19) becomes $\int_{y(k)}^{\infty} \frac{dy(p)}{p^2\sqrt{g(p)}} = \frac{y(k)}{k^2\sqrt{g(k)}}$ and a derivative with respect to $y(k)$ leads to

$$\frac{dg}{dy} - 4\frac{g}{y} + 4 = 0. \tag{20}$$

The solution corresponds to the Kolmogorov spectrum in the inertial range as shown by Chandrasekhar [42]. Since the above equation corresponds to a solution $g(k) \propto k^{4/3}$, the condition $\xi \ll 1$ holds for wave numbers k which are larger than k_2, which is proportional to $(\zeta S^2/\sigma)^{3/2}$.

The limit $\xi \gg 1$, on the other hand, leads to $\int_{y(k)}^{\infty} \frac{dy}{p^{3/2}g(p)^{3/4}} = \frac{y(k)}{k^{3/2}g(k)^{3/4}}$. The analog of Eq. (20) is

$$\frac{dg}{dy} - \frac{8g}{3} = -2. \tag{21}$$

Remembering $\frac{dy}{dk} = \frac{g}{k}$ implies $\ln k = \int \frac{dy}{g}$, we can integrate the above equation to obtain $g(k) = \frac{6k^{6/5}}{5(1+\beta k^2)^{8/5}}$ where β is a constant of integration. Since our concern is with the inertial range, the wave number k can be considered smaller than the scale $\beta^{-1/2}$ (dissipative scale for the concentration fluctuations) and we have $g(k) \propto k^{6/5}$. From Eq. (16), this yields $E(k) \propto k^{-7/5}$ for $\zeta S^2/\sigma \gg 1$. More accurately, we need $\frac{\zeta S^2}{\sigma}\frac{g(k)}{k^2} \gg 1$, which requires $k \ll k_1$ where $k_1 \propto (\zeta S^2/\sigma)^{5/4}$. Clearly for $\zeta S^2/\sigma > 1$, we have $k_2 > k_1$. Hence, the final picture that emerges is that for wave numbers $k < k_1$, the spectrum is purely of the $k^{-7/5}$ variety and for $k > k_2$ it is of the Kolmogorov variety. The region between k_1 and k_2 corresponds to the crossover from one scaling to another.

We conclude by pointing out that if measurements are done in real space, then the relevant quantity is the two-point correlation function $S_2(r) = \langle[\vec{u}(\vec{r} + \vec{x}) - \vec{u}(\vec{x})]^2\rangle$. This can be obtained from the Fourier transform of $E(k)$ by the relation

$$S_2(r) = 4\int_0^{\infty} E(k)\left[1 - \frac{\sin kr}{kr}\right]dk. \tag{22}$$

The scaling behavior of the energy spectrum leads to the conclusion that at short length scales the correlation function will behave as $r^{2/3}$ (Kolmogorov), while at larger length scales the scaling relation will be $S_2(r) \propto r^{2/5}$. In the case of buoyancy driven turbulence, the Kolmogorov regime is obtained at large length scales and the short scales lead to the Bolgiano-Obukhov scaling of $r^{6/5}$. This crossover was found numerically and experimentally by Kunnen *et al.* [43]. For this case of driven active matter turbulence, we predict that the Kolmogorov behavior will be seen at short spatial scales and an $r^{2/5}$ behavior at large length scales.

IV. CONCLUSION

We have considered an active matter suspension described by a model H stress tensor [13]. We envisage a situation where a gradient of the active matter concentration is maintained across a fluid layer. It was shown earlier that this can show convective instabilities [26] as the gradient reaches a threshold value. It was further shown [27] that increasing the gradient can lead to period-doubling instabilities. Here we show that increasing the gradient further (keeping the Reynolds number low) can induce a turbulent flow of the suspension where the usual kinetic energy spectrum will have a non-Kolmogorov spectrum given by $E(k) \propto k^{-7/5}$. At low gradients and high Reynolds number, the usual Kolmogorov ($k^{-5/3}$) spectrum holds. An unusual feature of the crossover is that it is determined not by the gradient alone but by a combination of the gradient S and a function of the wave number k.

We would like to point out that the recent experimental advances for generating linear concentration gradients of chemoattractants in a channel [44–46] makes it possible to test our predictions. It is possible that experimentally it would be easier to probe the concentration fluctuations and thus the correlations of the $\vec{B}$ field. To this end, we point out that a simple dimensional argument yields the "potential energy" spectrum, $E_B(k)$, defined by $\int E_B(k)dk = V^{-1}\int d^3r\langle B_j(r)B_j(r)\rangle$. We find that $E_B(k) \propto k^{-7/5}$ if the kinetic energy flux dominates and $E_B(k) \propto k^{-9/5}$ if the kinetic energy flux dominates. The crossover is now reversed. The Kolmogorov mechanism now holds at the higher wave numbers. In real space the two-point function will scale as $r^{2/5}$ at short distance scales and as $r^{4/5}$ at larger ones.

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